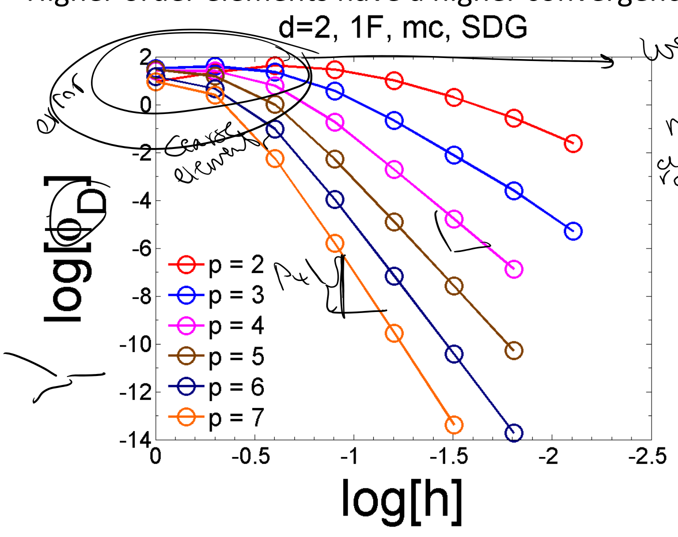
2022/12/06

Tuesday, December 6, 2022 11:17 AM

Higher order elements in 2D and 3D

Motivation

Higher order elements have a higher convergence rate



we have

not reached asymptotic range

element order

error : Ch^{p+1}

more general

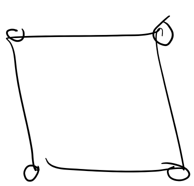
$Ch^{\alpha p + b}$

$\log \text{error} = \log C + (p+1) \log h$

Y X

Higher order element is better because convergence rate is p+1

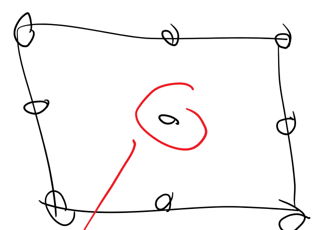
FEM



$p = 1$

Q4

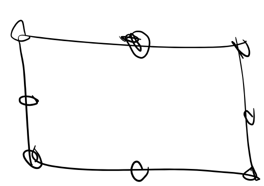
Ch^2



Q9

$p = 2$

in Final example you'll see how we get rid of the internal node



Q8 serendipity

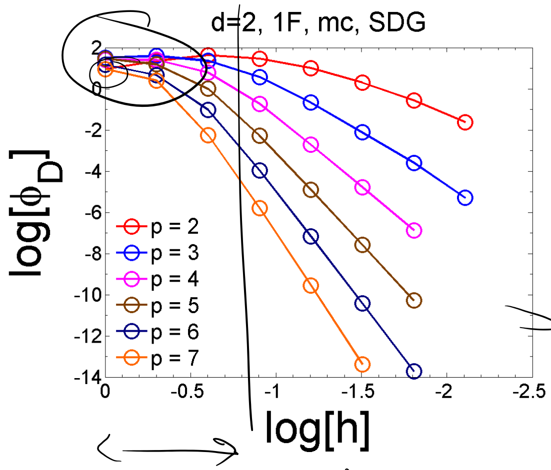
Ch^3

Efficiency : for the same resource (wall clock time, memory, etc) we'll get a lower error

If the solution is smooth enough (not dealing with strong and weak discontinuities such as crack tip, sharp wave fronts, shocks in fluid mechanics, …) it is often beneficiary to use higher order elements as we benefit from higher convergence rate.

BUT the solution should be accurate enough (already using small elements) so we are in the asymptotic convergence rate

Bad for high order elements

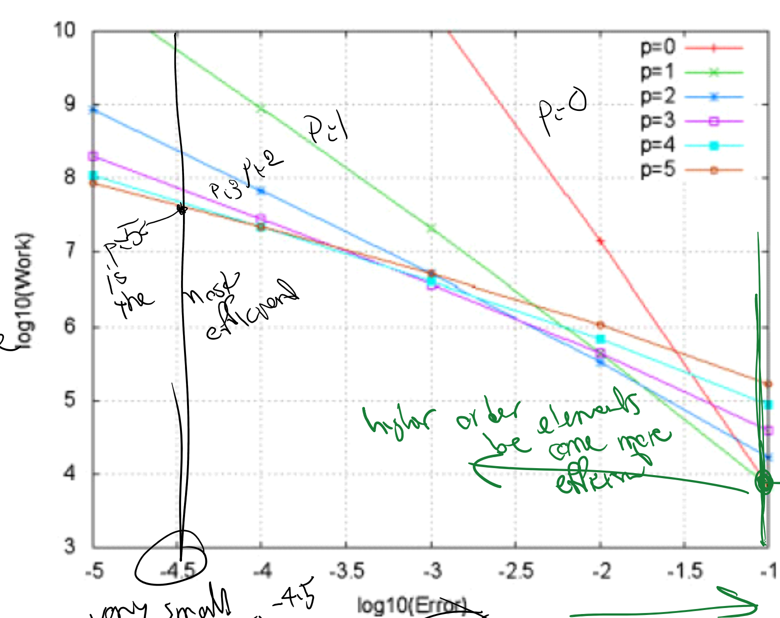


finer element

coarse element

Sample efficiency plot

Solution scale



p=0 & p=1

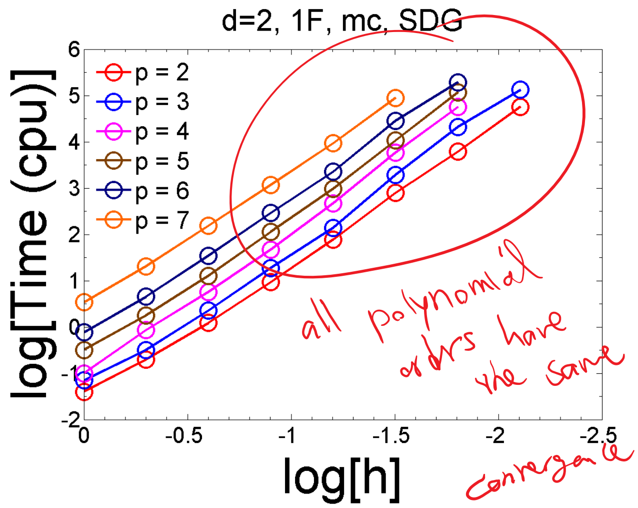
10 %, error

10^{-1}

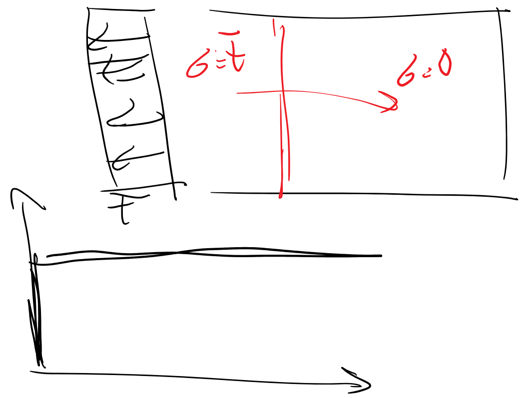
many eng. problems are here

Figure 1. Asymptotic work estimates for 3-D linear problems.

if the problem is not smooth enough, wouldn't get to optimal convergence rate



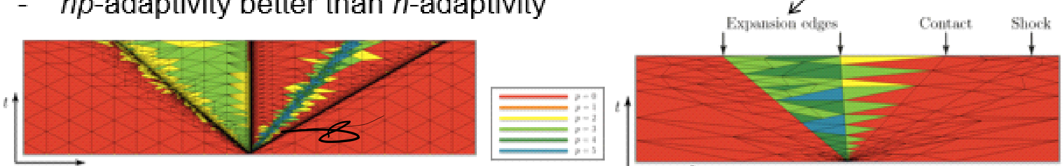
problem with nonsmooth solution



- Adaptive operations in spacetime:
  - Front-tracking better than shock capturing
  - *hp*-adaptivity better than *h*-adaptivity

Sod's shock tube problem

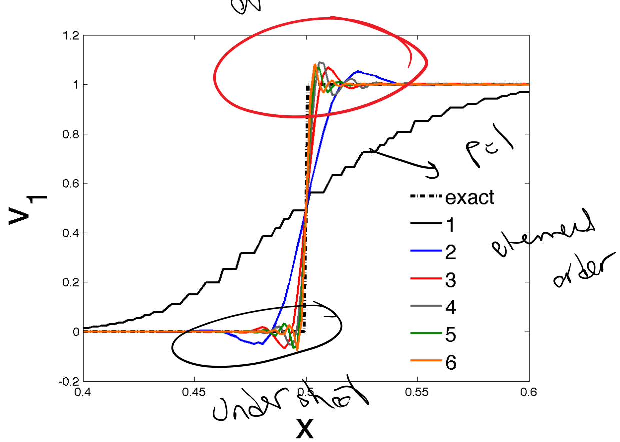
Results by Scott Miller



Shock capturing: 473K elements

Shock tracking: 446 elements

overshoot

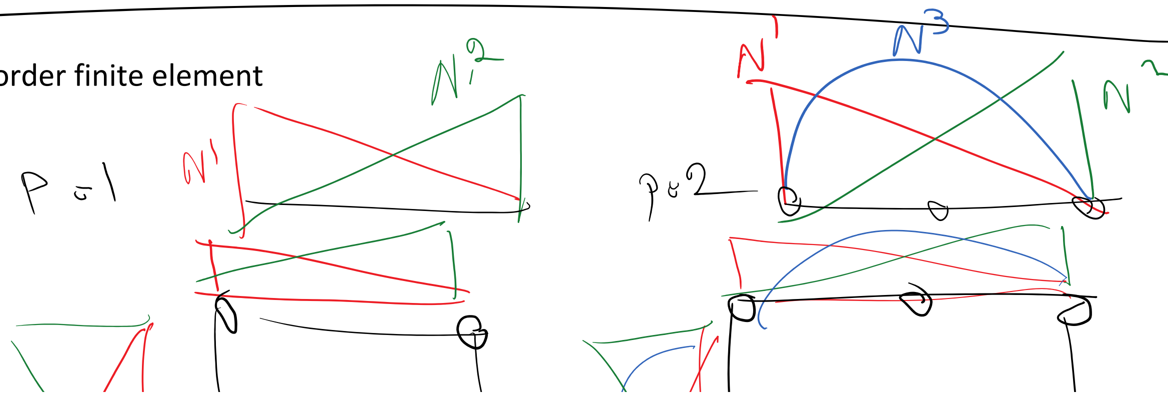


Formulation of a higher order finite element

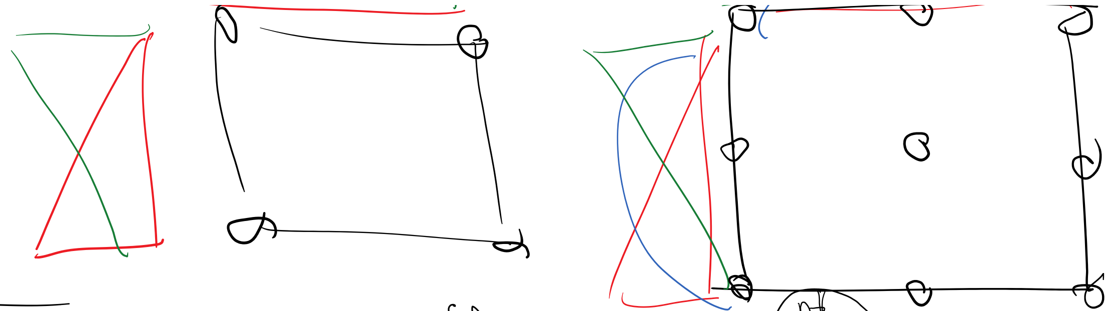
1D p=1

p=2

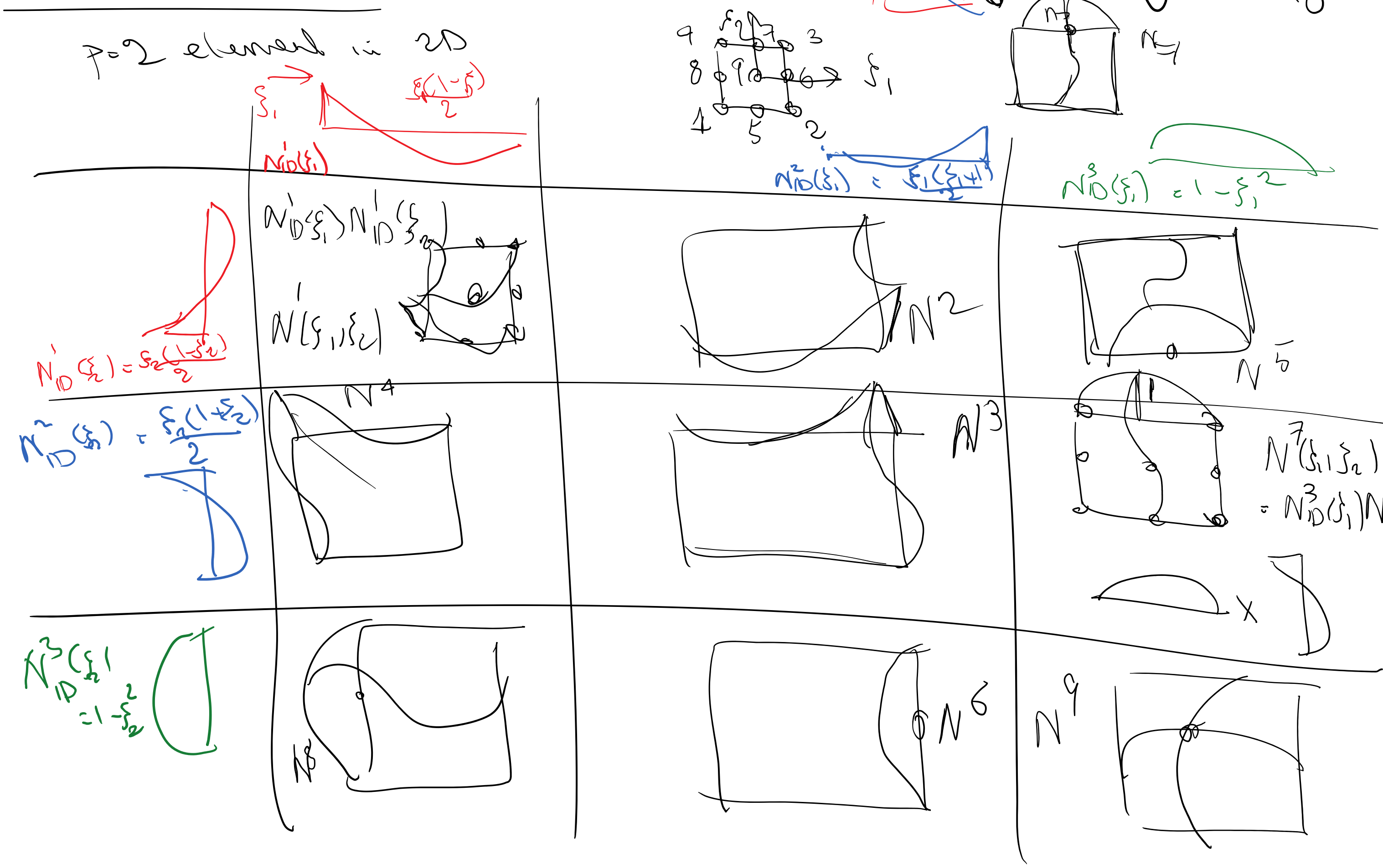
2D p=1



2D p=1



p=2 element in 2D



$$k = \int_{\xi_1=-1}^{\xi_1=1}\int_{\xi_2=-1}^{\xi_2=1} B_\xi^t \left(J^{-1} k J^{-t} |J|\right) B_\xi \, d\xi_1 \, d\xi_2$$

So the element is 2nd order in ξ_1 & ξ_2

$$N = [N^1(\xi_1,\xi_2) \quad \cdots \quad N^9(\xi_1,\xi_2)]$$

$$N = [\, N^1(\xi_1,\xi_2) \quad - - - \quad N^1(\xi_1,\xi_2)$$

$$N^1(\xi_1,\xi_2) = \frac{\xi_1(1-\xi_1)\,\xi_2(1-\xi_2)}{4}$$

For 2D will be different

our element is order 2 in ξ_1, order 2 in ξ_2

1D: B_p : $\frac{d}{d\xi_1} N$ → order p; → $p-1$

$$B_\xi = \begin{bmatrix} \frac{\partial N^1}{\partial \xi_1} & \frac{\partial N^2}{\partial \xi_1} & - - - & \frac{\partial N^9}{\partial \xi_1} \\ \frac{\partial N^1}{\partial \xi_2} & \frac{\partial N^2}{\partial \xi_2} & - - - & \frac{\partial N^9}{\partial \xi_2} \end{bmatrix}$$

$$B_\xi = \begin{bmatrix} (1-2\xi_1)\xi_2(1-\xi_2)/4 & \partial N^2/\partial\xi_1 & - - & \partial N^9/\partial\xi_1 \\ \xi_1(1-\xi_1)(1-2\xi_2)/4 & \partial N^2/\partial\xi_2 & & \partial N^9/\partial\xi_2 \end{bmatrix}$$

$$K = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} B^t J^{-1} \kappa J^{-t} |J| B = \iint \underbrace{\begin{bmatrix} \frac{(1-2\xi_1)\xi_2(1-\xi_2)}{4} & \frac{\xi_1(1-\xi_1)(1-2\xi_2)}{4} \\ \partial N^2/\partial\xi_1 & \partial N^2/\partial\xi_2 \\ | & \\ | & \\ \partial N^9/\partial\xi_1 & \partial N^9/\partial\xi_2 \end{bmatrix}_{B_\xi^t} \underbrace{J^{-1}\kappa J^{-1}|J|}_{\text{Ignore this}} \begin{bmatrix} \frac{(1-2\xi_1)\xi_2(1-\xi_2)}{4} & \dots & \frac{\partial N^9}{\partial\xi_1} \\ \frac{\xi_1(1-\xi_1)(1-2\xi_2)}{4} & \dots & \frac{\partial N^9}{\partial\xi_2} \end{bmatrix}_{B_\xi}}\, d\xi_1 d\xi_2$$

$$I(1,1) = \left(\frac{(1-2\xi_1)\xi_2(1-\xi_2)}{4}\right)^2 + \left(\frac{\xi_1(1-\xi_1)(1-2\xi_2)}{4}\right)^2 \quad I$$

order 2 in ξ_1 — order ~~4~~ in ξ_1

order 4 in ξ_2 — " 2 " ξ_2

I is order $2p$ in ξ_1, ξ_2

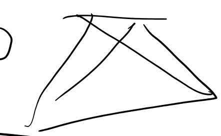
2D, 3D — for element order p, heat conduction

$p \to 2p$ integrand

$p \to 2(p-1)$ { 1D ; 2D simplex elements ; 3D }

tri

3D



Integration of K

$$K = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} \underbrace{\left(B^T J^{-1} \alpha J^{-T} |J| B\right)}_{I(\xi_1,\xi_2)} \, d\xi_1 \, d\xi_2$$

$I(\xi_1,\xi_2)$ order 4 in ξ_1, order 4 in ξ_2

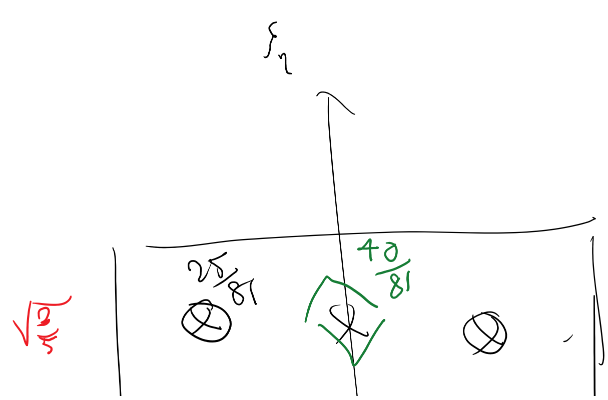
$$o_{\xi_1} = 4 \longrightarrow n_{\xi_1} = \text{ceil}\left(\frac{o_{\xi_1}+1}{2}\right) = 3$$

$$o_{\xi_2} = 4 \longrightarrow n_{\xi_2} = 3$$

Go to the Gauss table

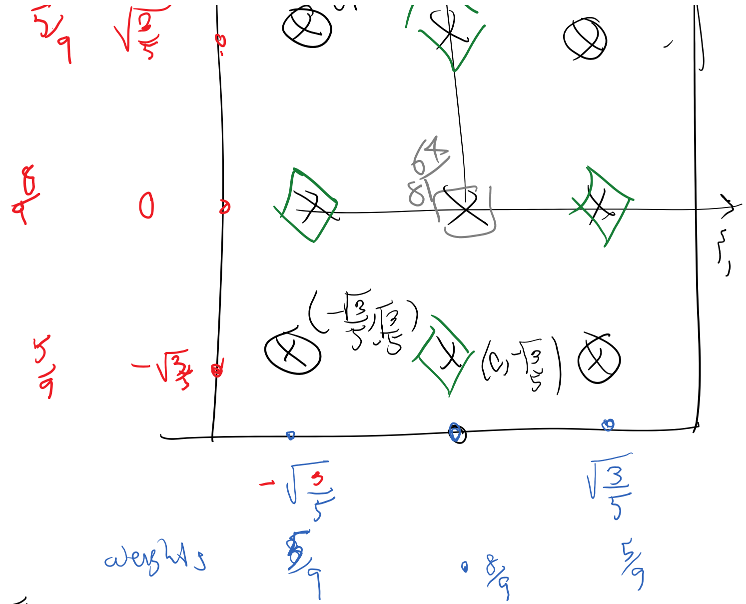
Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555

$\sqrt{3/5}$



$\sqrt{3/5}$

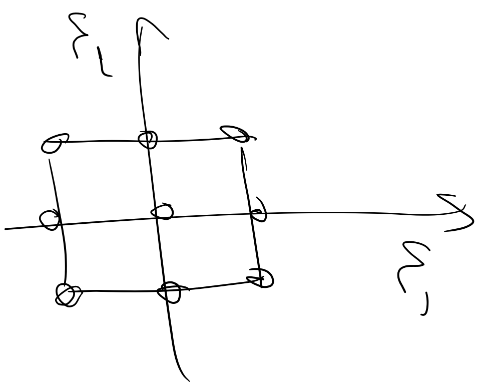
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

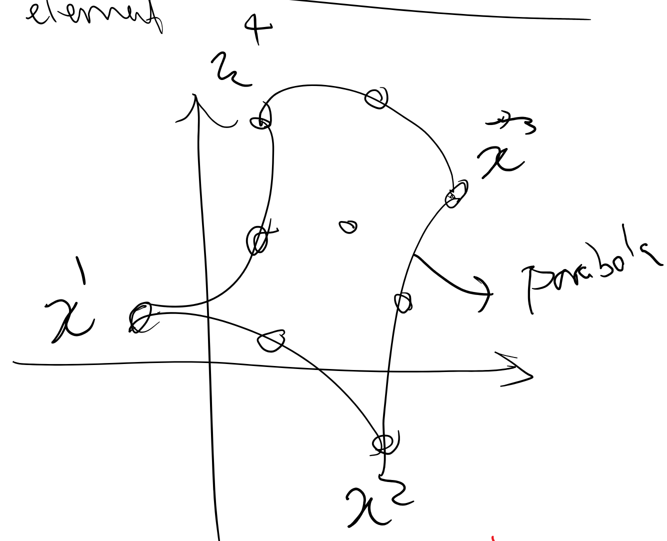


$$K = \sum_{i=1}^{9} \omega^i I(\xi_1^i, \xi_2^i)$$

$$\omega^1 = \frac{25}{81} \qquad (\xi_1^1, \xi_2^1) = \left(-\sqrt{\tfrac{3}{5}}, -\sqrt{\tfrac{3}{5}}\right)$$

things we can do with p = 2 element





$$\vec{x} = (x, y) = (x(\xi_1, \xi_2), y(\xi_1, \xi_2))$$

Iso parametric element

$$x(\xi_1, \xi_2) = \sum_{i=1}^{9} x^i N^i(\xi_1, \xi_2)$$

$$y(\,,\,) = \sum_{i=1}^{9} y^i N^i(\xi_1, \xi_2)$$

solution is the same order as geometry

same interpolation for solution & geometry

solution $T(\xi_1, \xi_2) = \sum_{i=1}^{9} T^i N^i(\xi_1, \xi_2)$

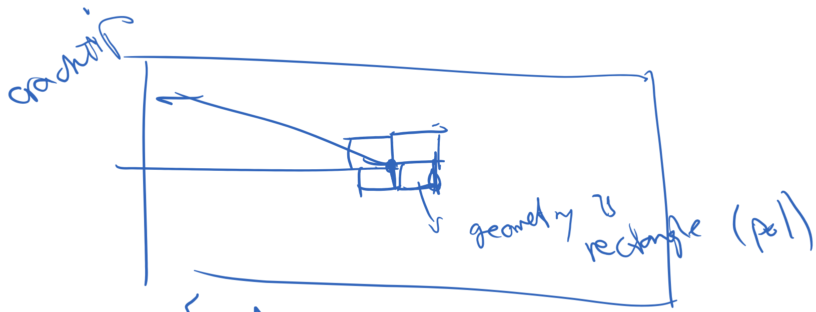
Application: modeling complex geometries well

has a much ... in capturing the geometry

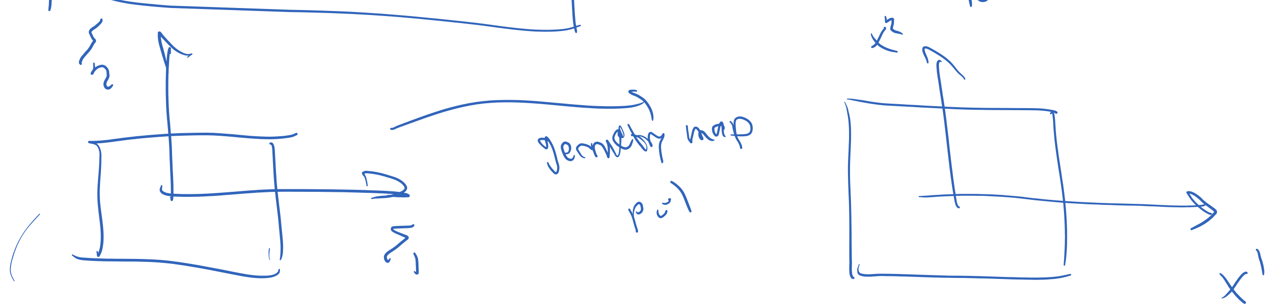
Application in modeling comp...

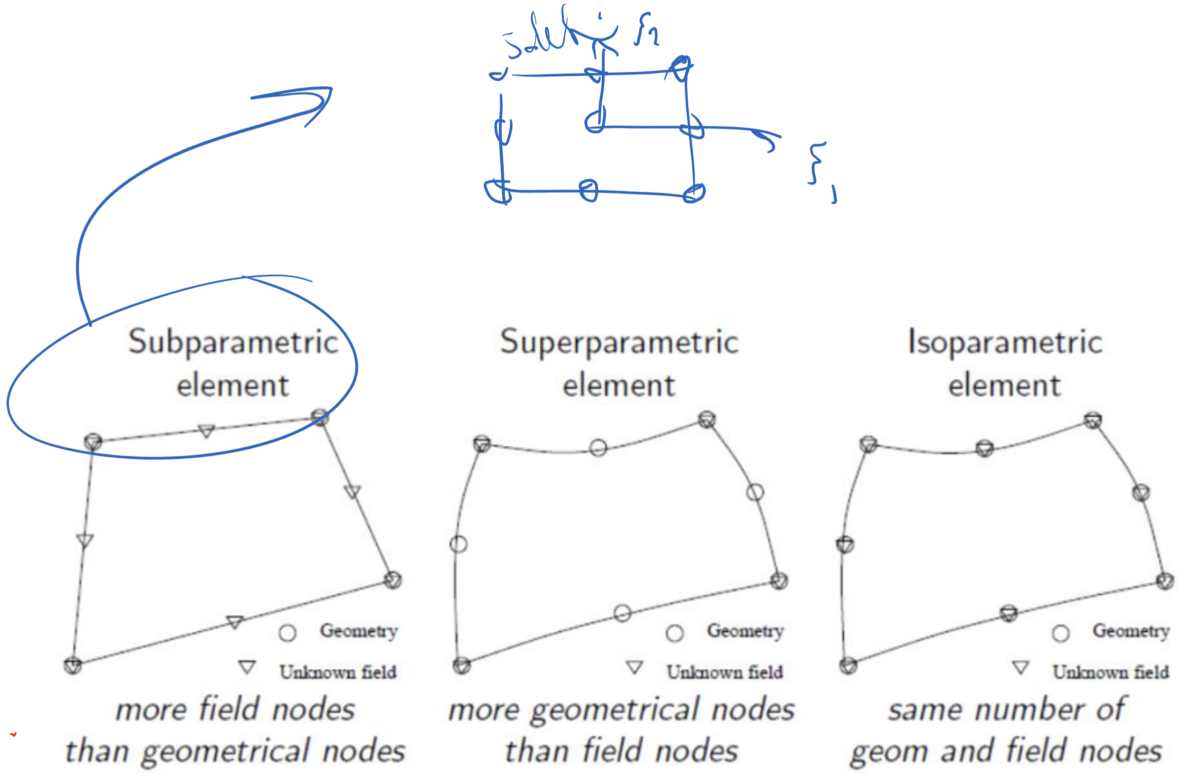
does a much better job capturing the geometry

There are cases that we only deal with simple geometries
but the solution should be accurate



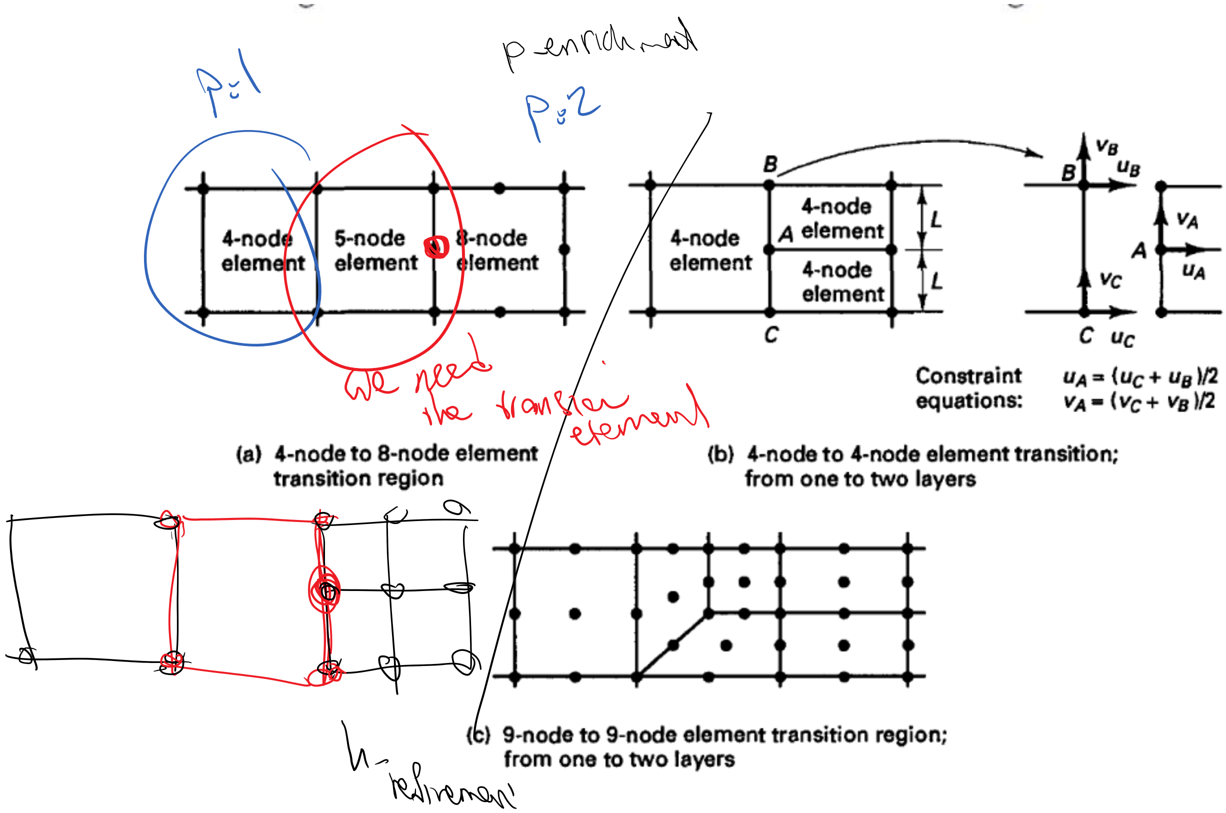
but the solution can be more accurate





p=1

p-enrichment

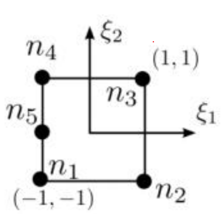


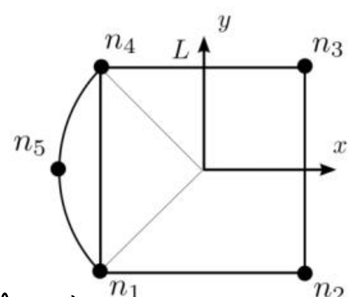
(a) 4-node to 8-node element transition region

(b) 4-node to 4-node element transition; from one to two layers

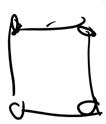
(c) 9-node to 9-node element transition region; from one to two layers

The questions for this element are.





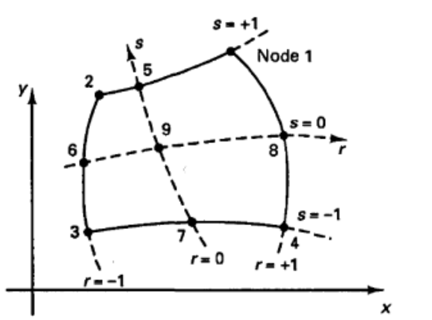
connecting element

start with  basis functions & modify it

N^1	$\frac{(1-\xi_1)(1-\xi_2)}{4}$	$-\frac{1}{2}N^5$
N^2	$\frac{(1+\xi_1)(1-\xi_2)}{4}$	
N^3	$\frac{(1+\xi_1)(1+\xi_2)}{4}$	
N^4	$\frac{(1-\xi_1)(1+\xi_2)}{4}$	$-\frac{1}{2}N^5$
N^5	$\frac{(1-\xi_1)}{2}$ $(1-\xi_2^2)$	$\frac{(1-\xi_1)(1-\xi_2^2)}{2}$

n_4 n_3 n_5 n_1 n_2

$N^1 = \frac{(1-\xi_1)(1-\xi_2)}{4} - \frac{1}{2}N^5$



(a) 4 to 9 variable-number-nodes two-dimensional element

		Include only if node i is defined				
		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					

(b) Interpolation functions