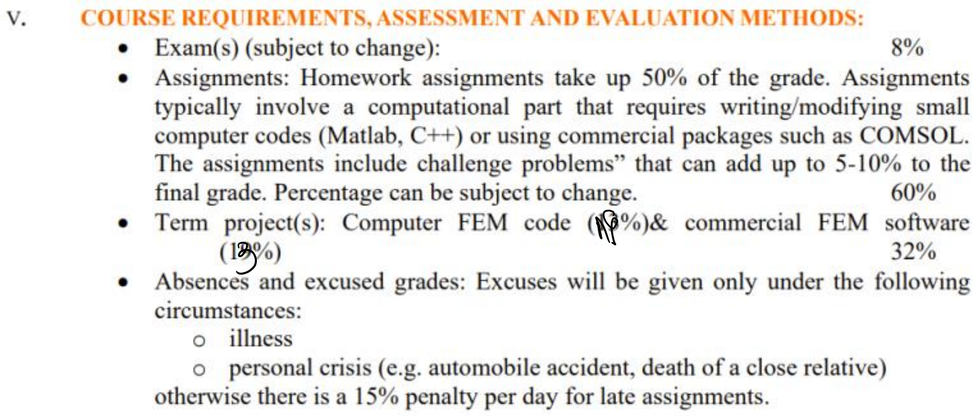
FEM2023/08/23

Wednesday, August 23, 2023 11:21 AM

https://rezaabedi.com/teaching/me-517-finite-elements/



V. **COURSE REQUIREMENTS, ASSESSMENT AND EVALUATION METHODS:**

- Exam(s) (subject to change): 8%
- Assignments: Homework assignments take up 50% of the grade. Assignments typically involve a computational part that requires writing/modifying small computer codes (Matlab, C++) or using commercial packages such as COMSOL. The assignments include challenge problems" that can add up to 5-10% to the final grade. Percentage can be subject to change. 60%
- Term project(s): Computer FEM code (19%)& commercial FEM software (13%) 32%
- Absences and excused grades: Excuses will be given only under the following circumstances:
  - illness
  - personal crisis (e.g. automobile accident, death of a close relative)

  otherwise there is a 15% penalty per day for late assignments.

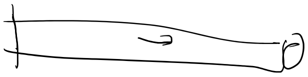
Code Project
- near the end of the course
- suggestion:
  use an object-oriented language like Python or C++
  (up to 10% credit)
  groups of up to 3 students

Brief course outline:

- Mathematical formulation (40%)
  - Long but more general approach
    - Balance laws
    - Differential equations
    - Weighted Residual Statement (WRS)
    - Weak statement (FEM, Discontinuous Galerkin, spectral method, …)
  - Energy methods, more limited but right away we get the weak statement

$\Sigma F = 0$

$(AEu')' + q = 0$



$$\int_0^L \underbrace{w}_{\text{weight}} \Big(\underbrace{(AEu')' + q}_{\text{Residual}} \Big) dx = 0 \quad x = 0$$

(WRS)

Energy = - - -

minimize the energy

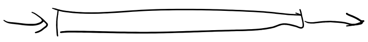
$$\int_0^L w' AE u' \, dx = \int_0^L w q$$

no BC

Main objective of the course is to be able to formulate and implement a finite element method

- Special 1D elements

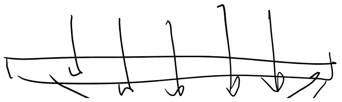
bar



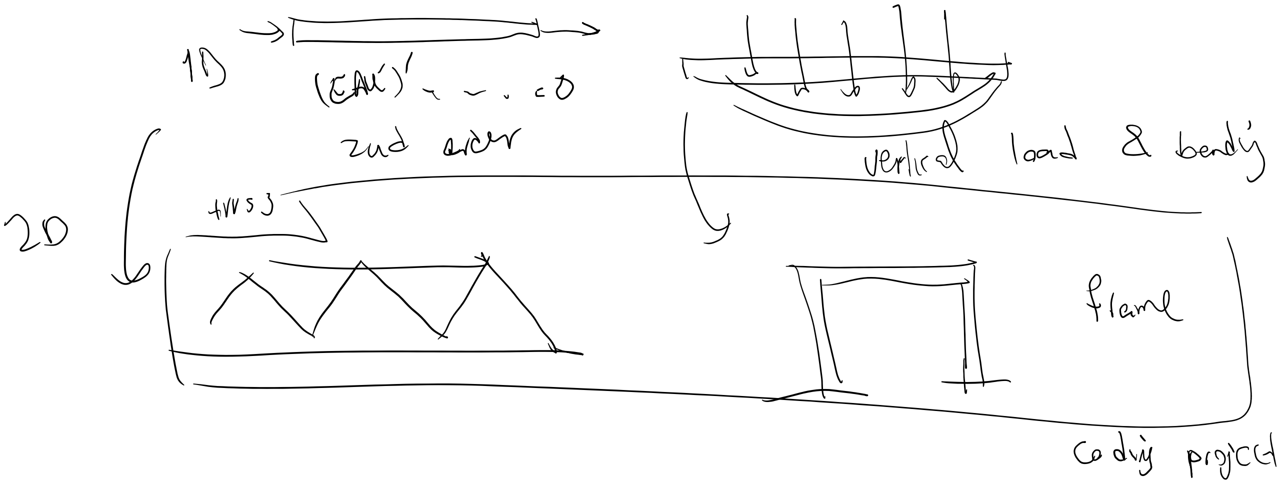
1D

$(EAu')' - - = 0$

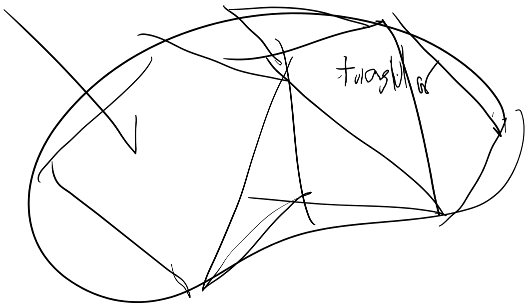
beam



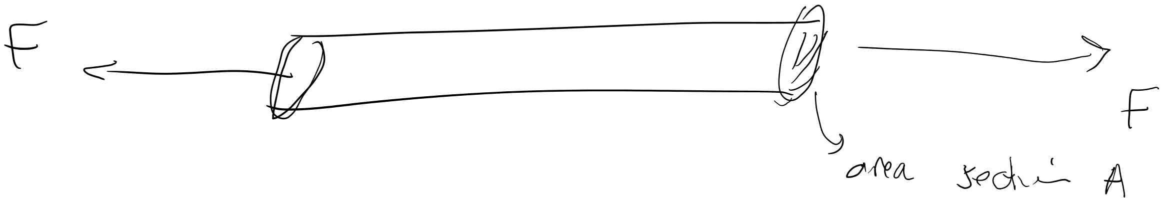
$(EIy'')'' + q = 0$



- Quadrature and 2D/3D elements:
    - Quadrature (numerical integration)
        - Newton-Cotes
        - Quass Quadrature
    - 2D/3D elements:
        - Heat equation
        - Elastostatics
- Coding of a FEM



Bar problem

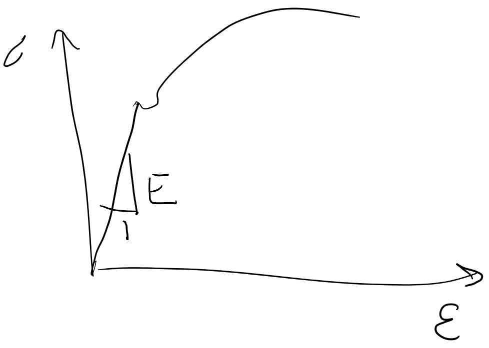


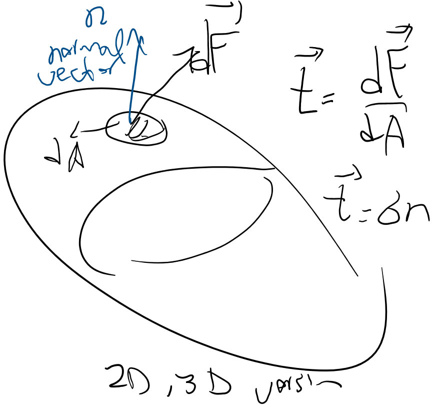
stress = intensity of force = force per unit area

$$\sigma = \frac{F}{A}$$

$$\sigma = E\varepsilon$$

strain



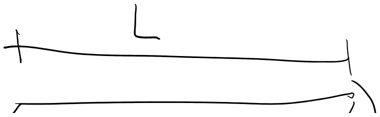


$$\vec{t} = \frac{d\vec{F}}{dA}$$

$$\vec{t} = \sigma n$$

$$F = AE\varepsilon$$

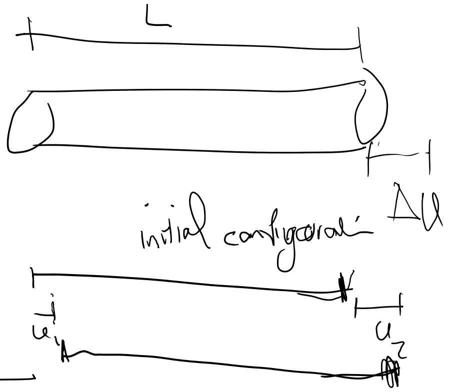
change of



$$\varepsilon = \frac{\Delta u}{L}$$

change of length

$$= \frac{u_2 - u_1}{L}$$

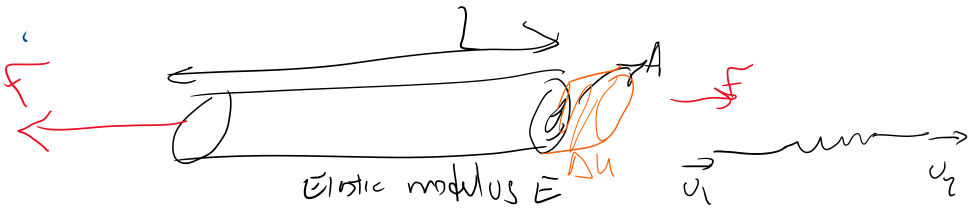


$$\Delta u = u_2 - u_1$$

$$F = AE\varepsilon = \frac{AE}{L}\Delta u$$

$$\Delta u = u_2 - u_1$$

Ⓘ



$$F = k\,\Delta u$$

k spring stiffness $= \frac{AE}{L}$

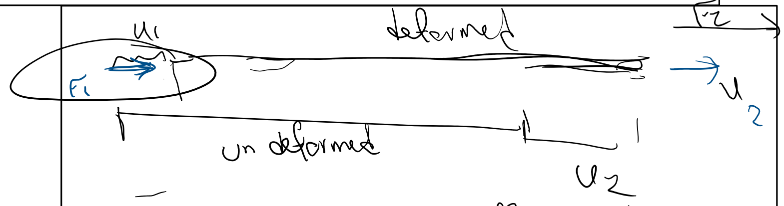
$$\Delta u = u_2 - u_1$$

$$F = k(u_2 - u_1)$$

$$F_2 = F$$

$$F_1 = -F$$

②

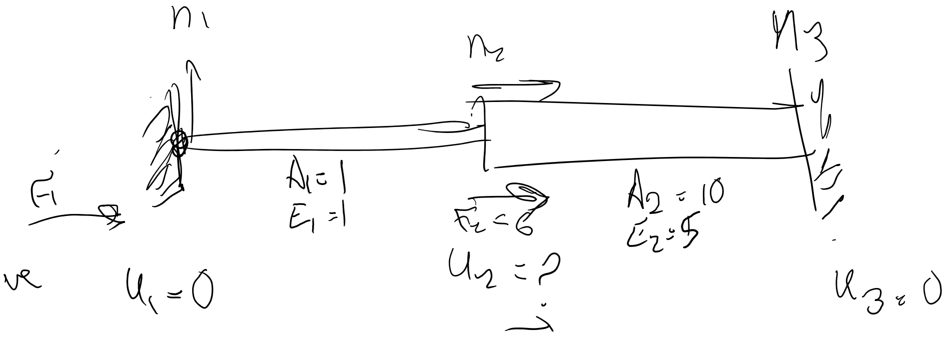


$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = k \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}^{\text{stiffness matrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

force vector

$$k = \frac{AE}{L}$$

displacement vector



x: don't know the value

X: don't know the value

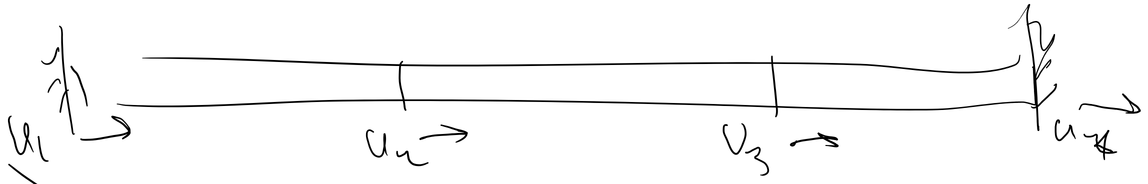
$u_1 = 0$ $\qquad$ $u_2 = ?$ $\qquad$ $u_3 = 0$

$\vec{F_2}$ $\qquad$ $\vec{F_3}$

$$\begin{array}{c}X\\ \checkmark\\ X\end{array}\begin{bmatrix}F_1\\F_2\\F_3\end{bmatrix} = \begin{bmatrix}K_{11}&K_{12}&K_{13}\\K_{21}&K_{22}&K_{23}\\K_{31}&K_{32}&K_{33}\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{array}{l}=0\\ X\\ 0\end{array}$$

stiffness for this structure

$$\checkmark\ F_2 = \cancel{K}\, u_2^{?} \qquad u_2 = K^{-1} F_2$$



$$\begin{bmatrix}F_2\\F_3\end{bmatrix} = K_{2\times2}\begin{bmatrix}u_2\\u_3\end{bmatrix}^{?} \rightarrow \begin{bmatrix}u_2\\u_3\end{bmatrix} = K^{-1}\begin{bmatrix}F_2\\F_3\end{bmatrix}$$

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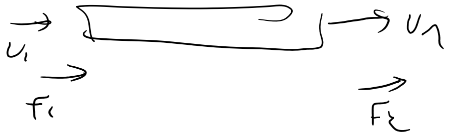
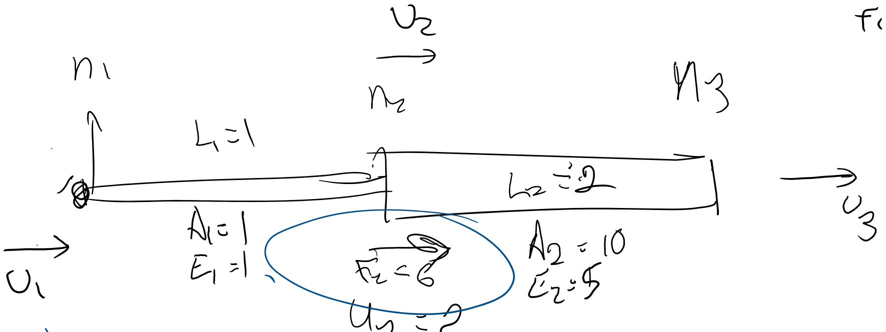
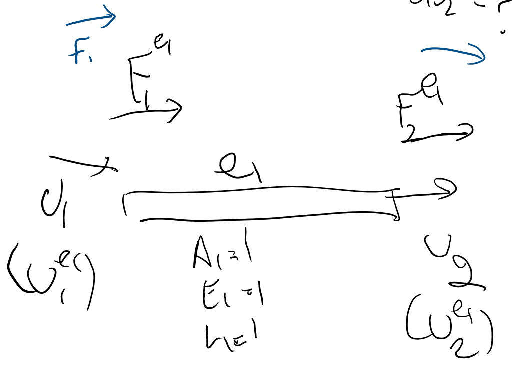
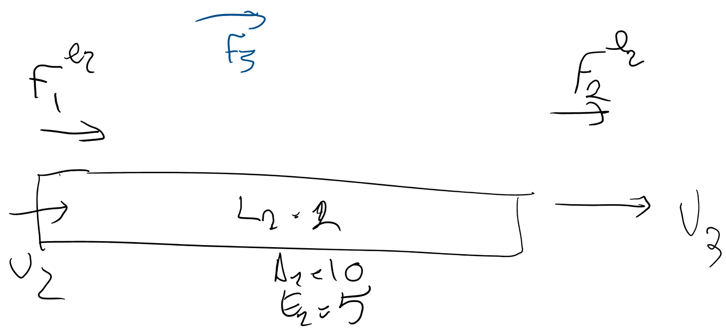
why $\begin{bmatrix}F_1\\F_2\end{bmatrix} = k\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$ is much better

than $F = k' \Delta u$







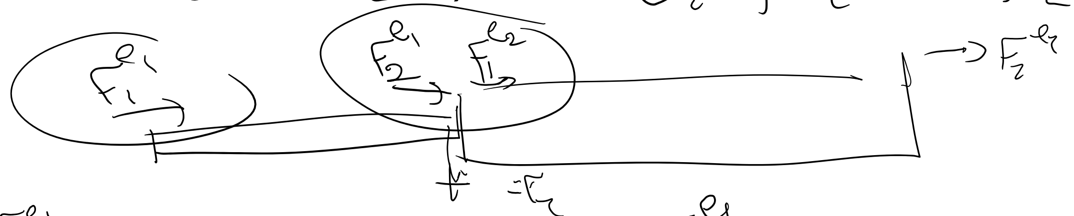


$(\omega_1^{e_1})$ $E_1=1$ $h=1$ $(\omega_2^{e_2})$ U_2 $A_2=10$ $E_2=5$

$$k_1 = \frac{A_1E_1}{L_1} = 1 \qquad k_2 = \frac{A_2E_2}{L_2} = 25$$

$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = 1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad \begin{bmatrix} F_1^{e_2} \\ F_2^{e_2} \end{bmatrix} = k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad \begin{bmatrix} F_1^{e_2} \\ F_2^{e_2} \end{bmatrix} = \begin{bmatrix} 25 & -25 \\ -25 & 25 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$



$F_1^{e_1} = U_1 - U_2$

$F_2^{e_1} = -U_1 + U_2$

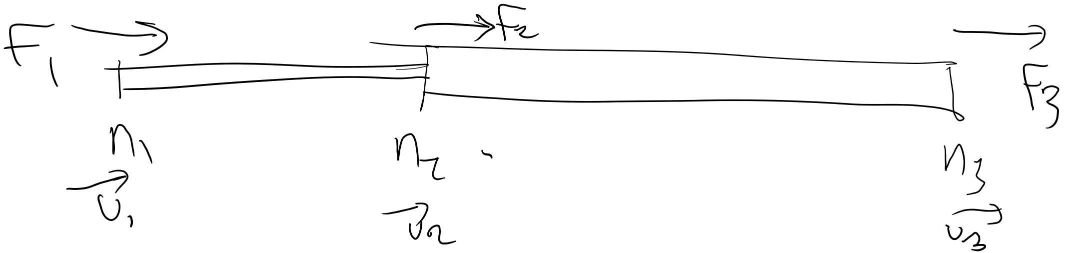
$F_1^{e_2} = 25U_2 - 25U_3$

$F_2^{e_2} = -25U_2 + 25U_3$

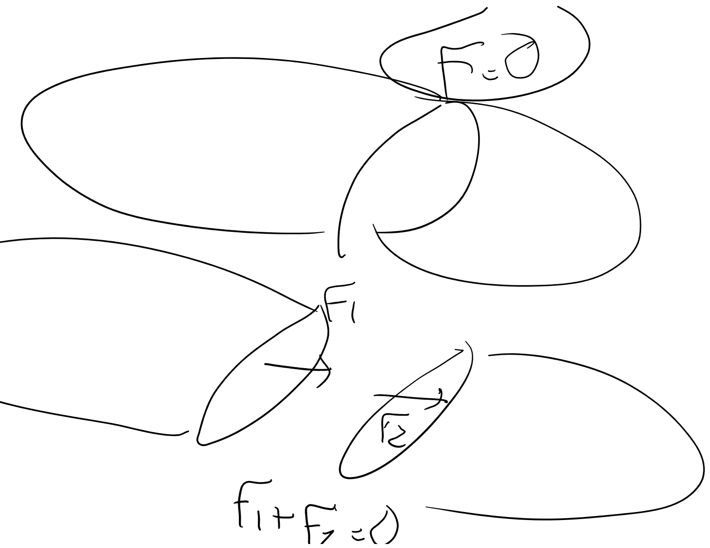
$F_1 = F_1^{e_1} = U_1 - U_2$

$F_2 = F_2^{e_1} + F_1^{e_2} = -U_1 + U_2 + 25U_2 - 25U_3$

$F_3 = F_2^{e_2} = -25U_2 + 25U_3$



$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 26 & -25 \\ 0 & -25 & 25 \end{bmatrix}}_{\text{stiffness of the structure}}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$



$$\begin{array}{c|cc} & ① & ② \\ \hline ① & 1 & -1 \\ ② & -1 & 1 \end{array}$$

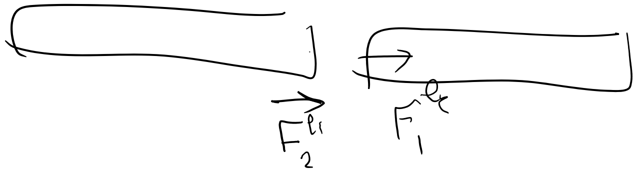
$$\begin{array}{c|cc} & ② & ③ \\ \hline ② & 25 & -25 \\ ③ & -25 & 25 \end{array}$$

$F_1 + F_2 = 0$

at each

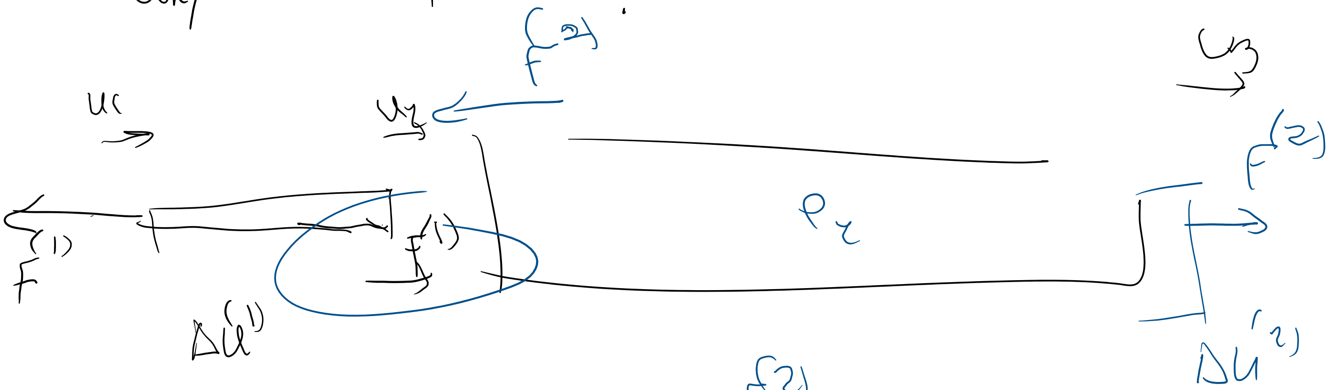
$$\begin{bmatrix} +1 & -1 & \\ -1 & 1+25 & -25 \\ & -25 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 26 & 25 \\ 0 & -25 & 25 \end{bmatrix}$$



$\vec{F_2} = F_2^{e_1} + F_1^{e_2}$

Why this is powerful?

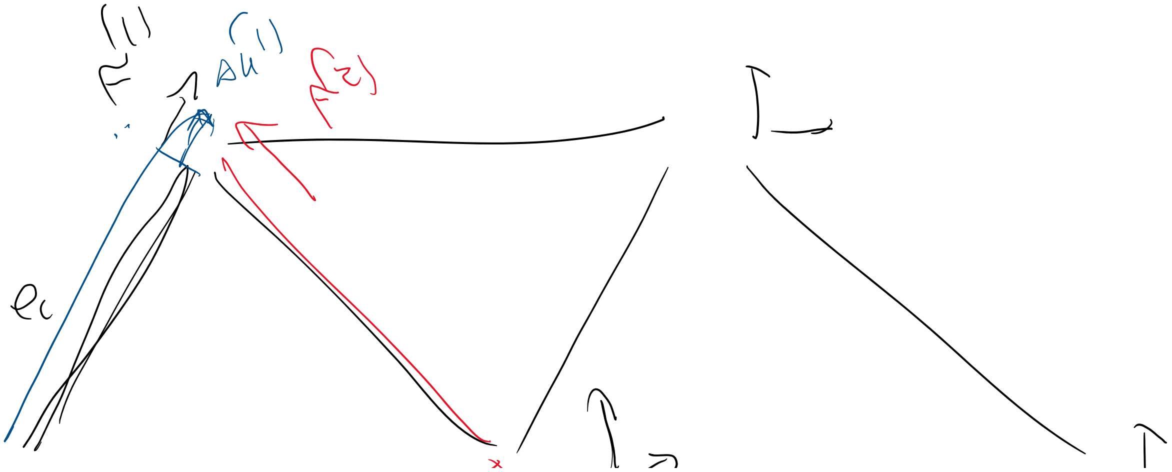


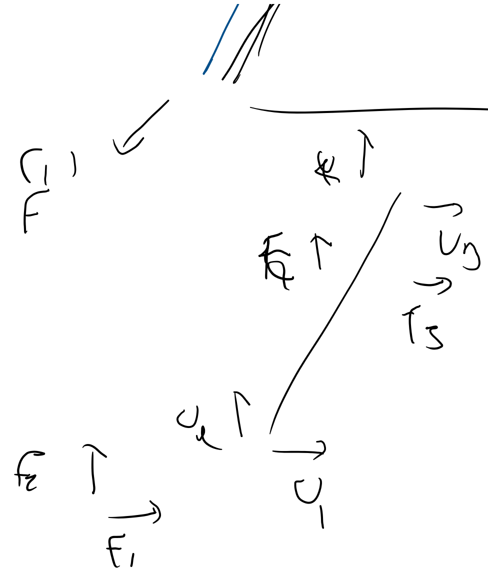
$F^{(1)} = K^{(1)} \Delta u^{(1)}$

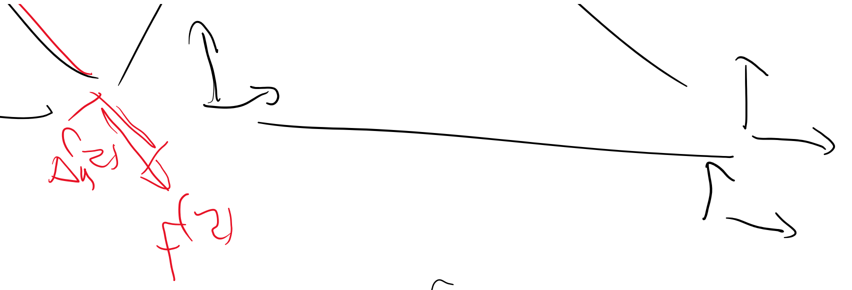
$\Delta u^{(1)} = u_2 - u_1$

$F^{(2)} = k^2 \Delta u^{(2)}$

$\Delta u^2 = u_3 - u_2$







$$\begin{pmatrix} F_1 \\ \vdots \\ F_4 \end{pmatrix} = K_{4x4} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$F = k \Delta u$$