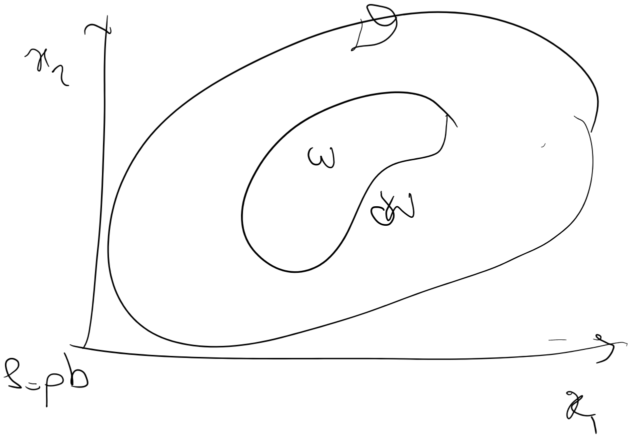
2023/09/06

Wednesday, September 6, 2023 11:17 AM

Balance laws:

- Slides 6-13 for dynamic case are FYI



(1) $-\int_{\partial\omega} f_x \cdot \vec{n}\, ds + \int_{\omega} g\, dV = 0$

Examples:
- Elastostatic $f_x = \sigma$ $\quad g = \rho b$
- Heat conduction $f_x = q$ $\quad g = Q$ $\quad$ HW2

want to change this to an interior integral

we'll use divergence theorem (2) $\int_{\partial\omega} f_x \cdot n\, ds = \int_{\omega} \nabla \cdot f_x\, dV$

(2) → (1) $\Longrightarrow \int_{\omega} (-\nabla\cdot f_x + g)\, dV = 0$

$\boxed{-\nabla\cdot f_x + g = 0}$ (3) Differential eqn

Explanation of (1) → (3)

Balance law → Differential eqn

(4)

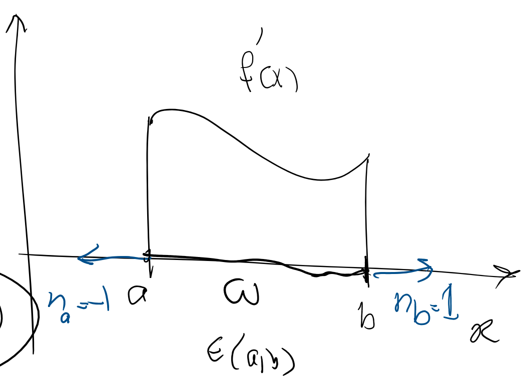
$\int_{\omega} f'(x)dx = \int_a^b f'(x)\,dx = f(b) - f(a)$

$\omega = [a,b]$

$f(b)n_b + f(a)n_a$

$= \sum_{\partial\omega} f(x)n(x)$

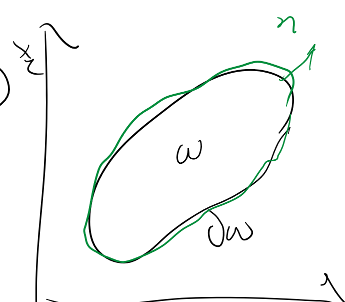
1D



$\in (a,b)$

$\partial\omega = \{a,b\}$

2D & 3D

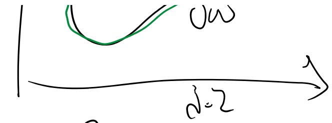


$\int_{\omega} \underline{\nabla\cdot f(x)}\, dV = \int_{\partial\omega} f(x)\cdot\vec{n}\, ds$

$\parallel$

$\partial f_1(x_1,x_2)$ $\quad \partial f(x_1,x_2)$

$$\omega \quad \frac{\partial f_1(x_1,x_2)}{\partial x_1} + \frac{\partial f(x_1,x_2)}{\partial x_2} \qquad \partial\omega$$



vector ← $f = (f_1(x_1,x_2), f_2(x_1,x_2))^T$

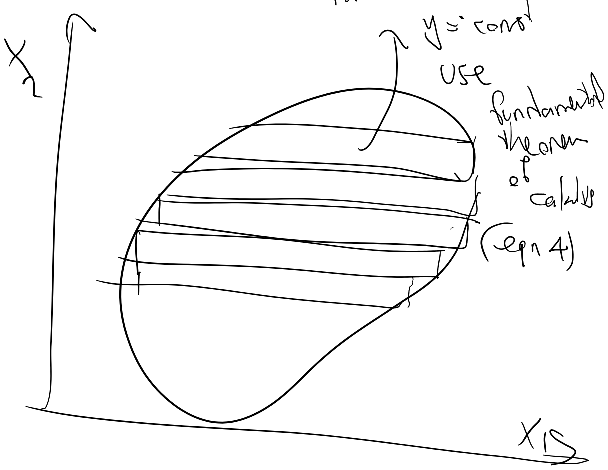
eg. $\vec{q} = (q_1(x_1,x_2), q_2(x_1,x_2))$

if f was a scalar function

$$\forall i \in \{1,2,3\} \int \frac{\partial f}{\partial x_i}\, dV = \int_{\partial\omega} f(x)\, n_i\, dS$$

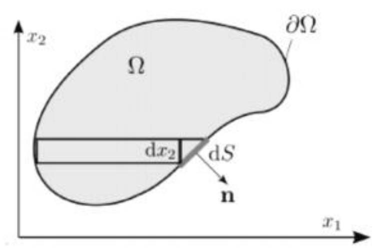
$i = 1$

$$\int \frac{\partial f}{\partial x_1}\, dV = \int_{\partial\omega} f(x)\, n_1\, dS$$



Transfer of boundary to interior integral higher dimensions

- Ω is compact and closed.
- $\partial\Omega$ is piecewise smooth.
- Normal vector $\mathbf{n}$ is defined almost everywhere (a.e.) and is pointing outward.
- tensor field (scalar, vector, matrix, ...):
  - $\mathbf{F}_{,i} = \partial\mathbf{F}/\partial x_i$ exists everywhere and
  - is continuous.



$$\boxed{\int_{\partial\Omega} \mathbf{F}.n_i\, \mathrm{d}S = \int_{\Omega} F_{,i}\, \mathrm{d}V} \qquad (18)$$

This is the generalization of the 1D version:

$$1.F(b) + (-1).F(a) = F(b) - F(a) = \int_{[a,b]} F'(x)\, \mathrm{d}x$$

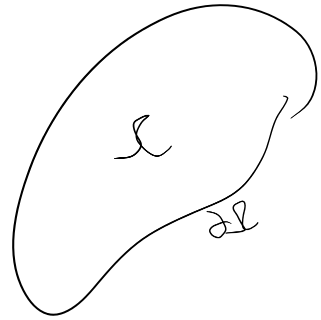
Divergence Theorem

C^1 function

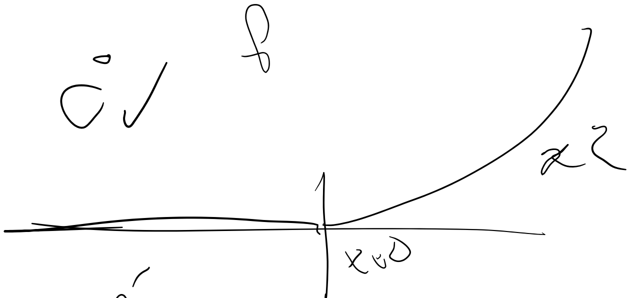
Let tensor field $\mathbf{F}$ have continuous partial derivatives, $F_{,i}$, everywhere for all i, Ω be closed, compacted, and has piecewise smooth boundary with well-defined normal vector a.e. Then, Let tensor field $\mathbf{F}$ have continuous partial derivatives, $F_{,i}$, everywhere for all i, Ω be closed, cómpacted, and has piecewise smooth boundary with well-defined normal vector a.e. Then,

$$\boxed{\int_{\partial\Omega} \mathbf{F}.\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \mathrm{div}\mathbf{F}\,\mathrm{d}V} \qquad (19)$$

eg F vector $\quad \mathrm{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$

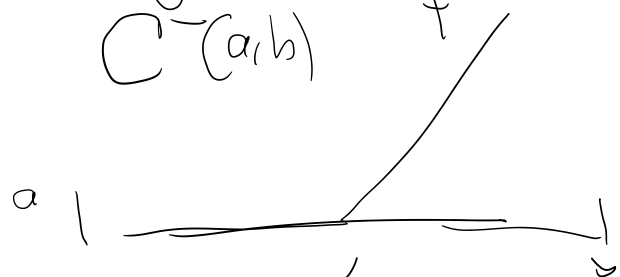


C^0 ✓ $\quad f$



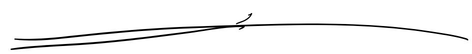
f' is continuous

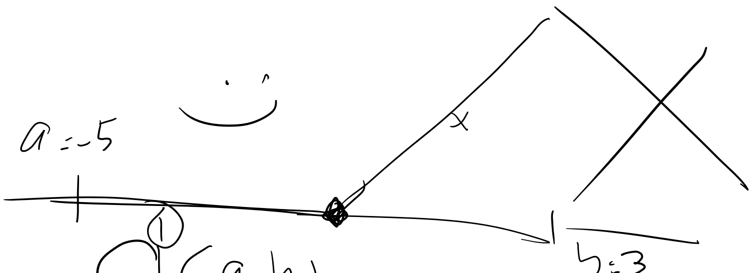
$C^0(a,b) \quad f$



f'

not continuous

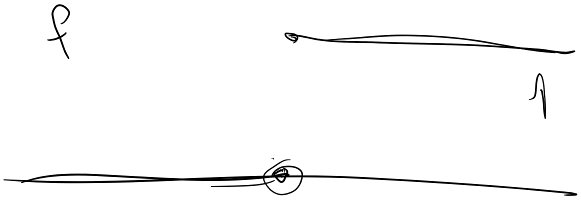




$C^1(a,b)$

has 1st derivative &
this derivative is continuous

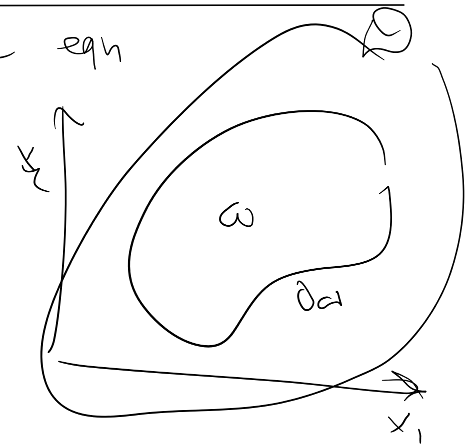
f''



f is NOT a C^2 function

See slide 18 why continuity of f' is necessary

Summary of Balance law + Divergence eqn



Balance law

$$-\int_{\partial\omega} \vec{f_x}\cdot\vec{n}\, ds + \int_{\omega} s\, dV = 0$$

$$-\int_{\omega} \nabla\cdot f_x\, dV + \int_{\omega} s\, dV = 0$$

$$\int_{\partial\omega} h\cdot n\, ds = \int_{\omega} \nabla\cdot h\, dV$$

we'll come back to ☆

$$\int_{\omega} \Big(\underbrace{\overbrace{-\nabla\cdot f_x + s}^{g\,(x)}}\Big)\, dV = 0$$

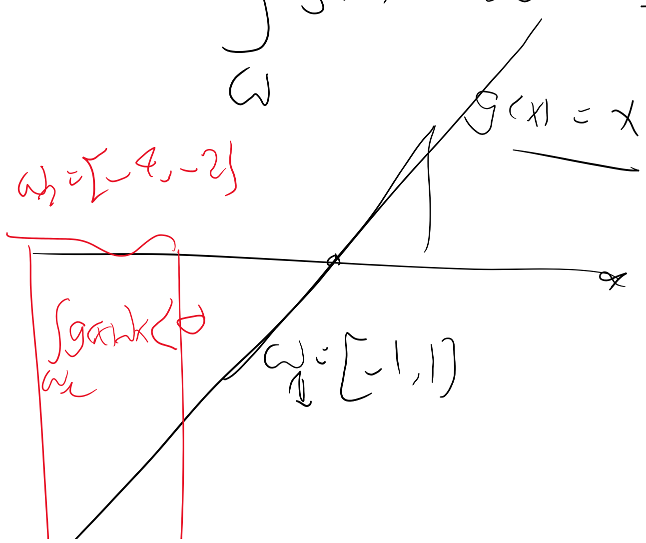
f_x : spatial flux

Elastostatic $f_x = -\sigma$

Heat conduction $f_x = q$

I'll claim that $g(x) = 0$

$$\int_{\omega} g(x)\, dV = 0 \xrightarrow{?} \text{Can we say } g(x) = 0$$



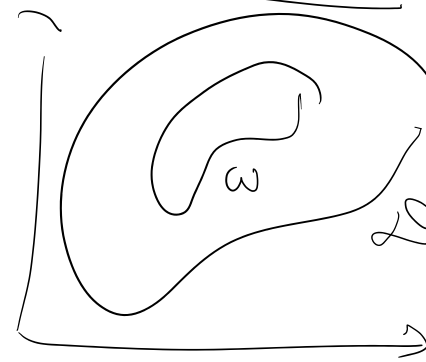
$\int_{\omega} g(x)\, dx = 0$ but $g(x) \neq 0$

$\omega = [-1,1]$

Localization theorem

function g is continuous

$\forall \omega \subseteq \mathcal{D} \quad \int_\omega g(x)\,dV = 0$



$\Longrightarrow g(x) = 0 \quad \forall x \in \mathcal{D}$

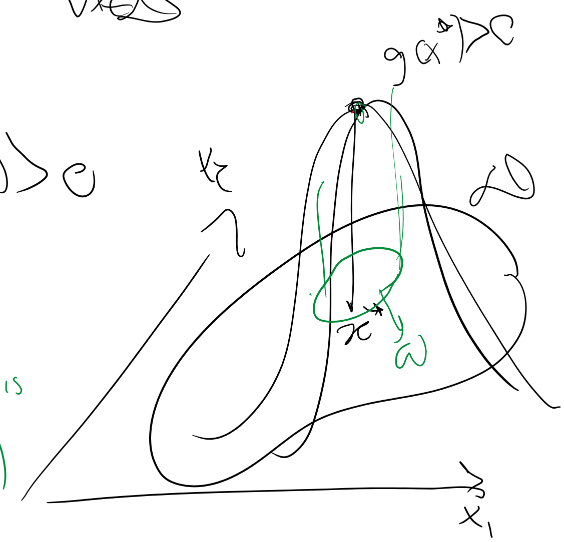
Counter proof

say there is a x^* for which $g(x^*) > 0$

choose a ω around x^* such that

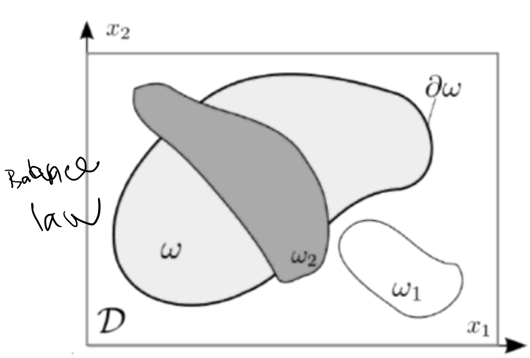
g stay's positive (because g is continuous)

$\int_\omega g(x)\,dx > 0$



Use localization to obtain Differential equation

Balance law



Balance law said

$\forall \omega \subseteq \mathcal{D} \quad -\int_{\partial\omega} f_x \cdot n\, ds + \int_\omega s\, dV = 0$

Divergence theorem

$\forall \omega \subseteq \mathcal{D} \quad \int (-\nabla \cdot f_x + s)\, dV = 0$

$\forall \omega \subseteq D \quad \int_{\omega} \left(-\nabla \cdot f_x + s \right) dV = 0$

$\xrightarrow{\text{Localization}}$

Differential equation

$$\boxed{-\nabla \cdot f_x + s = 0} \quad (DE)$$

Some points:

1. Balance law is more general that the differential equation:
   a. For balance law, the function fx should just be integrable $\int_{\partial\omega} f_x \, n \, ds < \infty$
   b. For DE, derivatives of fx should exist AND be continuous $\rightarrow \int_{\omega} \nabla \cdot f_x \, dV < \infty$

   So balance law always holds BUT DE only holds when spatial flux is smooth enough (C1)

2. The power of a balance law is that it holds for all sets omega inside D -> This enables us to use localization and prove that DE = 0.

Examples

1 Elastostatics

2D

$$f_x = -\sigma = -\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \qquad s = \rho b = \begin{bmatrix} \rho b_x \\ \rho b_y \end{bmatrix}$$

$$-\nabla \cdot f_x + s = \nabla \cdot \sigma + \rho b = \nabla \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} + \begin{bmatrix} \rho b_x \\ \rho b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \end{bmatrix}}_{\nabla \cdot \sigma} + \begin{bmatrix} \rho b_x \\ \rho b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

HW 9

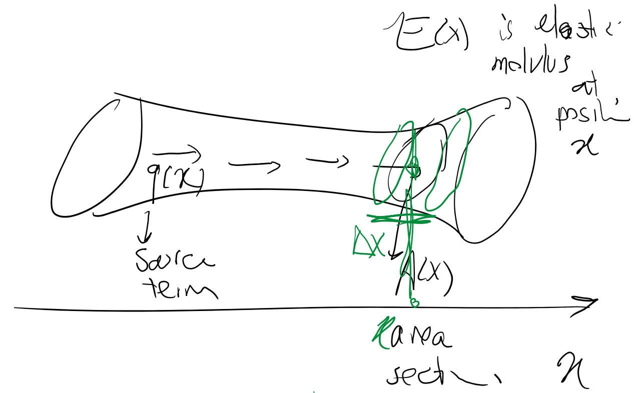
$$f_x = \begin{bmatrix} q_x \\ q_y \end{bmatrix} \qquad s = Q$$

$$-\nabla \cdot f_x + s = \nabla \cdot \begin{bmatrix} q_x \\ q_y \end{bmatrix} + Q = 0$$

$$-\nabla \cdot f_x + s = \nabla \cdot [q_y] + Q = 0$$

$$\boxed{-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right) + Q = 0}$$

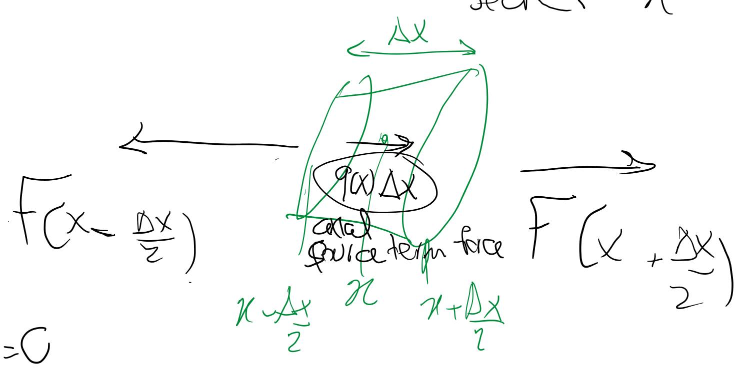
1D example: 1D elastostatics



① Balance law

Sum of forces is equal to zero

$$\sum F = 0$$



$$-F\left(x-\frac{\Delta x}{2}\right) + F\left(x+\frac{\Delta x}{2}\right) + q(x)\Delta x = 0$$

Divide by Δx

$$\frac{F\left(x+\frac{\Delta x}{2}\right) - F\left(x-\frac{\Delta x}{2}\right)}{\Delta x} + q(x) = 0$$

Similar to use of Divergence theorem + Localization for 2D/3D

let $\Delta x \to 0$

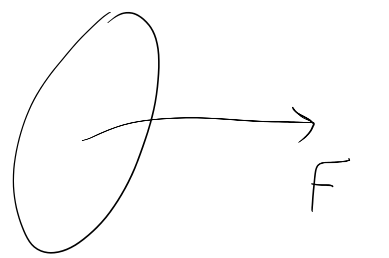
$$\lim_{\Delta x \to 0} \frac{F\left(x+\frac{\Delta x}{2}\right) - F\left(x-\frac{\Delta x}{2}\right)}{\Delta x} + q(x) = 0$$

Differential eqn

$$\boxed{\frac{dF(x)}{dx} + q(x) = 0 \qquad F' + q = 0} \quad \text{(DE)}$$

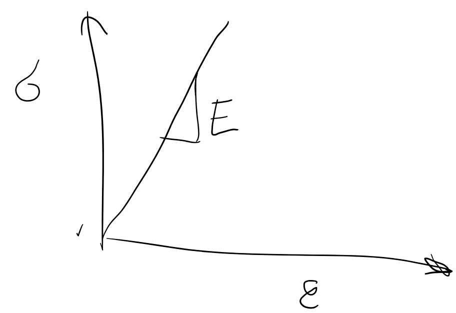
σ vs F

$\sigma = \frac{F}{A}$ or $F = A\sigma$



plug in DE

$(A\sigma)' + q = 0$



$\sigma = E\,\varepsilon$ Constitutive eqn

force like quantity

kinematic quantity

$$\boxed{F = AE\varepsilon}$$ "constitutive eqn" for 1D bar

$\frac{dF}{dx} + q = 0$

$$\boxed{\frac{d\,AE\varepsilon}{dx} + q = 0}$$

ε : strain $= \frac{\text{change of length}}{\text{original length}} = \frac{du}{dx}$ Compatibility equal

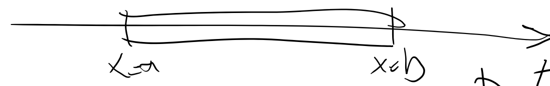
displacement

$\frac{d}{dx}\left(EA\frac{du}{dx}\right)$ ... Final form

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + q = 0$$

Final form of differential equation

$$(EAu')' + q = 0$$



$\forall x \in (a,b)$

DE

Balance law $\sum f = 0$