2022/11/22

Tuesday, November 22, 2022 11:22 AM

$x \to \xi$

Easier way to find Gauss quadrature points

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

$\xi = 0$

$\frac{1}{2}(3\xi^2-1) = 0 \rightarrow \xi = \mp \frac{1}{\sqrt{3}}$

$\xi = 0 \;\&\; \xi = \mp\sqrt{\frac{3}{5}}$

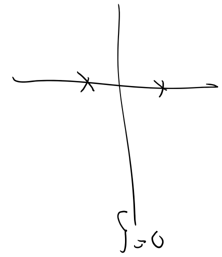


Figure 4: Legendre polynomials (Source: http://en.wikipedia.org/wiki/Legendre_polynomials

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

$\mp \frac{1}{\sqrt{3}}$ $\qquad 8/9 \qquad \sqrt{\frac{3}{5}} \qquad 5/9$

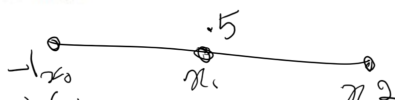
$\xi_1 = -.577 \ldots \qquad \omega_1 = 1$

$\xi_2 = + \;\; " \qquad \omega_2 = 1$

ME, AE, BME517: Finite Elements for Engineering Applications HW5: 1D Quadrature

1. **50 Points** Use a 3 point Gauss and 5 point Newton-Cotes quadrature rule to evaluate the following integral and obtain their respective errors with respect to exact value of the integral $I_e = \tan^{-1}(2) - \tan^{-1}(-1)$. Quadrature points and weights are given in fig. 1.

$$I = \int_{-1}^{2} \frac{dx}{1+x^2} \qquad f(x) = \frac{1}{1+x^2}$$



In the class   3 point NC $\rightarrow (2-(-1))\left(\frac{1}{6}f(x_0)+\frac{4}{6}f(x_1)+\frac{1}{6}f(x_2)\right)$

2 point G $\rightarrow$

Intervals, i	No. of Points, n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	2	1/2	1/2			(trapezoid rule)		
2	3	1/6	4/6	1/6		(Simpson's 1/3 rule)		
3	4	1/8	3/8	3/8	1/8	(Simpson's 3/8 rule)		
4	5	7/90	32/90	12/90	32/90	7/90		
5	6	19/288	75/288	50/288	50/288	75/288	19/288	
6	7	41/840	216/840	27/840	272/840	27/840	216/840	41/840

$\xi_1 = -1 \rightarrow x_1 = -1$ ξ_1 ξ_2

$\xi_2 = 1 \rightarrow x_2 = 2$

$x = x_1 N_1(\xi) + x_2 N_2(\xi)$

$= -1\left(\frac{1-\xi}{2}\right) + 2\left(\frac{1+\xi}{2}\right) = .5 + 1.5\xi$

$J = \frac{dx}{d\xi} = 1.5$

$$I = \int_{-1}^{1} f(x)\, dx = \qquad \frac{dx}{d\xi}d\xi \;\; J$$

$$= \int_{-1}^{1} f(x(\xi))\, 1.5\, d\xi = 1.5\int_{-1}^{1} g(\xi)\, d\xi$$

$g(\xi) = \frac{1}{\ldots}$

$\int_{-1}^{1}$

$$g(\xi) = \frac{1}{1+(.5+1.5\xi)^2}$$

$$= 1.5\left(\omega_1 g(\xi_1) + \omega_2 g(\xi_2)\right)$$

$\xi_1 = -x$, $\xi_2 = x$ $\quad \omega_1 = \omega_2$

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
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0.90617 98459 38664	0.23692 68850 56189

$\xi_1 = \frac{-1}{\sqrt{3}}$ $\quad \xi_2 = \frac{1}{\sqrt{3}}$

Summary of NC & Gauss quadrature

O: polynomial order

n: # of quadrature pts $\quad n \longrightarrow O$

	# quadrature		polynomial order that can be integrated exactly
Newton Cotes	n	$\xrightarrow{n \text{ unknowns}}$	$O = \begin{cases} n-1 & \text{even} \\ n & \text{odd} \end{cases}$
Gauss Quadrature	n	$\xrightarrow{2n \text{ unknowns}}$	$O = 2n-1$

polynomial order O: $\underline{\alpha_0 + \alpha_1\xi + \cdots + \alpha_O \xi^O}$ $\quad$ O+1 coefficients

We generally want to go $O \rightarrow n$

eg polynomial order given $\longrightarrow$ find the # quadrature points

NC

	n	O
odd	1	O = n = 1
even	2	O = n-1 = 1
odd	3	O = n = 3
	4	O = n-1 = 3

$O \rightarrow n$: $\quad n = \begin{cases} O & O \text{ odd} \\ O+1 & O \text{ even} \end{cases}$

Gauss Quadrature

O	n

Gauss Quadrature

$$o = 2n - 1 \longrightarrow n = \text{Ceiling}\left(\frac{o+1}{2}\right)$$

o	n
1	1
2	~~1.5~~ → 2
3	2

Important $\quad o \longrightarrow n$

NC $\quad n = \begin{cases} o & o \text{ odd} \\ o+1 & o \text{ even} \end{cases}$

in FEM often we deal with even order

(*) G $\quad n = \text{ceil}\left(\frac{o+1}{2}\right)$

How do we use * in FEM integrations:
From last time we had

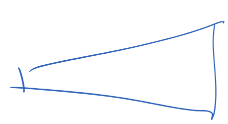
$$k^e = \int_{-1}^{1} \frac{EA(\xi)}{J(\xi)} \underbrace{\begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi - \frac{1}{2} & \xi + \frac{1}{2} & -2\xi \end{bmatrix}}_{\text{why order} = 2} d\xi$$

$J(\xi) \rightarrow 2\xi(x_{ave} - x^3) + \frac{L^e}{2}$

Component 11 of this matrix
$= (\xi - \frac{1}{2})(\xi - \frac{1}{2}) \qquad o = 1+1 = 2$

When AE is not constant?

A:



E: $\quad E(x)$ inhomogeneous

$J(\xi) = 2\xi(x_{ave} - x^3) + \frac{L^e}{2}$

$= \frac{L^e}{2}$ if $x^3 = x_{ave}$ (Element is not distorted)

If we ignore the form of AE and J, we deal with a matrix of polynomials of order o = 2.

NC

$o = 2 \qquad o$ even $\qquad n = o+1 = 3$

of polynomials of order o = 2.

NC

o = 2   o even $n = o+1 = 3$

G.

$n = \text{ceil}\left(\frac{o+1}{2}\right) = 2$

Full integration order

Order of the integrand when AE and J are constant **(treating the material as homogeneous, linear, constant section, not distorted, …)**

For this example, full integration order = 2

If these assumptions hold, the integral is EXACT.

AE = constant

J = const ($x^3 = x_{ave}$)

$$K = \frac{AE}{L}\begin{bmatrix} 7/3 & -8/3 & 1/3 \\ & 7/3 & -8/3 \\ \text{sym} & & 16/3 \end{bmatrix}$$

Q1: How many quadrature points are needed to integrate K exactly when J is not constant?

With no number of points, we can integrate K exactly in that case

Q2: When **AE and/or J are not constant**, how many quadrature points should we use then?

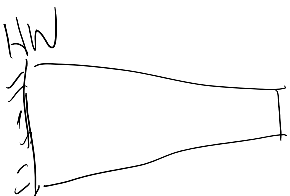
- The **full integration order is sufficient** and used in practice unless the element is highly distorted (here x3 far from xave), or we deal with nonlinearities, …
- Reason: Because the error we introduce by not integrating k exactly is not going to dominate discretization error (C h^(p + 1))

Q3: Are their benefits to using fewer quadrature points than full integration order:
Yes

- Reducing assembly computational time because this cost is directly proportional to the number of quadrature points in the domain.
- We make the problem (structure) less stiff (more compliant)

FEM stiffer than the exact solution

HW



FEM $k = 1.5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $k_{exact} = 1.4\ldots\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ exact

$> 1.4\ldots$

real deflection is higher

FEM based design may not be safe / acceptable

Reduced order integration addresses this to some extend by reducing the stiffness

I'll discuss how low we can go with reduced order integration later.

Now lets calculate k^e by quadrature

$$k^e = \underbrace{\int_{-1}^{1} \frac{AE(\xi)}{J(\xi)} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \underbrace{\left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi \right]}_{B_\xi}}_{I(\xi) \text{ integrand}} d\xi$$

$$J(\xi) \rightarrow 2\xi(x_{are} - \bar{x}) + \frac{L^e}{2}$$

What's the full integration order here & in general for polynomial order p:

			general
Here $p=2$	$N = \left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2 \right]$	order 2	p
$L_m = \frac{d}{dx} \rightarrow B_\xi = \frac{dN}{d\xi}$	$= \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi \right]$	order $2-1$	$p-1$

Integrand

$$\int \frac{AE}{J}(\xi) \underbrace{\underset{\text{order } p-1}{B_\xi^t(\xi)} \; \underset{\text{order } p-1}{B_\xi(\xi)}}_{\text{order} = 2(p-1)} d\xi$$

$$\boxed{\begin{array}{l} \text{Full integration order} = 2(p-1) \\ n \text{ for Gauss Quadrature} = \text{ceil}\left(\frac{\text{order}+1}{2}\right) \\ \qquad = \underline{p} \qquad (\text{eg. } 2 \text{ for } p=2) \\ n \text{ for NC} : \text{order} + 1 \\ \qquad = \underline{2p-1} \qquad (\text{eg } 3 \text{ for } p=2) \end{array}}$$

Calculate K for p $\longrightarrow \begin{cases} n=2 & \text{Gauss} \\ n=3 & \text{NC} \end{cases}$

Gauss quadrature

$$k^e = \int_{-1}^{1} \underbrace{\frac{AE(\xi)}{J(\xi)} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \underbrace{\left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi \right]}_{B_\xi}}_{I(\xi) \text{ integrand}} d\xi$$

$$J(\xi) \rightarrow 2\xi(x_{are} - \bar{x}) + \frac{L^e}{2}$$

$n = 2$ $\quad \xi_1 = \frac{-1}{\sqrt{3}}$ $\quad w_1 = 1$

$\xi_2 = \frac{1}{\sqrt{3}}$ $\quad w_2 = 1$

$$k^e = w_1 I(\xi_1) + w_2 I(\xi_2) = \underbrace{I\left(-\frac{1}{\sqrt{3}}\right) + I\left(\frac{1}{\sqrt{3}}\right)}_{\text{Gauss}} \quad \text{(Gauss)}$$

NC $\quad n=3$ $\quad$ Simpson's rule

$w_1 = \frac{1}{6}$ $\quad w_2 = \frac{4}{6}$ $\quad w_3 = \frac{1}{6}$

$\xi_1 = -1$ $\quad \xi_2 = 0$ $\quad \xi_3 = 1$

$$k^e = \underbrace{(1-(-1))}_{\xi\text{- domain size}} \left(w_1 I(\xi_1) + w_2 I(\xi_2) + w_3 I(\xi_3) \right)$$

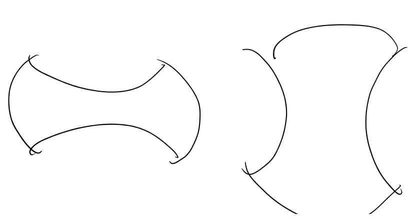
$\rightarrow$ Not included in Gauss quadrature

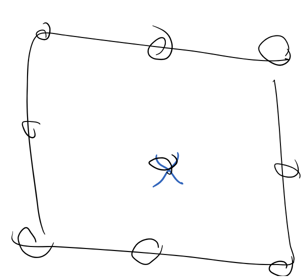
$$k^e = 2\left(\frac{1}{6} I(-1) + \frac{4}{6} I(0) + \frac{1}{6} I(1) \right) = \underbrace{\frac{1}{3} I(-1) + \frac{4}{3} I(0) + \frac{1}{3} I(1)}_{\text{NC}}$$

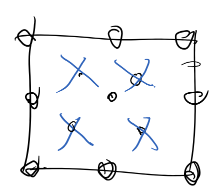
Going back to reduce integration order, why sometimes it doesn't work

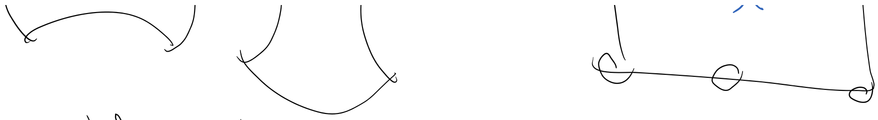
full quadrature

It can result in nonphysical solutions by introducing nonphysical ZERO ENERGY MODES

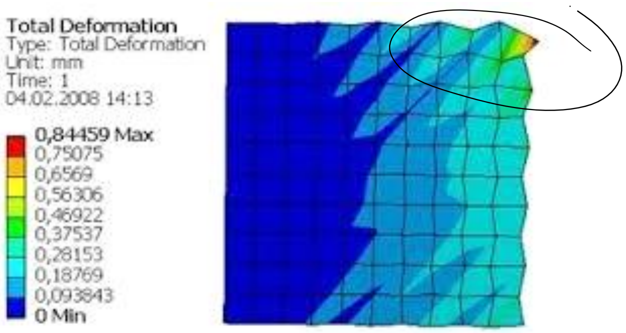


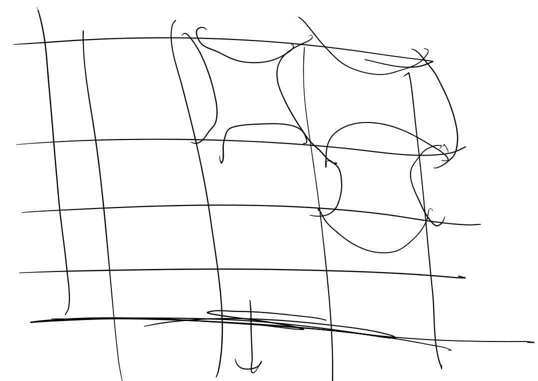






deformations
result in zero energy





zero
modes
limited