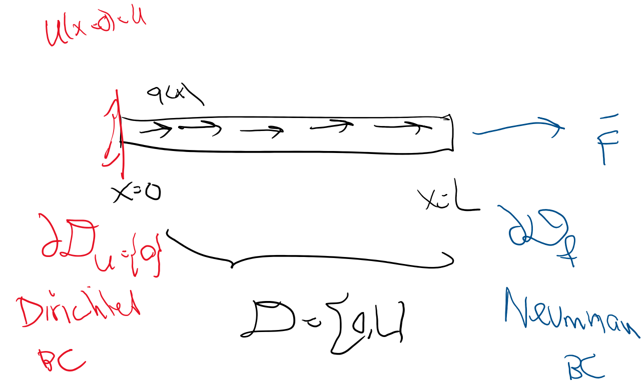
2023/09/13

Wednesday, September 13, 2023 11:13 AM

Section of "Weak Form"



(1)

$\forall x \in (0,L)$ $\quad (A\sigma)_{,x} + q = (AE u_{,x})_{,x} + q = 0$

Differential equation

$\forall x \in \partial D_u \quad \bar{u} - u = 0 \quad$ Dirichlet BC $\quad$ here $x=0$

$\forall x \in \partial D_f \quad \bar{F} - F = 0 \quad$ Neumann BC $\quad$ here $x=L$

Define Residuals

$\forall x \in D = [0,L) \qquad R_i(x) = (A(x)\sigma(x))_{,x} + q(x) = (AE u_{,x})_{,x} + q$

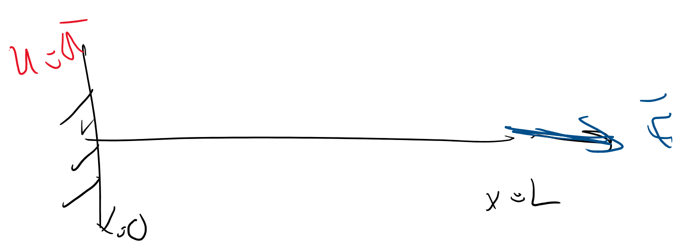
interior

for Dirichlet BC $x=0$ $\qquad R_u(x=0) = \bar{u} - u(0)$

for Neumann BC $\qquad R_f(x=L) = \bar{F} - F(L) = \bar{F} - AE\, u_{,x}(L)$

if all residuals are zero we have the exact solution

Weighted Residual Statement (WRS)



Find u $\ni$ (such that) $\qquad \forall w \rightarrow$ weight function

$$\text{WRS: } \int_0^L w(x) R_i(x)\,dx + w(L) R_f(L) + w(0) \underbrace{R_u(0)}_{\bar{u}-u} = 0 \qquad (2)$$

WRS: $\int_0 w(x) R_i(x)\,dx$ + ... = 0 (2)

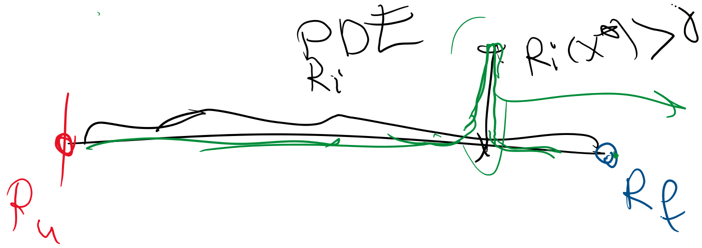
weighted residual

$R_i = 0$ $R_f = 0$ $R_u = 0$ $\Longrightarrow$ WRS = 0 :)

$\Longleftarrow$ $\forall w$

more tricky

sketch of proof



Weak: Any integral statement like Balance law or WRS
Strong: a point-wise equation like the differential equation

We could not easily solve the balance law -> We derived DE + BC but again this cannot be solve -> multiplied residuals by weights and got WRS which again is another weak statement but a much easier one that balance law to numerically solve.

Find u $\ni$ (such that) $\forall w$ (weight under)

$$\text{WRS: } \int_0^L w(x) R_i(x)\,dx + w(L) R_f(L) + w(0) R_u(0) = 0 \quad (2)$$

weighted residual

$\bar{u} - u$

$R_u = \bar{u} - u$ $\qquad$ $R_f = \bar{F} - F = \bar{F} - AEu'$

in FEM we'll start off with solution spaces that this condition is satisfied
P.L.'s are continuous

in FEM we'll start off with solution spaces that the conditions is satisfied a priori

f, f', f'' are continuous

A more common WRS is: Find $u \in U = \{ \underbrace{f \in C^2([0,L])}_{\text{because } (EAu')'} \mid \underbrace{f(0) = \bar{u}}_{\text{a priori need to satisfy essential BC}} \}$

$\Rightarrow \forall w \in W = \{ C^0([0,L]) \quad \}$

$$\int_0^L w R_i \, dx + w R_f \Big|_{x=L} = 0$$

$$\underbrace{\int_0^L w \Big(\underbrace{(EAu')'}_{F = (A\sigma)'} + q \Big) dx}_{} + w(L) \Big(\bar{F} - \underbrace{AE u'(L)}_{F = A\sigma} \Big) = 0$$

Basically we satisfy the Dirichlet BC (Essential BC) strongly but PDE (Ri) and Neumann BC (Natural BC) weakly

Problem of the WRS is that it has a high continuity constraint on u but not much on w. -> We want to fix this
Integration by parts fixes the issue:

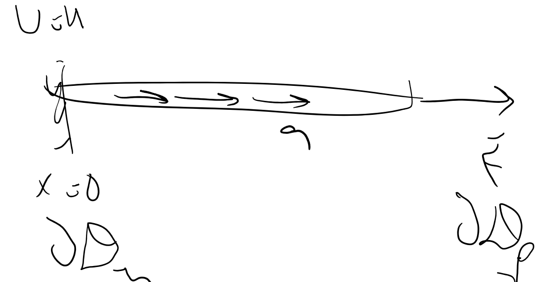
$$\forall w \quad \int_0^L w (F' + q) \, dx + w(L) (\bar{F} - F(L)) = 0 \qquad F(x) = EAu'(x)$$

$$\Big(\underbrace{wF \Big|_0^L - \int_0^L w' F \, dx}_{\int_0^L w F' dx} \Big) + \int_0^L q \, dx + w(L)\bar{F} - w(L)F(L) = 0$$

$$\cancel{w(L)F(L)} - \underbrace{\cancel{w(0)}F(0)}_{\text{in general } \neq 0} - \int_0^L w' F \, dx + \int_0^L q \, dx + w(L)\bar{F} \cancel{- w(L)F(L)} = 0$$

$w(0) = 0$

I want to make this term zero



$\partial\Omega_u$ $\partial\Omega_f$

I choose weight functions that are zero @ essential BC (here $x=0$)

$$\int w' F\,dx = \int_0^L q\,dx + w(L)\bar{F}$$

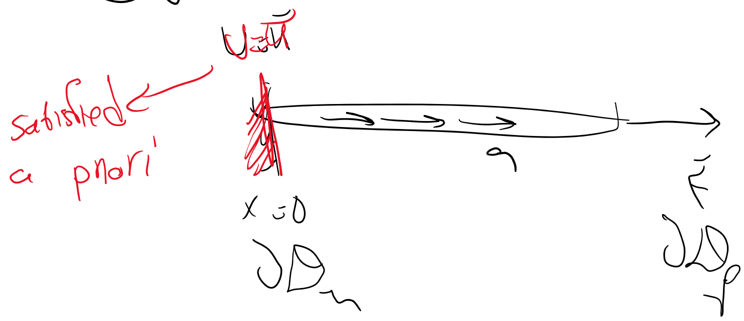
Recall $F = EAu'$

Find $u \in \mathcal{V} = \{f \in C^1(0,L) \mid f(0) = \bar{u}\}$ $\ni$

$\forall w \in \mathcal{W} = \{f \in C^1(0,L) \mid f(0) = 0\}$ & : orders are balanced & sln & weight satisfy essential BC (for weight the homogeneous version of it)

$$\int_0^L w' EA u'\,dx = \int_0^L q\,dx + w(L)\bar{F}$$

the weak statement

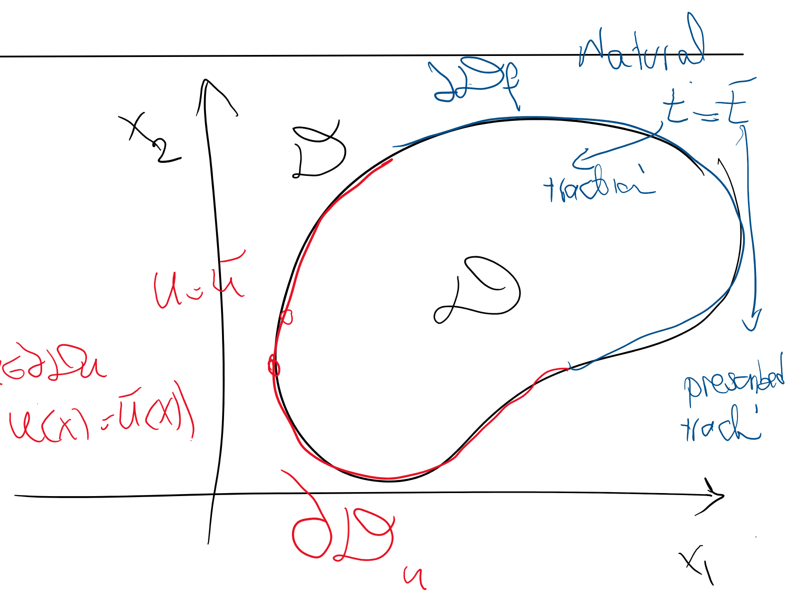


2D elasticity

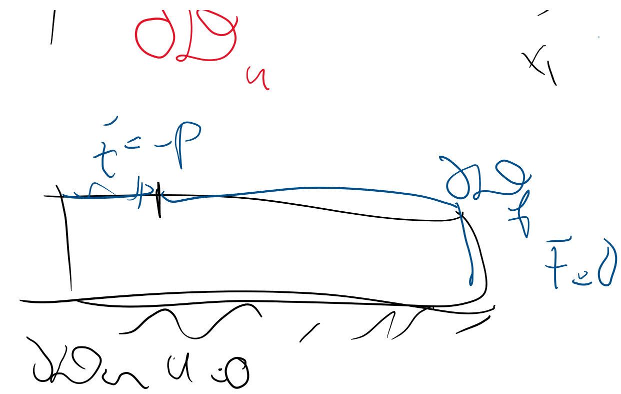
$$R_i = \nabla\cdot\sigma + \rho b \quad \forall x \in \Omega$$

$$R_u = \bar{u} - u \quad \forall x \in \partial\Omega_u \quad (\forall x \in \partial\Omega_u \; u(x) = \bar{u}(x))$$

$$R_f = \bar{t} - t \quad \forall x \in \partial\Omega_f$$



$$R_f = t - t \quad \forall x \in \partial\Omega_f$$
$$= \bar{t} - \sigma \cdot n \qquad (t = \sigma \cdot n)$$



WRS

$$\forall w \quad \int_\Omega w R_i \, dv + \int_{\partial\Omega_f} w R_f \, ds + \underbrace{\int_{\partial\Omega_n} w R_u \, ds}_{\text{we chose to satisfy this a priori}} = 0$$

WRS

find $u \in V = \{ f \in C^2(\Omega) \mid \forall x \in \partial\Omega_n \; u(x) = \bar{u}(x) \}$

$\ni \forall w \in W = \{ f \in C^0(\Omega) \mid \quad \}$

$$\int_\Omega w R_i \, dv + \int_{\partial\Omega_f} w R_f \, ds = 0$$

$$\int_\Omega w (\nabla \cdot \sigma + \rho b) + \int_{\partial\Omega_f} w(\bar{t} - \sigma n) \, ds = 0$$

$$\nabla \cdot (\underbrace{C \nabla u}_{\sigma})$$

WRS $\longrightarrow$ WK (Weak statement)

$$\int_\Omega w \, \nabla \cdot \sigma = ?$$

$$w = \begin{bmatrix} w_1(x_1, x_2) \\ w_2(x_1, x_2) \end{bmatrix} \quad \text{in 2D}$$

side note

$\sigma = C\varepsilon$

$\varepsilon = \frac{\nabla u + \nabla u^T}{2}$

(2D/3D version)

$\sigma = E\varepsilon$

$\varepsilon = u'$

1D

$\sigma = C \nabla u$

$$\left[\omega_2(x_1, x_2) \right]$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \qquad \nabla \cdot \sigma = \begin{bmatrix} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{21,1} \quad \sigma_{22,2} \end{bmatrix} \qquad (\cdot)_{,1} = \frac{\partial(\cdot)}{\partial x_1}$$

$$\omega \nabla \cdot \sigma = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{21,1} + \sigma_{22,2} \end{bmatrix} = (\omega_1 \sigma_{11,1} + \omega_1 \sigma_{12,2}) + (\omega_2 \sigma_{21,1} + \omega_2 \sigma_{22,2})$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \omega_i \sigma_{ij,j}$$

next time

$$\omega_i \sigma_{ij,j} \longrightarrow \omega_{i,j} \sigma_{ij}$$