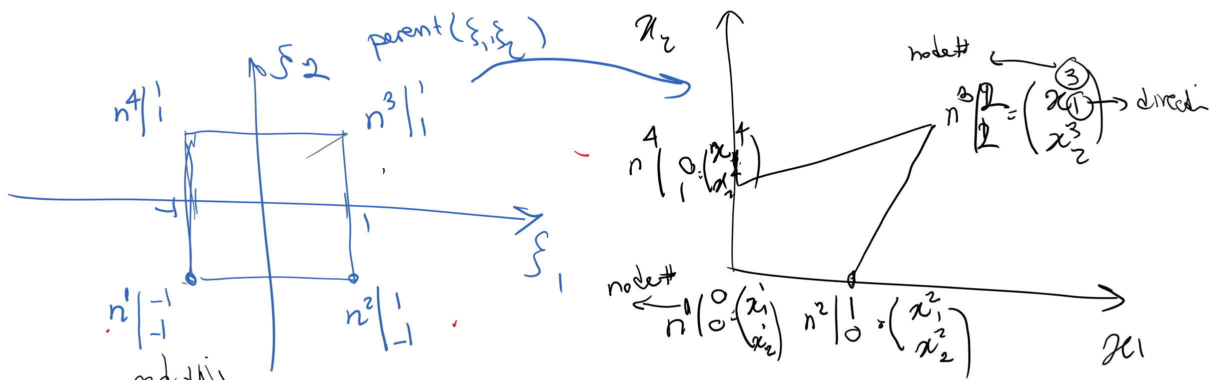
2022/12/01

Thursday, December 1, 2022 11:17 AM



we need this

$$B = \begin{bmatrix} \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_1} & \frac{\partial N^4}{\partial x_1} \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^3}{\partial x_1} & \frac{\partial N^4}{\partial x_2} \end{bmatrix} \qquad B_\xi = \begin{bmatrix} \frac{\partial N^1}{\partial \xi_1} & — & — & \frac{\partial N^4}{\partial \xi_1} \\ \frac{\partial N^1}{\partial \xi_2} & & & \frac{\partial N^4}{\partial \xi_2} \end{bmatrix}$$

this can be calculated

$$K^e = \int_e B^t \kappa B \, dV$$

$$B_\xi \longrightarrow B$$

How about using the chain rule:

node #i

$$\frac{\partial N^i}{\partial x_1} = \frac{\partial N^i}{\partial \xi_1} \frac{\partial \xi_1}{\partial x_1} + \frac{\partial N^i}{\partial \xi_2} \frac{\partial \xi_2}{\partial x_1}$$

But there is a problem with this

Recall $x_1(\xi_1, \xi_2)$, $x_2(\xi_1, \xi_2)$ is given (last time) but we don't have $\xi_1(x_1, x_2)$, $\xi_2(x_1, x_2)$

Let's go the opposite direction:

$$\frac{\partial N^i}{\partial \xi_1} = \frac{\partial N^i}{\partial x_1} \frac{\partial x_1}{\partial \xi_1} + \frac{\partial N^i}{\partial x_2} \frac{\partial x_2}{\partial \xi_1}$$

$$\frac{\partial N^i}{\partial \xi_2} = \frac{\partial N^i}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial N^i}{\partial x_2} \frac{\partial x_2}{\partial \xi_2}$$

$i = 1, \dots, 4$
node #

we'll write this in matrix form

(1)

$$\begin{bmatrix} \frac{\partial N^i}{\partial \xi_1} \\ \frac{\partial N^i}{\partial \xi_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}}_{J^t} \begin{bmatrix} \frac{\partial N^i}{\partial x_1} \\ \frac{\partial N^i}{\partial x_2} \end{bmatrix}$$

we have it

we don't have it, and want to obtain this

$$J = J_{x/\xi} = \nabla_{x/\xi} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}$$

since we have $\begin{cases} x_1(\xi_1, \xi_2) \\ x_2(\xi_1, \xi_2) \end{cases} \rightarrow J$ can be calculated

$$\begin{bmatrix} \frac{\partial N^i}{\partial \xi_1} \\ \frac{\partial N^i}{\partial \xi_2} \end{bmatrix} = J^t \begin{bmatrix} \frac{\partial N^i}{\partial x_1} \\ \frac{\partial N^i}{\partial x_2} \end{bmatrix} \rightarrow \underline{\begin{bmatrix} \frac{\partial N^i}{\partial x_1} \\ \frac{\partial N^i}{\partial x_2} \end{bmatrix} = J^{-t} \begin{bmatrix} \frac{\partial N^i}{\partial \xi_1} \\ \frac{\partial N^i}{\partial \xi_2} \end{bmatrix}}$$

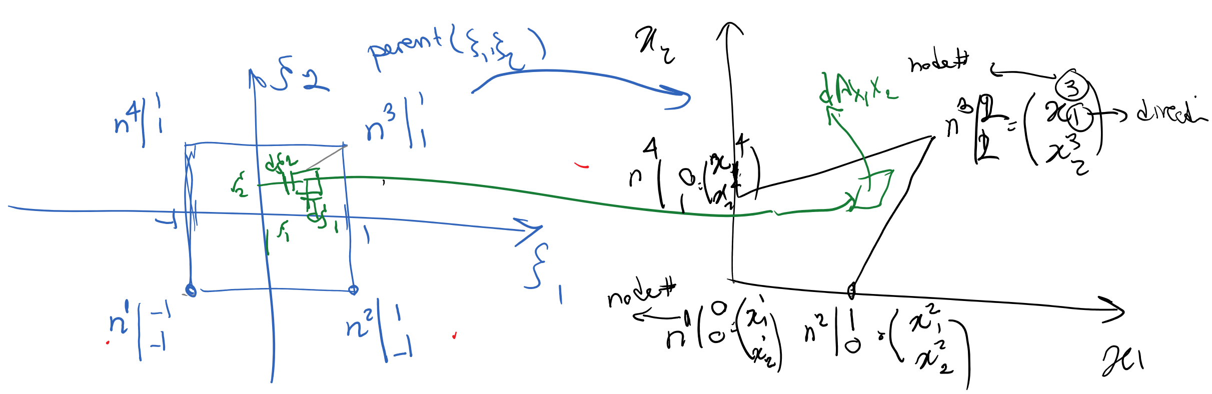
$$J^{-t} = \left(J^t\right)^{-1} = \left(J^{-1}\right)^t$$

Repeat equation (1) for $i = 1, \dots, 4$

(2)

$$\underbrace{\begin{bmatrix} \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_1} & \frac{\partial N^4}{\partial x_1} \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^3}{\partial x_2} & \frac{\partial N^4}{\partial x_2} \end{bmatrix}}_{B = \nabla_x N} = J^{-1} \underbrace{\begin{bmatrix} \frac{\partial N^1}{\partial \xi_1} & \frac{\partial N^2}{\partial \xi_1} & \frac{\partial N^3}{\partial \xi_1} & \frac{\partial N^4}{\partial \xi_1} \\ \frac{\partial N^1}{\partial \xi_2} & \frac{\partial N^2}{\partial \xi_2} & \frac{\partial N^3}{\partial \xi_2} & \frac{\partial N^4}{\partial \xi_2} \end{bmatrix}}_{B_\xi = \nabla_\xi N}$$

$$B = J^{-1} B_\xi$$



$$K = \int_e B^t \kappa B \, \underbrace{dA}_{dx_1 dx_2} = \int_{\xi_1=-1}^{1} \int_{\xi_2=-1}^{1} \left(J^{-t} B_\xi\right)^t \kappa \left(J^{-1} B_\xi\right) \underbrace{\det J \, d\xi_1 d\xi_2}_{dA_{x_1 x_2}}$$

$$k = \int_e B^t \kappa B \underbrace{dx_1 dx_2}_{} = \int_{\xi_1=-1} \int_{\xi_2=-1} \underbrace{(J^{-t} B_\xi)^t \kappa (J^{-t} B_\xi)}_{B} \underbrace{(\det J \, d\xi_1 d\xi_2)}_{dA_{x_1x_2}}$$

From eqn 2

$$\boxed{dA_{x_1x_2} = \det J \; dA_{\xi_1\xi_2}} \quad (3)$$

$dA_{\xi_1\xi_2}$

I'll discuss this below

note $(AB)^t = B^t A^t$

$$k = \int_{\xi_1=-1}^{1} \int_{\xi_2=-1}^{1} B_\xi^t \left(J^{-t}\right)^t \kappa J^{-t} B_\xi \det J \, d\xi_1 d\xi_2$$

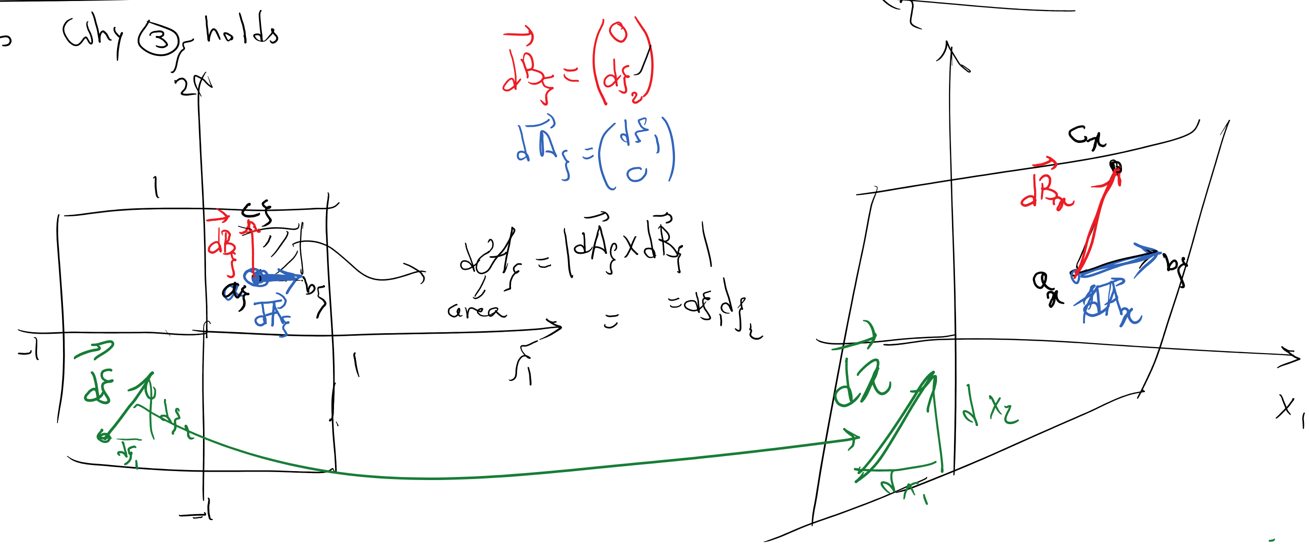
used $(A^t)^t = A$

$$(4) \quad \boxed{k = \int_{-1}^{1}\int_{-1}^{1} B_\xi^t \left(J^{-1} \kappa J^{-t}\right) B_\xi \; \det J \, d\xi_1 d\xi_2}$$

General stiffness (Conductivity) matrix for ANY Quad element

Some notes why (3) holds

$$\vec{dB}_\xi = \begin{pmatrix} 0 \\ d\xi_2 \end{pmatrix} \qquad \vec{dA}_\xi = \begin{pmatrix} d\xi_1 \\ 0 \end{pmatrix}$$



$$\nabla_{x/\xi} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}$$

$$dx_1 = \frac{\partial x_1}{\partial \xi_1} d\xi_1 + \frac{\partial x_1}{\partial \xi_2} d\xi_2 \quad \text{chain rule}$$

$$dx_2 = \frac{\partial x_2}{\partial \xi_1} d\xi_1 + \frac{\partial x_2}{\partial \xi_2} d\xi_2$$

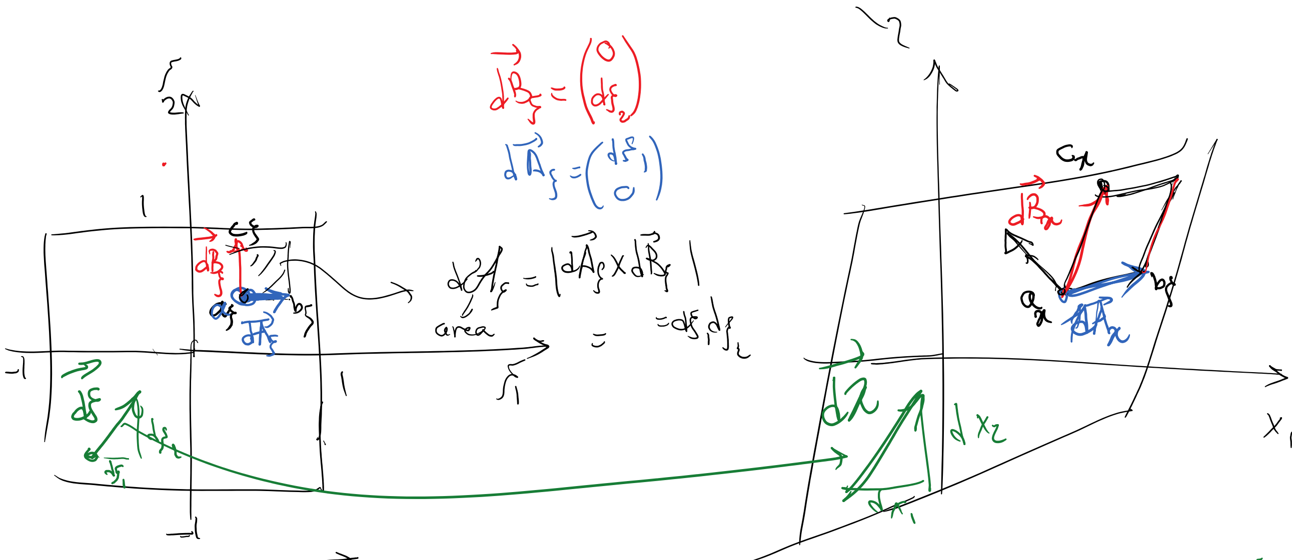
$$\vec{dx} = \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix}$$

$$d\vec{x} = \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}}_{\nabla_{x/\xi}} \begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix}$$

$$\boxed{\underset{\text{funct}}{d\vec{x}(\xi)} = \nabla_{x/\xi}\, \underset{\text{argument}}{d\vec{\xi}}} \qquad (5)$$

Meaning of grad operator

change of funct $x(\xi)$ is equal to $\text{grad}_{x/\xi}$ times change of argu



Apply eq 5 to $d\vec{A}_\xi$ & $d\vec{B}_\xi$

$$\overrightarrow{dA_x} = \nabla_{x/\xi}\, d\vec{A}_\xi = \nabla_{x/\xi} \begin{bmatrix} d\xi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} d\xi_1 \\ 0 \end{bmatrix} = d\xi_1 \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} \\ \frac{\partial x_2}{\partial \xi_1} \end{bmatrix}$$

$$\overrightarrow{dB_x} = \nabla_{x/\xi}\, d\vec{B}_\xi = \nabla_{x/\xi} \begin{bmatrix} 0 \\ d\xi_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} 0 \\ d\xi_2 \end{bmatrix} = d\xi_2 \begin{bmatrix} \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}$$

$$\underset{\text{area in } x}{dA_x} = d\vec{A}_x \times d\vec{B}_x = \underbrace{d\xi_1 d\xi_2}_{dA_\xi} \det \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}}_{J = \nabla_{x/\xi}}$$

$$\boxed{dA_x = \det J \; dA_\xi}$$ this is eqn (3) above

Now that we have fully transformed equations from x to ξ
Let's now calculate k

equation 4 $$k = \int_{-1}^{1}\int_{-1}^{1} B_\xi^t \left(J^{-1} k J^{-t} |J|\right) B_\xi \, d\xi_1 \, d\xi_2$$

(a) calculate B_ξ

$$N = [N_1, N_2, N_3, N_4] = \left[\frac{(1-\xi_1)(1-\xi_2)}{4}, \frac{(1+\xi_1)(1-\xi_2)}{4}, \frac{(1+\xi_1)(1+\xi_2)}{4}, \frac{(1-\xi_1)(1+\xi_2)}{4}\right]$$

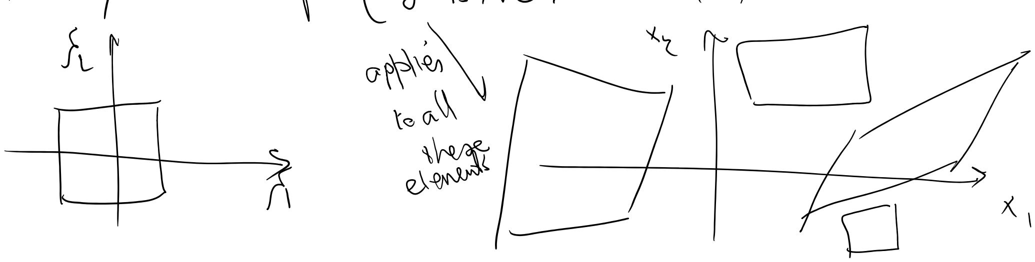
$$\boxed{B_\xi = \begin{bmatrix} \frac{\partial N^1}{\partial \xi_1} & \frac{\partial N^2}{\partial \xi_1} & \frac{\partial N^3}{\partial \xi_1} & \frac{\partial N^4}{\partial \xi_1} \\ \frac{\partial N^1}{\partial \xi_2} & \frac{\partial N^2}{\partial \xi_2} & \frac{\partial N^3}{\partial \xi_2} & \frac{\partial N^4}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} -\frac{(1-\xi_2)}{4} & \frac{1-\xi_2}{4} & \frac{1+\xi_2}{4} & -\frac{(1+\xi_2)}{4} \\ -\frac{(1-\xi_1)}{4} & -\frac{(1+\xi_1)}{4} & \frac{1+\xi_1}{4} & \frac{(1-\xi_1)}{4} \end{bmatrix}} \quad (6.)$$

plug B_ξ in equation for k

$$k^e = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & -\frac{(1-\xi_1)}{4} \\ \frac{1-\xi_2}{4} & -\frac{(1+\xi_1)}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{(1+\xi_2)}{4} & \frac{(1-\xi_1)}{4} \end{bmatrix}}_{B_\xi^t} J^{-1} k J^{-t} \det J \underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & \frac{(1-\xi_2)}{4} & \frac{1+\xi_2}{4} & -\frac{(1+\xi_2)}{4} \\ -\frac{(1-\xi_1)}{4} & -\frac{(1+\xi_1)}{4} & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi}}_{I(\xi_1,\xi_2)} d\xi_1 \, d\xi_2 \quad (7)$$

integrand in terms of ξ_1, ξ_2

For any element shape (J is NOT inserted yet)



What is a full integration order for this element?

$$k^e = \int^{1}\int^{1} \begin{bmatrix} -\frac{(1-\xi_2)}{4} & -\frac{(1-\xi_1)}{4} \\ \frac{1-\xi_2}{4} & -\frac{(1+\xi_1)}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ \vdots & \vdots \end{bmatrix} \underbrace{J^{-1} k J^{-t} \det J}_{\text{in deciding}} \begin{bmatrix} -\frac{(1-\xi_2)}{4} & \frac{(1-\xi_2)}{4} & \frac{1+\xi_2}{4} & -\frac{(1+\xi_2)}{4} \\ -\frac{(1-\xi_1)}{4} & -\frac{(1+\xi_1)}{4} & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix} d\xi_1 \, d\xi_2$$

$$K^e = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\begin{bmatrix} \cdots & \cdots \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{(1+\xi_2)}{4} & \frac{(1-\xi_1)}{4} \end{bmatrix} \cdots \begin{bmatrix} \cdots & \boxed{-\frac{(1-\xi_1)}{4}} & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{I(\xi_1,\xi_2)} d\xi_1 d\xi_2$$

in bending
full integration order
we ignore these terms

The integrand I(ξ_1,ξ_2) is a 4x4 matrix

What is the $\underline{\text{FULL INTEGRATION}}$ ORDER for $I(\xi_1,\xi_2)$

$\mathbb{K} = 10$ nomcs

$\mathbb{K}(\xi_1,\xi_2) = 10 + .1\xi_1 + 3\xi_2$

inhomogeneous

$$I_{11} = \frac{-(1-\xi_1)(-(1-\xi_2))}{16} + \cancel{\frac{\left(-\frac{(1-\xi_1)}{4}\right)\left(-\frac{(1-\xi_1)}{4}\right)}{c}}$$

order 2 in ξ_2 order 2 in ξ_1

We need to the same order check for all the other 15 components of the stiffness, but a short look at the matrix components show that the order for terms is 2 in ξ_1 and 2 in ξ_2

$$\boxed{\begin{gathered}\text{Full integration order} = 2 \text{ in } \xi_1, \ 2 \text{ in } \xi_2 \\ O_{\xi_1} = 2, \quad O_{\xi_2} = 2\end{gathered}} \quad (8)$$

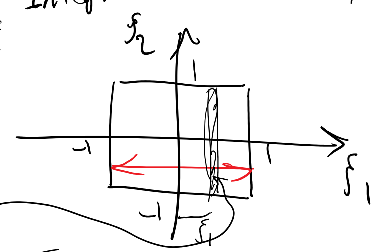
Now let's $\underbrace{\text{integrate}}_{\text{quadrature}}$ the element $\underline{\text{Numerically}}$

Equation (7)

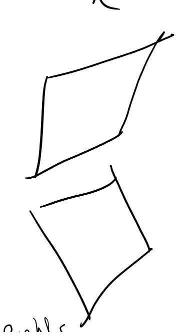
$$K^e = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & -\frac{(1-\xi_1)}{4} \\ \frac{1-\xi_2}{4} & -\frac{(1+\xi_1)}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{(1+\xi_2)}{4} & \frac{(1-\xi_1)}{4} \end{bmatrix}}_{B_\xi^t} J^{-1} \mathbb{K} J^{-t} \det J \underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & \frac{(1-\xi_2)}{4} & \frac{1+\xi_2}{4} & -\frac{(1+\xi_2)}{4} \\ -\frac{(1-\xi_1)}{4} & -\frac{(1+\xi_1)}{4} & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi}}_{I(\xi_1,\xi_2)} d\xi_1 d\xi_2$$

integrand in terms of ξ_1, ξ_2

Integrate this numerically:



any element



(A)

$$K^e = \int_{\xi_1=-1}^{1} d\xi_1 \left[\int_{-1}^{1} I(\xi_1,\xi_2)\, d\xi_2\right]$$

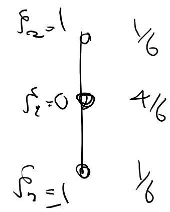
$O_{\xi_1} = 2$ $\quad O_{\xi_2} = 2$

$\int_{\xi_1=-1} d\xi_1 \left| \int_{-1} \ldots \right|$ $\quad o_{\xi_1}=2 \quad o_{\xi_2}=2$ ✓

1D integral in ξ_2

ξ_1 is fixed
order in $\xi_2 = 2$

weights



Newton-Cotes scheme $\quad o_{\xi_2}=2 \longrightarrow n_{\xi_2}=3$

$$k^e = \int_{\xi_1=-1}^{1} d\xi_1 \left(\underbrace{(1-(-1))}_{\text{length}} \left(\frac{1}{6} I(\xi_1,-1) + \frac{4}{6} I(\xi_1,0) + \frac{1}{6} I(\xi_1,1) \right) \right)$$

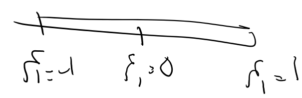
$$= \int_{\xi_1=-1}^{1} \Big(\underbrace{\frac{1}{3} I(\xi_1,-1) + \frac{2}{3} I(\xi_1,0) + \frac{1}{3} I(\xi_1,1)}_{I'(\xi_1)} \Big) d\xi$$

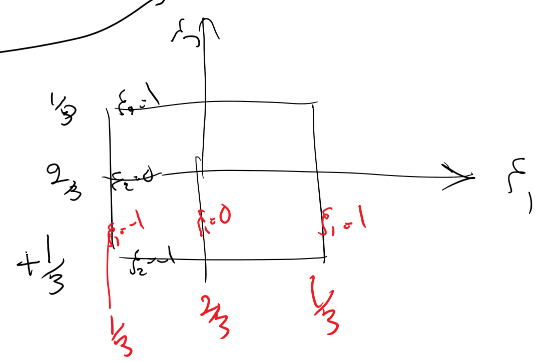
NC

Order of I is 2 in $\xi_1 \longrightarrow n_{\xi_1}=3 \quad (o_{\xi_1}+1)$

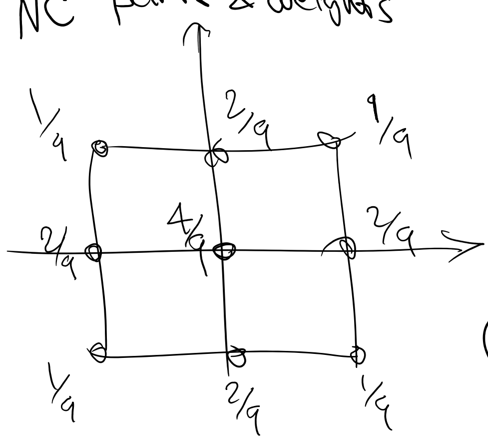
$$k^e = (1-(-1))\left(\frac{1}{6} I'(\xi_1=-1) + \frac{4}{6} I'(\xi_1=0) + \frac{1}{6} I'(\xi_1=1) \right)$$

$$= \frac{1}{3} I'(\xi_1=-1) + \frac{2}{3} I'(\xi_1=0) + \frac{1}{3} I'(\xi_1=1)$$





NC points & weights



this suffices

corners

$$k = \frac{1}{9}\Big(I(-1,-1) + I(-1,1) + I(1,-1) + I(1,1) \Big)$$

$$+ \frac{2}{9}\Big(I(0,1) + I(0,-1) + I(-1,0) + I(1,0) \Big)$$

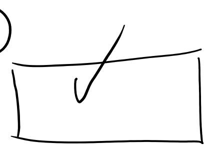
edge centers

9NC $\quad + \frac{4}{9} I(0,0)$

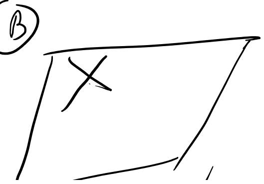
center

no need to write this in your HW

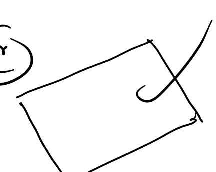
A



B



C



$k=12$ ✓

$\times \; k = 10 + .1\xi_1 + .2\xi_2$ $\quad k = 5$ ✓ $\quad k = 12$ ✓

For which one 9NC calculates k exactly? C

Do the same thing with Gauss Quadrature:

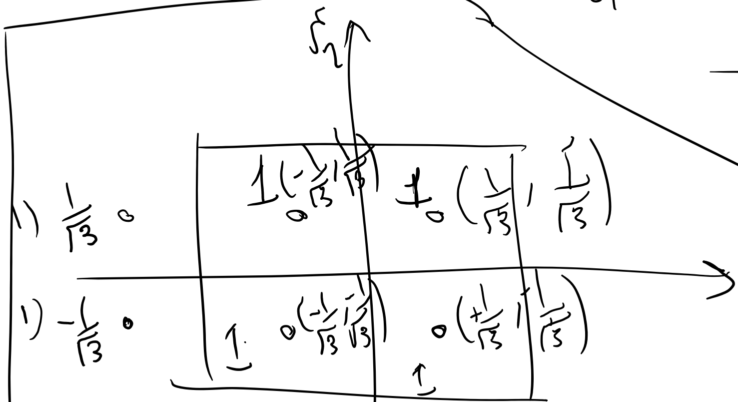
$o_{\xi_1} = 2 \longrightarrow n_{\xi_1} = \text{ceil}\left(\frac{o_{\xi_1}+1}{2}\right) = \text{ceil}\left(\frac{3}{2}\right) = 2$

$n_{\xi_2} = \text{ceil}\left(\frac{o_{\xi_2}+1}{2}\right) = 2$

Gauss quadrature $\int_{-1}^{1} I(\xi)\,d\xi = w_1 I(\xi_1) + w_2 I(\xi_2)$

$w_1 = 1 \quad w_2 = 1$

$\xi_1 = \frac{-1}{\sqrt{3}} \quad \xi_2 = \frac{1}{\sqrt{3}}$



positions	$-\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$
weight	1	1

Quad table for GQ

9G

Needs 4pts

9NC $\rightarrow$ 9 pts $\quad$ 2.25 x more expensive

We'll use Gauss Quadrature

Relating this to the specific geometry we have before

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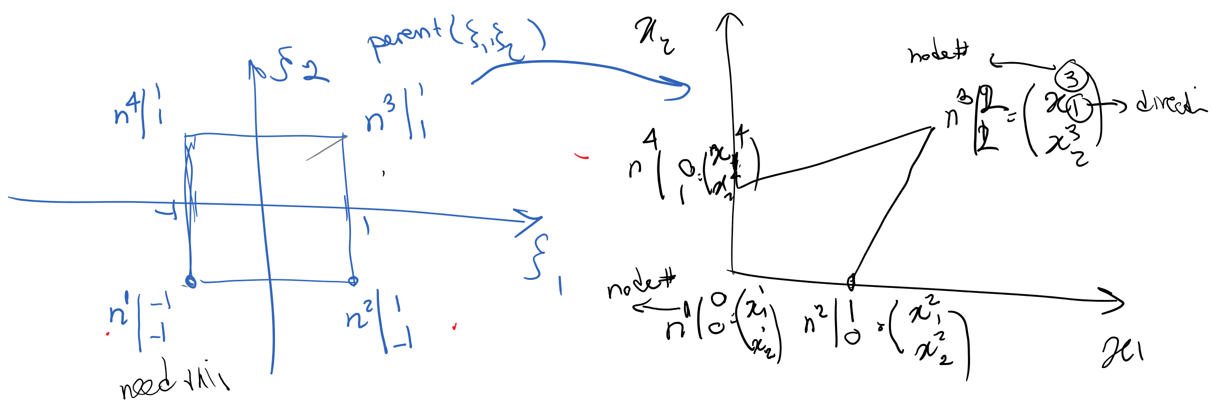


From last time

$$x_1(\xi_1,\xi_2) = \frac{1}{4}\left(3+3\xi_1+\xi_2+\xi_1\xi_2\right)$$

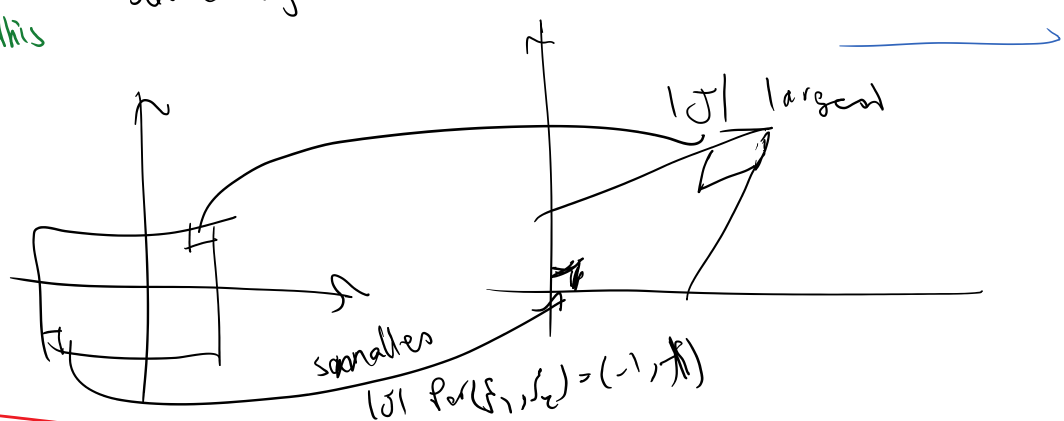
$$x_2(\xi_1,\xi_2) = \frac{1}{4}\left(3+\xi_1+3\xi_2+\xi_1\xi_2\right)$$

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} \frac{3+\xi_2}{4} & \frac{1+\xi_1}{4} \\ \frac{1+\xi_2}{4} & \frac{3+\xi_1}{4} \end{bmatrix}$$

$$\det J = |J| = \frac{(3+\xi_2)(3+\xi_1)}{16} - \frac{(1+\xi_1)(1+\xi_2)}{4} = \frac{4+\xi_1+\xi_2}{8}$$

same thing

Use this in



Equation (7)

$$I(\xi_1,\xi_2) = \underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & -\frac{(1-\xi_1)}{4} \\ \frac{1-\xi_2}{4} & -\frac{(1+\xi_1)}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{(1+\xi_2)}{4} & \frac{(1-\xi_1)}{4} \end{bmatrix}}_{B_\xi^T} J^{-1} \kappa J^{-t} \det J \underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & \frac{(1-\xi_2)}{4} & \frac{1+\xi_2}{4} & -\frac{(1+\xi_2)}{4} \\ -\frac{(1-\xi_1)}{4} & -\frac{(1+\xi_1)}{4} & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi} d\xi$$

fully known

Calculate k using 9NC or 9Gauss (preferred).