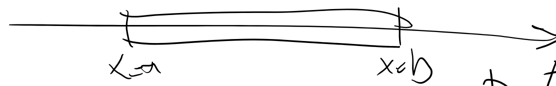
# 2023/09/11

Monday, September 11, 2023 11:23 AM

$$\boxed{\frac{d}{dx}\left(EA\frac{du}{dx}\right)+q=0 \qquad (EAu')'+q=0 \qquad \forall x\in(a,b)} \quad (1)$$

Final form of differential equation

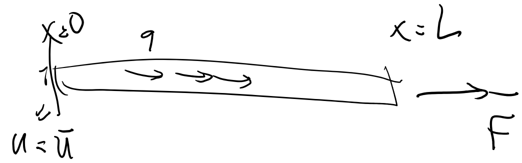



DE

Balance law $\quad \sum f=0$

Can we solve (1) now?

we need Boundary conditions (BCs)?




$u(x=0)=\bar{u} \qquad F(x=L)=\bar{F}$

$A\sigma=A(E\varepsilon)=$

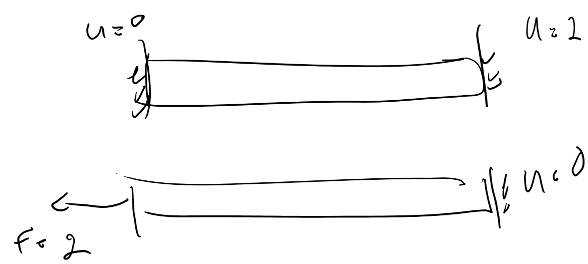
$AEu'(L)=F$

$\varepsilon$

PDE + BCs

Boundary value Problem (BVP) ] static (steady state)

other example




for dynamics + Initial conditions (IC)

Initial Boundary value Problem (IBVB)

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Continue on BVP for a bar problem

Bar Problem (2)

~~PDE~~ $\quad (EAu')'+q=0$


$u(x=0)=\bar{u}$

$F(x=L)=AEu'(x=L)=\bar{F}$

BC's

$M$ = order of differential eqn

$=2$

$m=\frac{M}{2}=1$ half the order of DE.

derivative order

$u(x=0)=\bar{u}$ (order 0 ... order $<\frac{m}{2}$)

Dirichlet BC

Solid Mechanics

kinematic

derivative order

$u(x=0) = \bar{u}$ (order 0 ... order $\frac{m}{2}$)

$\partial D_u$

0 | Dirichlet BC, Essential BC | kinematic quantity ($u$)

(M-1)

1 | Neumann BC, Natural BC | force-like quantity ($F$)

$F(x=L) = EAu'(x=L) = \bar{F}$

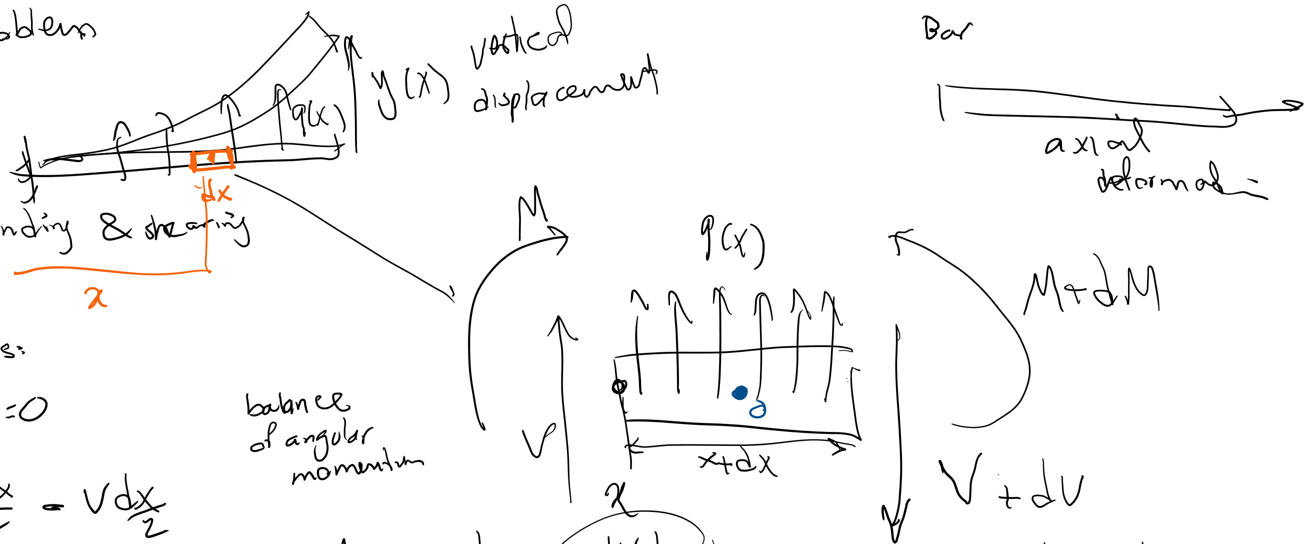
$\partial D_f$

order $\frac{m}{2}$ to $M-1$

general fx spatial flux

$M=2 \quad (EAu')' + q = 0$

Beam Problem




Balance laws:

(*) $\Sigma M_o = 0$ balance of angular momentum

$-(V+dV)\frac{dx}{2} - V\frac{dx}{2}$

$+(M+dM) - M = 0 \longrightarrow dM = Vdx + \frac{dVdx}{2}$

divide by $dx$ $\quad \frac{dM}{dx} = V + \cancel{\frac{dV}{2}}$

Higher order mods (HOM)

$\frac{dM}{dx} = V$ (i)

(*) $\Sigma F_y = 0 \implies$ Balance of forces

$-(V+dV) + V + q(x)dx = 0$

$dV = q\,dx \longrightarrow$

$\frac{dV}{dx} = q$ (ii)

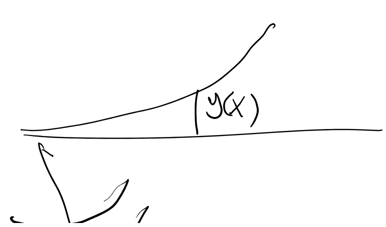
i, ii $\longrightarrow \frac{dV}{dx} = \frac{d}{dx}\left(\frac{dM}{dx}\right) = q \rightarrow \boxed{\frac{d^2M}{dx^2} = q \quad (M'' = q)}$ (3)

Differential equation in terms of M

(3) $\longrightarrow$ DE in terms of y

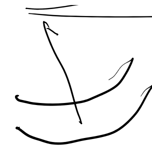
For Euler Bernoulli Beam model:

constitutive eqn $\quad M = EI\,k \quad ; \; k = $ curvature (4a)



constitutive eqn: $M = EI\kappa$ ; $\kappa$ = curvature (4a)

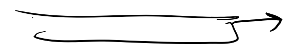
kinematic eqn: $\kappa = y''$ for small def. theory (4b)



compatibility eqn

4a, 4b $\rightarrow$ $\boxed{M = EI y''}$ (5)

compare these with bar problem



$F = \sigma A$, $\sigma = E\varepsilon$ } $F = EA\varepsilon$ Constitutive eqn

compatibility eqn $\varepsilon = u'$

(3) & (5) $\Rightarrow$

$$\boxed{\begin{gathered}\frac{d^2}{dx^2}M = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = q \\ (EIy'')'' = q\end{gathered}}$$

or shorthand

(6) Differential eqn for beams

$M = 4 \rightarrow m = \frac{M}{2} = 2$

order

0) $y(x)$

1) $\theta(x) = y'(x)$

correspond to Essential "Dirichlet" BC

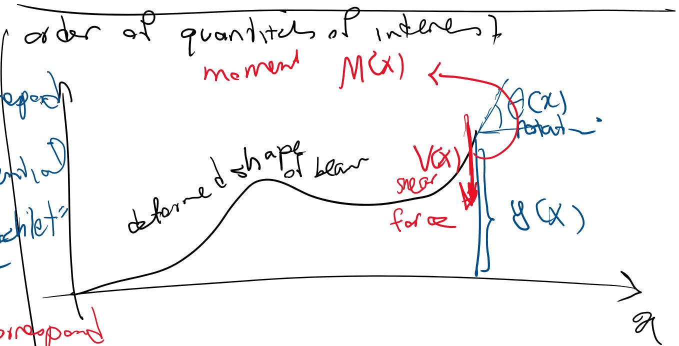
2) $M(x) = EIy''(x)$

3) $V(x) = \frac{d}{dx}M = (EIy'')'$

M-1

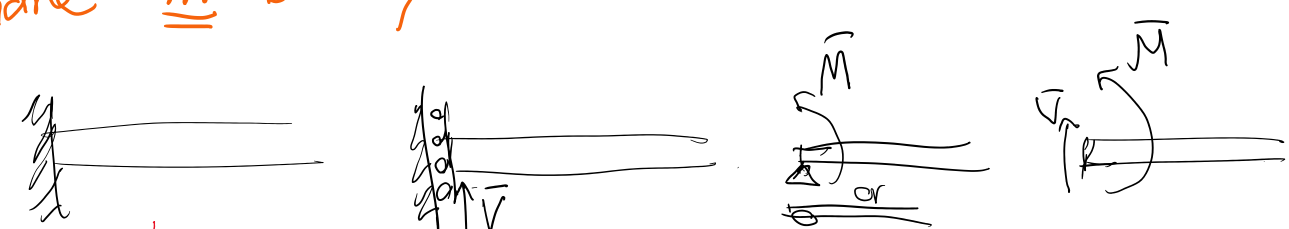
correspond to Natural "Neumann" BC

DE is order M=4




We have $\underline{\underline{m}}$ boundary conditions for each pt on the boundary

— essential
— natural



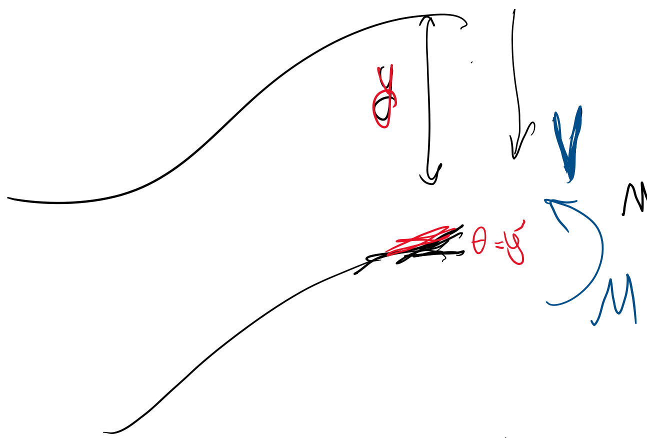

$y(x=0) = \bar{y}$ ; $\theta(0) = y'(x=0) = \bar{\theta}$ — all essential

$(EIy'')' = V(x=0) = \bar{V}$ ; $\theta(x=0) = \bar{\theta}$

$y(x=0) = \bar{y}$ ; $EIy'' = M(x=0) = \bar{M}$

$(EIy'')' = V(x=0) = \bar{V}$ ; $EIy'' = M(x=0) = \bar{M}$ — all natural

work $= y(-V) = -yV$




work $= y(-V) = -yV$

$M-1 = 3 =$ work: $\underset{\text{order 0}}{y} \longleftrightarrow \underset{\text{order 3}}{V = (EIy'')'}$ $+$ $\theta M$

$M-1 = 3 =$ $\underset{\text{order 1}}{y' = \theta} \longleftrightarrow \underset{\text{order 2}}{M = EIy''}$ $+$

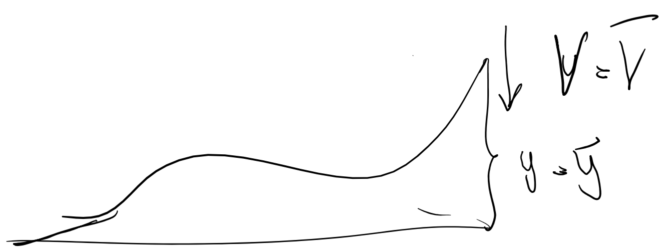
| order | |
|---|---|
| 0 | $y$ |
| 1 | $\theta = y'$ |
| 2 | $M = EIy''$ |
| $M-1 = 3$ | $V = \frac{dM}{dx} = (EIy'')'$ |

sum = 3

sum = 3 = M−1

work/energy pairs

$(EIy'')'' + q = 0$




can we have this BC?

take aways

- we have $m = \frac{M}{2}$ BCs at each point on the boundary of domain.

- order 0 $\longleftrightarrow$ order $M-1$

  " 1 $\longleftrightarrow$ $M-2$

  ⋮ ⋮

  only one from each can be specified

$\underbrace{\qquad\qquad}_{\text{order 0 to } m-1}$ Dirichlet / Essential

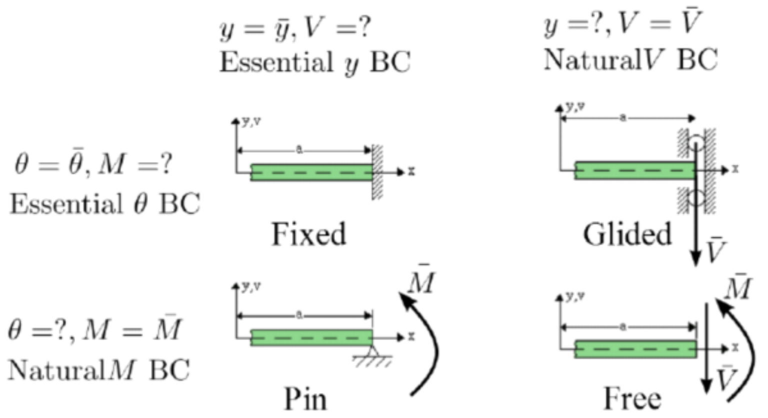
$\underbrace{\qquad\qquad}_{\text{order } m \text{ to } M-1}$ Neumann / Natural

(7)

## Euler Bernoulli beam: BCs

| Operator | Sample | elastostatics | operator order |
|---|---|---|---|
| $L_{2m}(\mathbf{u}) = \mathbf{r}$ | $\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = q$ | $\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2 (.)}{\mathrm{d}x^2}\right) = q$ | $m = 2 (M = 4)$ |
| $L_u(\mathbf{u}) = \mathfrak{u}$ | $u = \begin{bmatrix} \theta \\ y \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}y}{\mathrm{d}x} \\ y \end{bmatrix} = \begin{bmatrix} \bar{\theta} \\ \bar{y} \end{bmatrix} = \bar{u}$ | $L_u = \begin{bmatrix} \frac{\mathrm{d}(.)}{\mathrm{d}x} \\ (.) \end{bmatrix}$ | $M_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ |
| $L_f(\mathbf{u}) = \bar{\mathbf{f}}$ | $\begin{bmatrix} EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) \end{bmatrix} = \begin{bmatrix} \bar{M} \\ \bar{V} \end{bmatrix}$ | $L_f = \begin{bmatrix} EI\frac{\mathrm{d}^2 (.)}{\mathrm{d}x^2} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 (.)}{\mathrm{d}x^2}\right) \end{bmatrix}$ | $M_f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ |

- One and only one of the pair $M$ (Neumann) and $\theta$ (Dirichlet) is enforced at each end of the beam.
- One and only one of the pair $V$ (Neumann) and $y$ (Dirichlet) is enforced at each end of the beam.
- Neumann boundary conditions correspond to the flux terms ($M$ and $V$).
- Neumann boundary conditions fall in the upper half of derivatives ($[m, 2m-1] = [2, 3]$).
- Dirichlet boundary conditions fall in the lower half of derivatives ($[0, m-1] = [0, 1]$).
- There are two boundary conditions at each end point (equal to $m = M/2$).
- $M_u + M_f = M - 1$.




- For each pair of Neumann and Dirichlet (Natural and Essential) boundary conditions, one and only one is specified.



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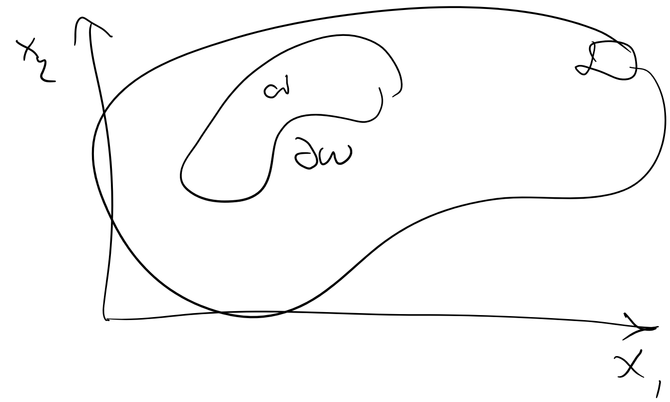
2D elasticity

– Balance law

$$\forall \omega \quad \int_{\partial\omega} \sigma \cdot n\, ds + \int_{\omega} \rho b\, dV = 0$$




Divergence theorem

$$\int_{\omega} \nabla \cdot \sigma\, dV + \int_{\omega} \rho b\, dV = 0 \qquad \forall \omega \int_{\omega} (\nabla \cdot \sigma + \rho b)\, dV = 0$$

localization ($\forall \omega$ ...)

(8)

$$\boxed{\text{PDE} \quad \nabla \cdot \sigma + \rho b = 0}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\sigma_{21} = \sigma_{12}$

$$\begin{aligned} \text{eq1} \quad & \sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0 \\ \text{eq2} \quad & \sigma_{21,1} + \sigma_{22,2} + \rho b_2 = 0 \end{aligned}$$

\# eq = 2

\# unknowns = 3 stresses

$$s = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \quad \text{Voigt notation}$$

1D bar

$$F_{,x} + q = 0$$

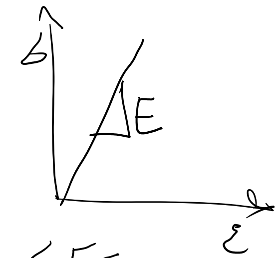
$$(A\sigma)_{,x} + q = 0$$

# eq = 2
# unknowns = 3
stresses

$$s = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$ Voigt notation

We need to close this system

1D




$\sigma = E\varepsilon$

— Constitutive equation

2D

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underbrace{C_{3\times3}}_{\text{elasticity stiffness matrix}} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

3 eqns added

3 unknowns added $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12})$

— Compatibility equation

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} = \frac{\nabla u + (\nabla u)^t}{2} = \frac{1}{2}\left\{\begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix} + \begin{bmatrix} u_{1,1} & u_{2,1} \\ u_{1,2} & u_{2,2} \end{bmatrix}\right\}$$

$\longrightarrow$

$$\boxed{\begin{aligned} \epsilon_{11} &= u_{1,1} \\ \epsilon_{22} &= u_{2,2} \\ \epsilon_{12} = \epsilon_{21} &= \frac{u_{1,2} + u_{2,1}}{2} \end{aligned}}$$

2 unknowns added $u_1, u_2$

3 eqns added