# 2022/11/29

Tuesday, November 29, 2022    11:15 AM

| Gauss Points (± $x_i$) | Weights ($w_i$) |
|---|---|
| n = 2 | |
| 0.57735 02691 89626 | 1.00000 00000 00000 |
| n = 3 | |
| 0.00000 00000 00000 | 0.88888 88888 88888 |
| 0.77459 66692 41483 | 0.55555 55555 55555 |
| n = 4 | |
| 0.33998 10435 84856 | 0.65214 51548 62546 |
| 0.86113 63115 94053 | 0.34785 48451 37454 |
| n = 5 | |
| 0.00000 00000 00000 | 0.56888 88888 88889 |
| 0.53846 93101 05683 | 0.47862 86704 99366 |
| 0.90617 98459 38664 | 0.23692 68850 56189 |

→ number

$\xi$

$\xi_1 = 0.339 \ldots\ldots 56$   $w_1 = .65 \ldots 46$

$\xi_2 = -0.339 \ldots\ldots 56$   $w_2 = "$

$\xi_3 = -0.86113 \ldots\ldots 94053$   $w_3 = .347 \ldots 54$

$\xi_4 = + \ldots$   $w_4 = "$

odd cases zero is not repeated

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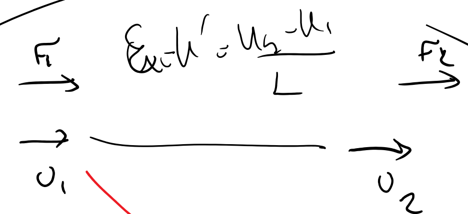
Physical and nonphysical zero modes

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{K} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$




det $K = 0$   we have exactly 1 zero eigenvalue

$$K \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

zero eigenvector   0 eigenvalue


$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \frac{AE}{L} \begin{bmatrix} u_1 - u_2 \\ u_2 - u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$$

zero mode $\boxed{u_1 = u_2}$

$\boxed{u_1 = u_2} \rightarrow \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

rigid displacement

n = size of K = 2

# zeros = 1

rank(K) = 2 - 1 = 1

$u_1 = u_2 = u$


( — rigid motion   $\begin{bmatrix} F_1 \\ \vdots \end{bmatrix} = K \begin{bmatrix} u \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \end{bmatrix}$

— rigid motion $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = K \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

— for $\vec{u} \neq 0 \rightarrow \vec{F} = 0$

— zero eigenvalue for $K$

Bar $\int \omega' EA \underbrace{u'}_{\varepsilon} dx$ mesh

$\downarrow$ $K = \int_e N'^t EA N' dx$

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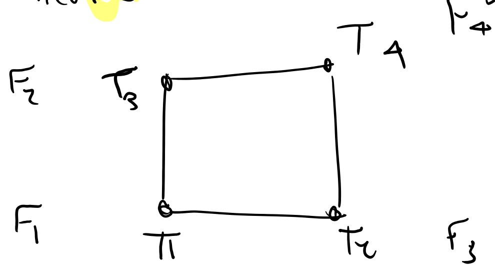
Another example

2D Heat conduction $F_4 \rightarrow$ Concentrated heat source




$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_4 \end{bmatrix} = \underset{4\times4}{K} \begin{bmatrix} T_1 \\ \vdots \\ T_4 \end{bmatrix}$$

$n = 4$

#zeros $= 1$

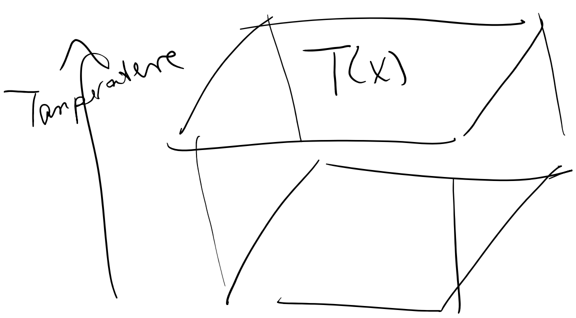
rank $= 4 - 1 = 3$

$$\underbrace{\int_{\Omega} \nabla w \, k \nabla T \, dv}_{\text{weak statement LHS}}$$

$$K = \int \nabla N^t \, k \, \nabla N \, dV$$

$$\nabla N \rightarrow \begin{bmatrix} \frac{dN_1}{dx_1} & - & \frac{dN_4}{dx_1} \\ \frac{dN_1}{dx_2} & & \frac{dN_4}{dx_2} \end{bmatrix}$$

if $T_1 = T_2 = T_3 = T_4$ $\quad \nabla T = 0$




$\nabla T(x) = 0$

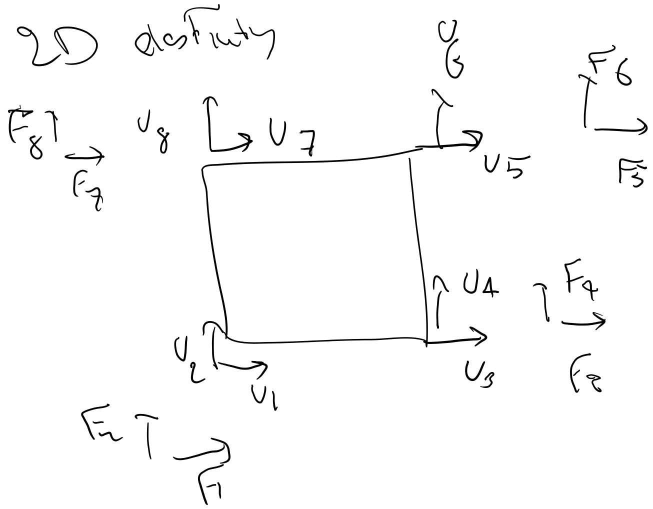
1 zero mode

$$\boxed{T_1 = T_2 = T_3 = T_4 = \overline{T}}$$

$$K \begin{bmatrix} \overline{T} \\ \overline{T} \\ \overline{T} \\ \overline{T} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

exactly 1 zero eigenvalue

2D elasticity




$$\begin{bmatrix} F_1 \\ | \\ F_8 \end{bmatrix} = K_{8\times 8} \begin{bmatrix} u_1 \\ | \\ u_8 \end{bmatrix}$$

$n = 8$

#zeros = 3

rank = 8 - 3 = 5

$$\int \varepsilon(w)\, C\, \varepsilon(u)\, dV \longrightarrow K = \int L_m(N)^t\, C\, L_m(N)\, dV$$

$D\, L_m(w)$

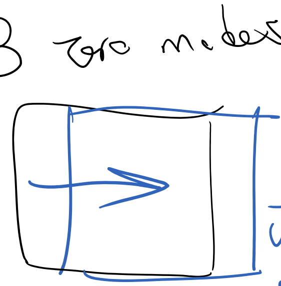
LHS weak statement

$\varepsilon(N)$

$$\varepsilon(u) = \frac{\nabla u + (\nabla u)^t}{2} \qquad L_m = \frac{\nabla + \nabla^t}{2}$$

sym part of grad

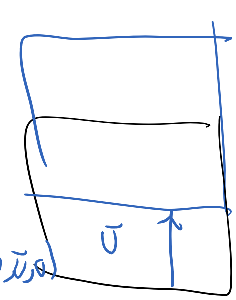
3 zero modes



$\vec{U} = [\bar{U},0,\bar{U},0,\bar{U},0,\bar{U},0]$

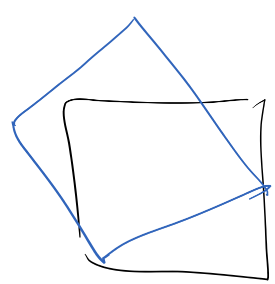
$F\vec{U} = 0_{8\times 1}$

— before displacement




$\vec{U} = [0,\bar{U},0,\bar{U},0,\bar{U},0,\bar{U}]$

$F\vec{U} = 0$



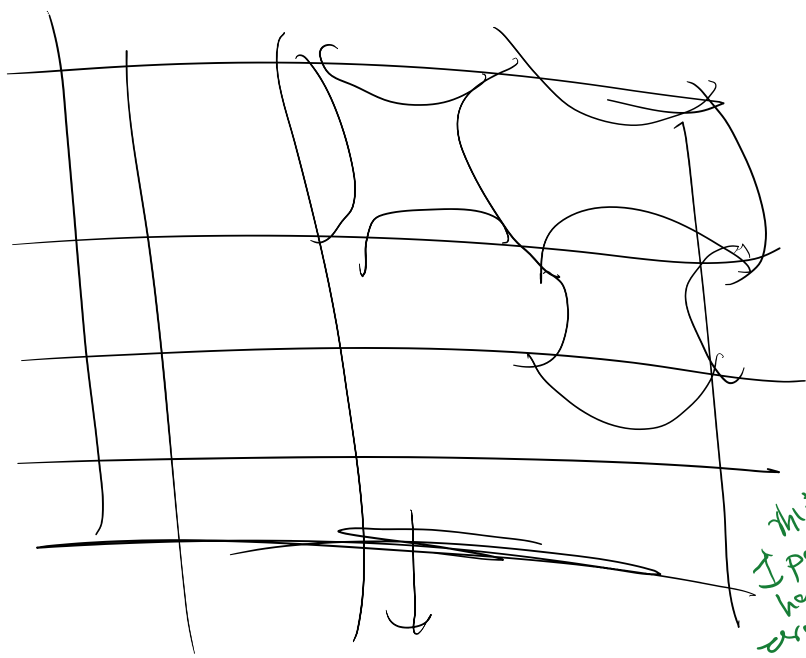
How to get the number of physical zero modes:
- Look at the weak statement and Lm operator
- See how many solutions should generate Lm(solution) = 0. -> number of physical zero modes

$$\boxed{\operatorname{rank}(A_{n\times n}) = n - \#\,\text{zero eigenvalues}(A)}$$

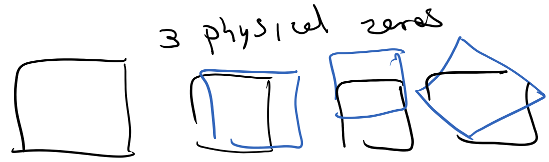
If a given element has the correct number of zero modes -> great!

BUT sometimes if we are too aggressive with reduced order integration, unfortunately, we introduce ADDITIONAL (nonphysical) zero modes

Last time

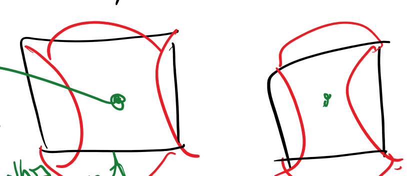


3 physical zeros

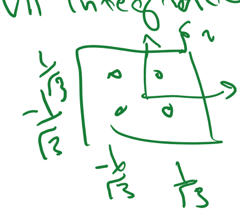


2 nonphysical (physically they should generate force energy, etc BUT because of reduced integration they don't :(

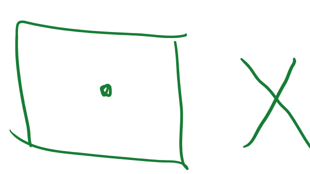
this 1 point has zero strain for this point



Full integration scheme needs 2 Gauss pt in each direction




Reduced order



zero mode

generates 2

How should we use this in practice:

1. Calculate the number of physical zeros that we should have (1, 1, 3) in examples above.
2. For the reduced integrated stiffness calculate numerical number of zeros:
   - Size of the matrix n
   - Rank(K): Matlab, ….
     numberofZeros(K) = n - rank(K)

     This number should be equal to physical number of zeros. If greater, we have reduced the number of quadrature points too much.

More on this in Hughes book (in the course references)

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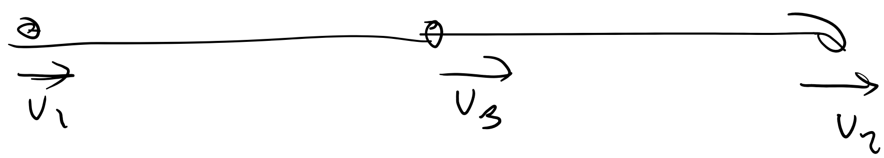
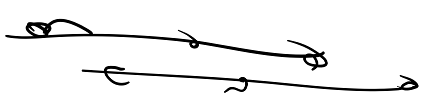
Example



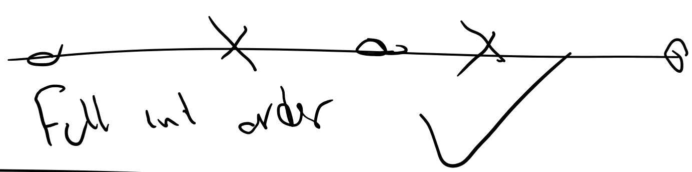

$n = 3$

$nZeros = 1$



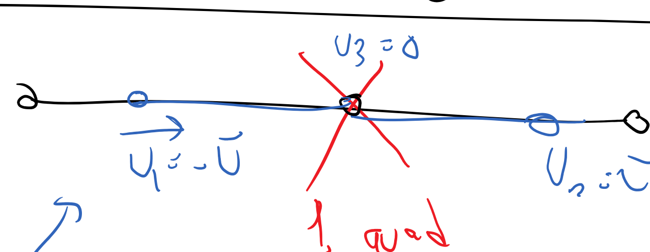
$\{u_1, u_2, u_3\}$ $\{\bar{u}, \bar{u}, \bar{u}\}$

rank $= 3 - 1 = 2$



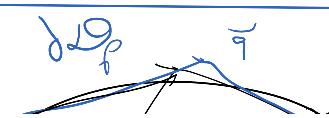

Bad

rank(K) = 1



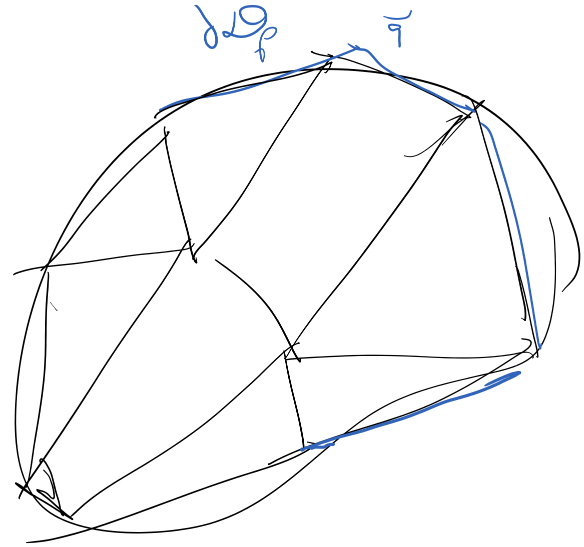

nonphysical zero $[-\bar{u}, \bar{u}, 0]$

2D ( & 3D elements)



2D (& 3D elements)

Heat conduction in 2D




WF:

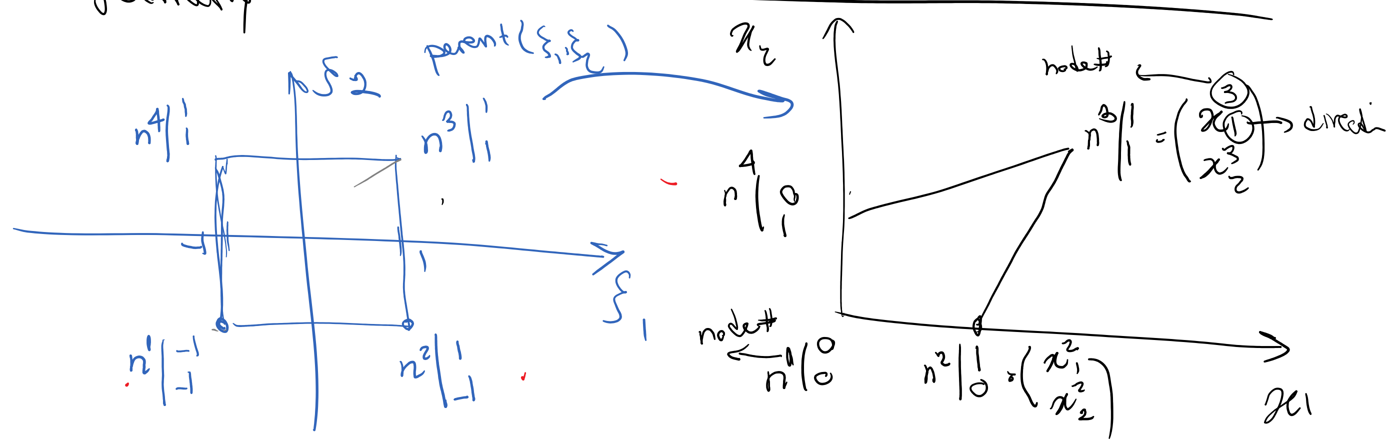
$$\int_\Omega \underbrace{\nabla w}_{\mathcal{L}_m(w)} \, \underbrace{k}_{D = k} \, \underbrace{\nabla T}_{\mathcal{L}_m(T)} \, dV = \int_\Omega w Q \, dV - \int_{\partial\Omega_f} w \bar{q} \, ds$$

$$\mathcal{L}_m = \nabla$$

$$k_e = \int_e B^t D B \, dV \qquad B = \mathcal{L}_m(N) \qquad \text{here } B = \nabla N, \; D = k$$

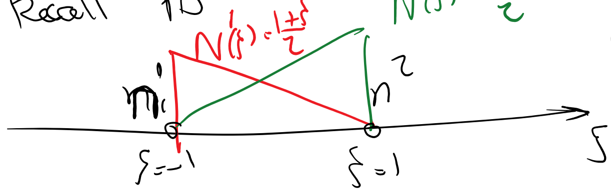
$$\boxed{\text{For heat conduction} \quad k_e = \int_e B^t k B \, dV \qquad B = \nabla N} \quad (1)$$

~~Element geometry~~




Shape functions

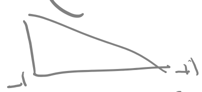
Recall 1D

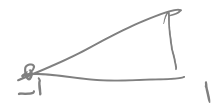



Challenges of working directly with the element geometry:

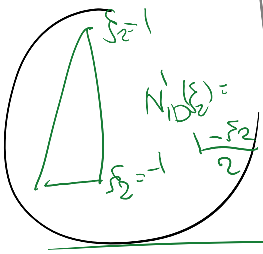
1. Integration of element
2. Forming the shape functions

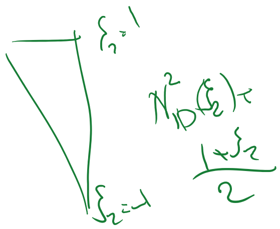

how the parent helps us here

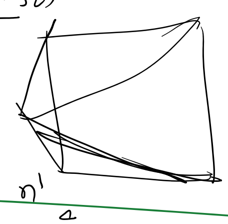


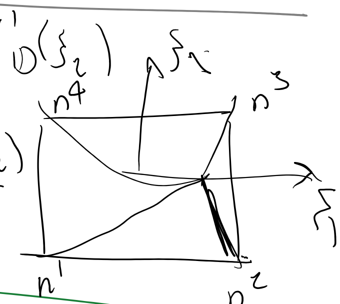



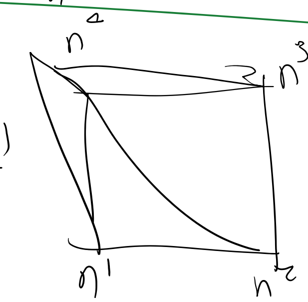

| $\xi_1$ \ $\xi_2$ | $N^1_{1D}(\xi_1)=\dfrac{1-\xi_1}{2}$ | $N^2_{1D}(\xi_1)=\dfrac{1+\xi_1}{2}$ |
|---|---|---|
| $N^1_{1D}(\xi_2)=\dfrac{1-\xi_2}{2}$ | $N^1(\xi_1,\xi_2)=\dfrac{(1-\xi_1)(1-\xi_2)}{4}$ | $N^2_{1D}(\xi_1)\,N^1_{1D}(\xi_2)$; $N^2(\xi_1,\xi_2)=\dfrac{(1+\xi_1)(1-\xi_2)}{4}$ |
| $N^2_{1D}(\xi_2)=\dfrac{1+\xi_2}{2}$ | $N^4(\xi_1,\xi_2)=\dfrac{(1-\xi_1)(1+\xi_2)}{4}$ | $N^3(\xi_1,\xi_2)=\dfrac{(1+\xi_1)(1+\xi_2)}{4}$ |

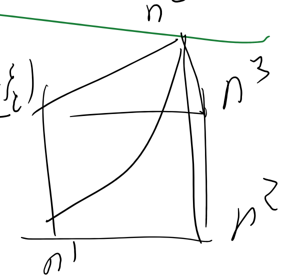


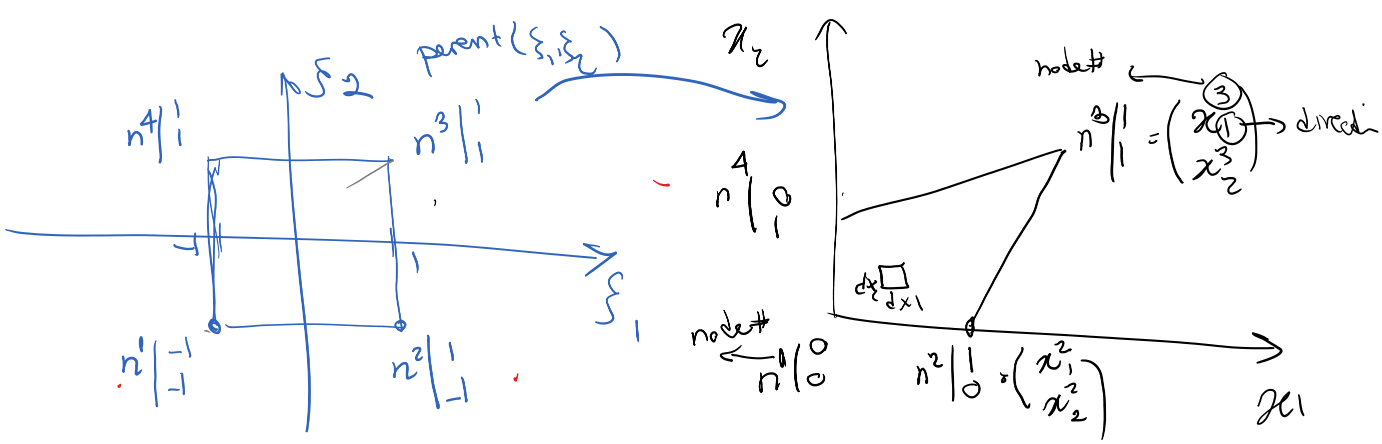




(2)

$$\boxed{N=[N^1,N^2,N^3,N^4]=\left[\frac{(1-\xi_1)(1-\xi_2)}{4},\ \frac{(1+\xi_1)(1-\xi_2)}{4},\ \frac{(1+\xi_1)(1+\xi_2)}{4},\ \frac{(1-\xi_1)(1+\xi_2)}{4}\right]}$$

$$K^e=\int_e \nabla N^t\, k\, \nabla N\, dv \qquad dx_1\,dx_2$$

$$\nabla N=\begin{bmatrix}\dfrac{\partial N^1}{\partial x_1} & - & - & \dfrac{\partial N^4}{\partial x_1}\\ \dfrac{\partial N^1}{\partial x_2} & & & \dfrac{\partial N^4}{\partial x_2}\end{bmatrix}$$

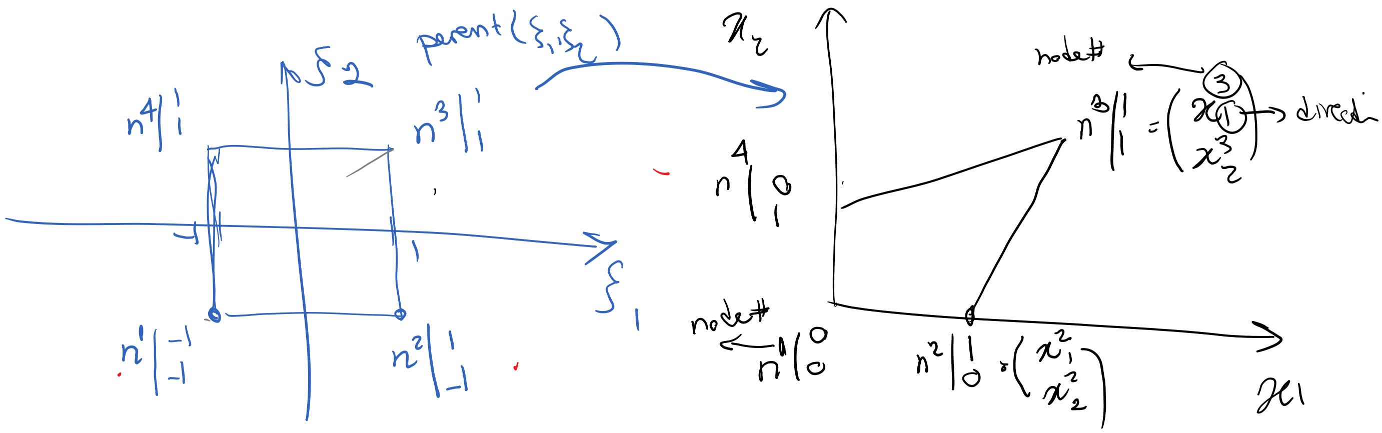
N's are functions of $\xi_1,\xi_2$

$K = \int_e \nabla N \, \kappa \, \nabla N \, d\cancel{x_1} d\cancel{x_2}$ ... $\left[ \frac{\partial N^1}{\partial x_2} \quad \frac{\partial N^4}{\partial x_2} \right]$ functions of $\xi_1, \xi_2$)

$(\det J)\, d\xi_1 d\xi_2$ use $J = \frac{\partial x}{\partial \xi}$

We need $x(\xi)$ How do we get this?

Again we use shape functions

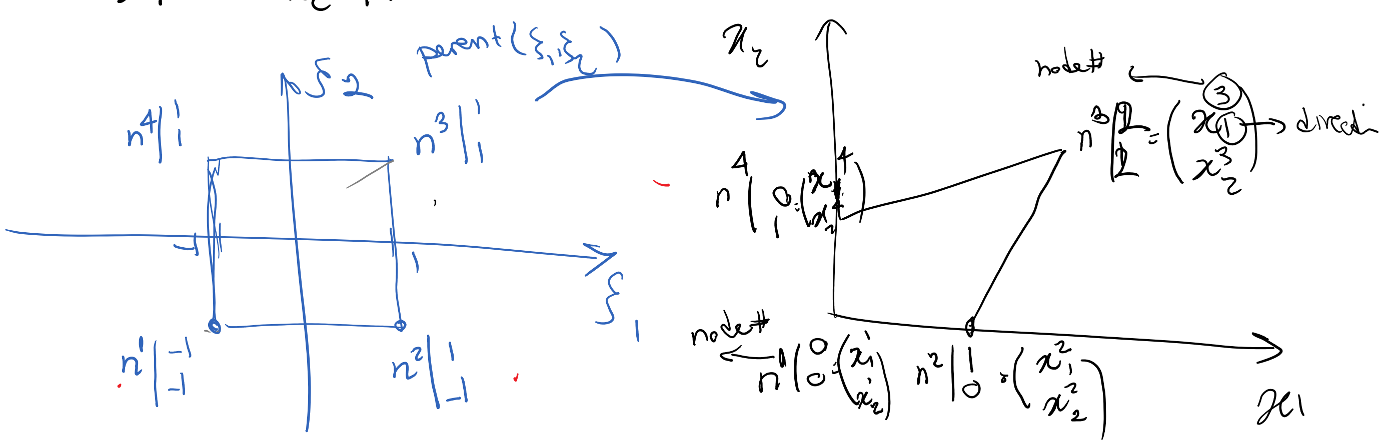



$$T^h(\xi_1, \xi_2) = T^1 N^1(\xi_1, \xi_2) + T^2 N^2(\xi_1, \xi_2) + T^3 N^3(\xi_1, \xi_2) + T^4 N^4(\xi_1, \xi_2)$$

$T^i$'s are unknown temperature solutions @ nodes 1 to 4

$$T^h = \vec{N} \vec{a} \qquad \vec{N} = \{N^1 \cdots N^4\} \qquad \vec{a} = \begin{bmatrix} T^1 \\ \vdots \\ T^4 \end{bmatrix} \qquad a = \begin{bmatrix} T^1 \\ T^2 \\ T^3 \\ T^4 \end{bmatrix}$$

Use $x_1$ & $x_2$ instead of $T$

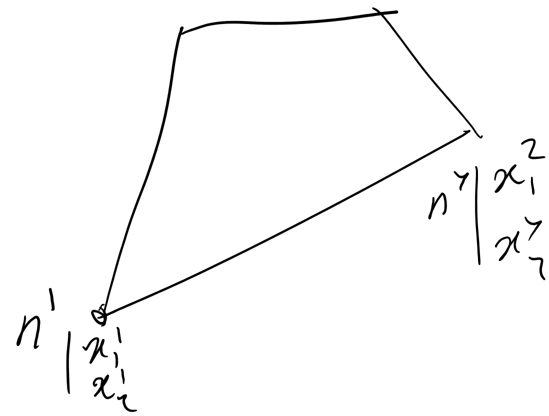



$$\boxed{x_1(\xi_1, \xi_2) = x_1^1 N^1(\xi_1, \xi_2) + x_1^2 N^2(\xi_1, \xi_2) + x_1^3 N^3(\xi_1, \xi_2) + x_1^4 N^4(\xi_1, \xi_2)}$$

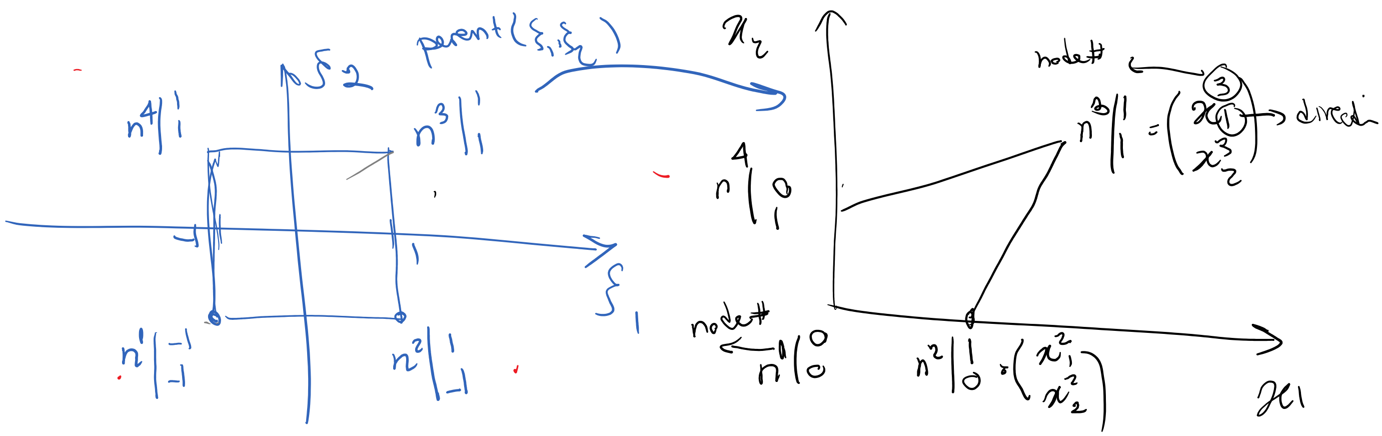
$$x_1(\xi_1,\xi_2) = x_1^1 N^1(\xi_1,\xi_2) + x_1^2 N^2(\xi_1,\xi_2) + x_1^3 N^3(\xi_1,\xi_2) + x_1^4 N^4(\xi_1,\xi_2)$$
$$x_2(\xi_1,\xi_2) = x_2^1 \text{ `` } + x_2^2 \text{ `` } + x_2^3 \text{ `` } + x_2^4 \text{ `` }$$

direction, node #

3a General for ANY Q4 element




How about equation 3 (x(xi)) for the geometry of the element we are dealing with?

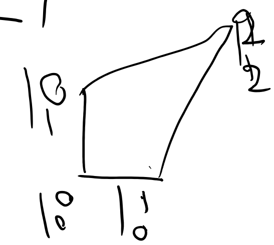



$$x_1(\xi_1,\xi_2) = 0\, N^1(\xi_1,\xi_2) + 1\, N^2(\xi_1,\xi_2) + 2\, N^3(\xi_1,\xi_2) + 0\, N^4(\xi_1,\xi_2)$$
$$x_2(\xi_1,\xi_2) = 0 \text{ `` } + 0 \text{ `` } + 2 \text{ `` } + 1 \text{ `` }$$
$$x_1(\xi_1,\xi_2) = 1\,\frac{(1+\xi_1)(1-\xi_2)}{4} + 1\,\frac{(1+\xi_1)(1+\xi_2)}{4} N^3$$
$$x_2(\xi_1,\xi_2) = \text{ `` } + 1\,\frac{(1-\xi_1)(1+\xi_2)}{4}$$

$$x_1(\xi_1,\xi_2) = \frac{1}{4}\left(3 + 3\xi_1 + \xi_2 + \xi_1\xi_2\right)$$

$$x_2(\xi_1,\xi_2) = \frac{1}{4}\left(3 + \xi_1 + 3\xi_2 + \xi_1\xi_2\right)$$

ONLY for this geom



3b