2023/09/18

Monday, September 18, 2023 11:22 AM

From last time

want to turn it to

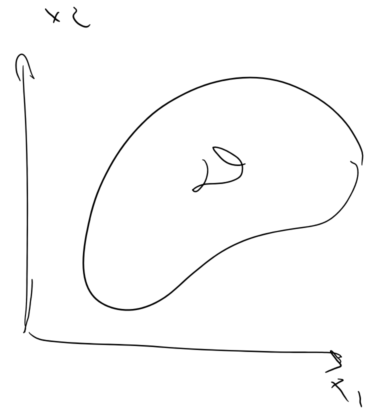
$$\int_{\mathcal{D}} \omega \, \nabla\cdot\sigma \;=\; \int_{\mathcal{D}} \nabla\omega \; \underbrace{\sigma}_{C\nabla u} + \ldots$$

$$\omega \, \nabla\cdot\sigma = \sum_{i=1}^{d} \sum_{j=1}^{d} \omega_i \, \sigma_{ij,j} \qquad \sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

dimension

from last time

$$\int_{\mathcal{D}} \omega \, \nabla\cdot\sigma \, dV = \int_{\mathcal{D}} \sum_{i=1}^{d} \sum_{j=1}^{d} (\underbrace{\omega_i}_{a} \underbrace{\sigma_{ij,j}}_{b}) \, dV \quad (1)$$



Recall

$$\boxed{\begin{aligned} &(2a) \quad \int_{\mathcal{D}} f_{,j} \, dV = \int_{\partial\mathcal{D}} f \, n_j \, dS \\ &\qquad\qquad \Downarrow \\ &(2b) \quad \int_{\mathcal{D}} \nabla\cdot f \, dV = \int_{\partial\mathcal{D}} f\cdot n \, dS \end{aligned}}$$

(2) Divergence theorem

$$\int_{\mathcal{D}} \omega \, \nabla\cdot\sigma \, dV = \int_{\mathcal{D}} \sum_{i=1}^{d} \sum_{j=1}^{d} \Big(\underbrace{(\omega_i \sigma_{ij})_{,j}}_{f_{,j}} - \omega_{i,j}\sigma_{ij} \Big) \, dV$$

$$ab' = (ab)' - a'b \qquad (\;)' = \frac{\partial}{\partial x_j}$$

use 2a for $f_{,j} \Rightarrow$

$$\int_{\mathcal{D}} \omega \, \nabla\cdot\sigma \, dV = \underbrace{\int_{\partial\mathcal{D}} \sum_{i=1}^{d} \sum_{j=1}^{d} \omega_i \, \sigma_{ij} \, n_j \, dS}_{\parallel} - \int_{\mathcal{D}} \underbrace{\sum_{i=1}^{d} \sum_{j=1}^{d} \omega_{ij} \, \sigma_{ij}}_{} \, dV$$

$$\underbrace{\begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix}}_{\omega} \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}}_{\sigma\cdot n}$$

$$\begin{bmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{bmatrix} : \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \omega_{1,1}\sigma_{11} + \omega_{1,2}\sigma_{12} + \omega_{2,1}\sigma_{21} + \omega_{2,2}\sigma_{22}$$

$$\boxed{\int \omega \, \nabla\cdot\sigma \, dV = \int \omega \, \sigma\cdot n \, dS - \int \nabla\omega : \sigma \, dV}$$

(3)

$$\int_{\mathcal{D}} \omega \underbrace{\nabla\cdot\sigma}_{\nabla\cdot C\nabla u} \, dV = \int_{\partial\mathcal{D}} \omega\, \sigma\cdot n \, dS - \int_{\mathcal{D}} \underbrace{\nabla\omega}_{\text{gradient of } \omega} : \sigma \, dV$$

0 der on ω — 2 der on u ☹

1 der on ω = 1 der on u ($\sigma \rightarrow C\nabla u$) ☺

For your HW you can directly use $q=-\kappa\nabla T$

(b) **Weak Statement**: Noting that $w\,(\nabla.\mathbf{q}) = \nabla.(w\mathbf{q}) - (\nabla w).\mathbf{q}$ (or alternatively $w\,(\nabla.(\kappa\nabla T)) = \nabla.(w\kappa\nabla T) - \nabla w.(\kappa\nabla T)$): ☹

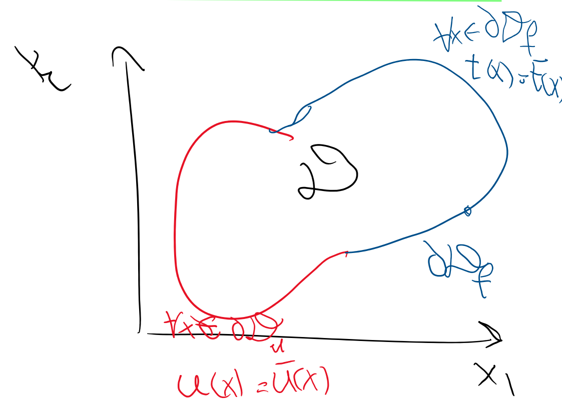
i. Use the Gauss (divergence) theorem to transform the weighted residual statement to the *weak statement*.
Hints: 1. It is better to keep the heat flux $\mathbf{q}$ all the way from its appearance in the WRS $\int_{\mathcal{D}} w(\nabla.\mathbf{q} - Q)dV$ to the form in the weak statement and eventually expressing q in terms of the (gradient of) temperature. This makes the process cleaner;
2. Make sure in the WR statement $\mathcal{R}_f$ is added with the right sign so that boundary terms generated by $w\mathbf{q}$ term cancel some of $\mathcal{R}_f$ terms;
3. After the application of Gauss theorem some boundary terms are generated on $\partial\mathcal{D}_u$. Make judicious choice for the spaces of the functions T or w so that those terms would disappear.

From last time

WRS

$\forall \omega$:

$$\int_{\mathcal{D}} w R_i \, dV + \int_{\partial\mathcal{D}_f} \omega R_f \, ds = 0$$

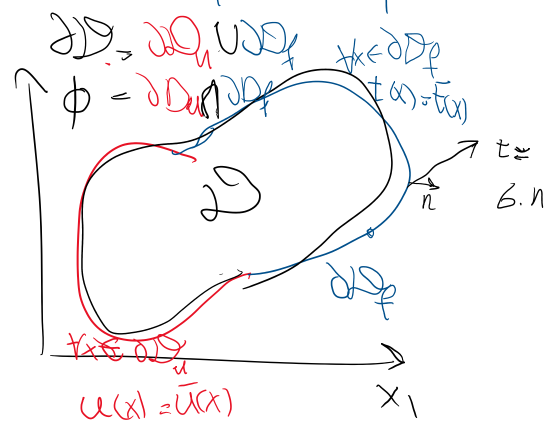


$$\int_{\mathcal{D}} \omega(\nabla\cdot\sigma + \rho b)\, dV + \int_{\partial\mathcal{D}_f} \omega(\bar{t} - t)\, dS = 0$$

from eqn 3 $\int_{\mathcal{D}} \omega \nabla\cdot\sigma \, dV = \int_{\partial\mathcal{D}} \omega\, \sigma\cdot n \, dS - \int_{\mathcal{D}} \nabla w\, \sigma \, dV$ $\Longrightarrow$

$$\left(\int \omega\, \sigma\cdot n \, ds - \int \nabla w\, \sigma \, dV \right) + \int w \rho b \, dV + \int \omega \bar{t} \, dS - \int w t \, dS = 0$$

$$\underbrace{\left(\int_{\partial\Omega} \omega\, \underbrace{\sigma\cdot n}_{t}\, ds - \int_{\Omega} \nabla\omega\,\sigma\, dV\right)}_{\int_{\Omega} \omega\nabla\cdot\sigma\, dV} + \int_{\Omega} \omega\rho b\, dV + \int_{\partial\Omega_f} \omega\bar{t}\, ds - \int_{\partial\Omega_f} \omega t\, ds = 0$$



$$\int_{\partial\Omega} \omega t\, ds - \int_{\Omega} \nabla\omega\,\sigma\, dV + \int_{\Omega} \omega\rho b\, dV + \int_{\partial\Omega_f} \omega\bar{t}\, ds$$

$$- \int_{\partial\Omega_f} \omega t\, ds = 0$$

$$\underbrace{\cancel{\int_{\partial\Omega_f} \omega t\, ds} + \int_{\partial\Omega_u} \omega t\, ds}_{\int_{\partial\Omega} \omega t\, ds} \underbrace{- \int_{\Omega} \nabla\omega\, \sigma\, dV}_{} + \int_{\Omega} \omega\rho b\, dV + \int_{\partial\Omega_f} \omega\bar{t}\, ds$$

$$\cancel{- \int_{\partial\Omega_f} \omega t\, ds} = 0$$

$t = \sigma n$

t: traction vector

σ: 2nd order tensor "matrix" stress

n: normal vector

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$$\int_{\Omega} \underbrace{\nabla\omega\, \sigma}_{\varepsilon(\omega):\sigma}\, dV = \int_{\Omega} \omega\rho b\, dV + \int_{\partial\Omega_f} \omega\bar{t}\, ds \quad \cancel{+ \int_{\partial\Omega_u} \omega t\, ds} \quad 0$$

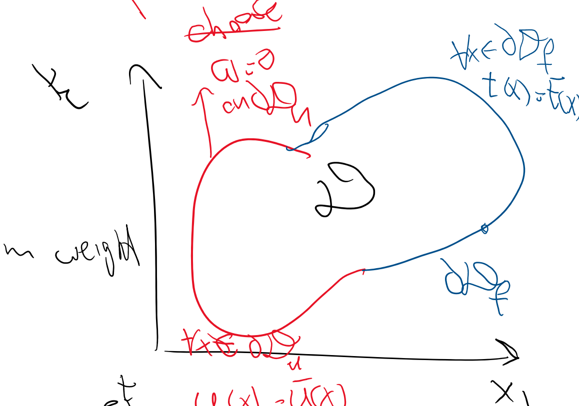
note

$$\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2}$$

$$\varepsilon(\omega) = \frac{\nabla\omega + \nabla\omega^T}{2}$$ strain calculated from weight

σ is symetric matrix

$$\sigma\nabla\omega = \underbrace{\underset{\text{sym}}{\sigma}\left(\underset{\text{sym}}{\frac{\nabla\omega + \nabla\omega^T}{2}}\right)}_{\varepsilon(\omega)} + \underbrace{\underset{\text{sym}}{\sigma}\left(\underset{\text{skew sym}}{\frac{\nabla\omega - \nabla\omega^T}{2}}\right)}_{0}$$



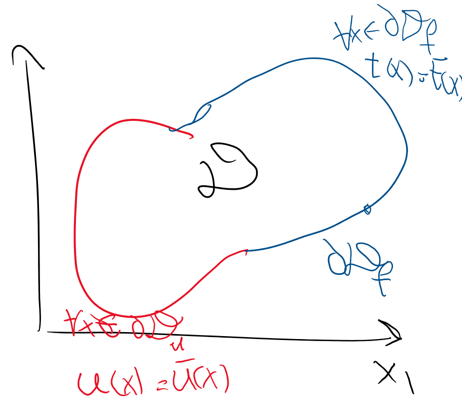
we can reduce continuity by 1 ($C^1 \to C^0$)

Find $u \in \mathcal{V} = \{ f \in \cancel{C^1}(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \;\; u(x) = \bar{u}(x) \}$

$\ni$ such that $\forall w \in \{ f \in C^1(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \;\; u(x) = 0 \}$

$$\int_{\mathcal{D}} \underline{\varepsilon(w)} \, (\underline{C\,\varepsilon(u)}) \, dV = \int_{\mathcal{D}} w \, \rho b \, dV + \int_{\partial\mathcal{D}_f} w \bar{t} \, ds \qquad \forall x$$

$\sigma(u) = C\,\varepsilon(u)$



recall

$$\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2}$$

$$\varepsilon(w) = \frac{\nabla w + \nabla w^T}{2}$$

Comment:

④ we can (and will) change $C^1 \to C^0$ for FEM

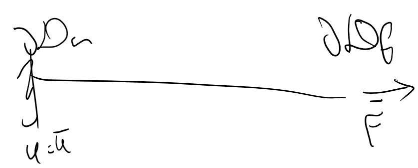
1D

$\int w \, EA u'$

$u \in V = \{ \cancel{C^1} C^0([0, L]) \mid \}$

compare with 1D

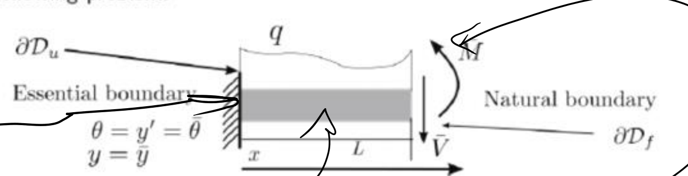
$$\int_{\mathcal{D}} w' E A u' \, dx = \int_0^L w q \, dx + w \bar{F} \big|_{x=L}$$



Please read slides 51 to 57 for the beam problem (WRS -> weak statement)

Weighted residual statement to Weak statement

To demonstrate the process of deriving the weak statement from the weighted residual statement consider the following problem:

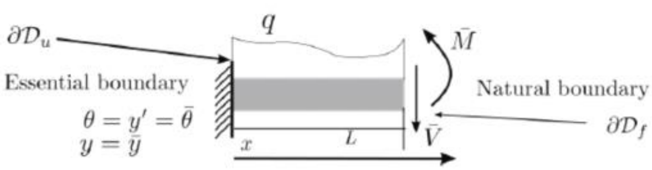


The residuals for this problem are:

$$\begin{aligned}
\mathcal{R}_i &= \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q && \text{Interior residual for} && \mathcal{D} = [0, L] \\
\mathcal{R}_f &= \begin{bmatrix} \bar{M} - M \\ \bar{V} - V \end{bmatrix} && \text{Natural BC residual for} && \partial\mathcal{D}_f = \{L\} \\
\mathcal{R}_u &= \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} && \text{Essential BC residual for} && \partial\mathcal{D}_u = \{0\}
\end{aligned} \tag{53}$$

As mentioned previously, we want to drop the weighted residual term for essential boundary condition (why?). Accordingly, we need to strongly enforce the essential boundary condition (This is why this is called "essential" boundary condition). That is, we require:

$$\mathcal{R}_u = \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} = 0 \quad \text{at } x = 0 \ (\partial\mathcal{D}_u). \tag{54}$$



Since we strongly enforce the essential boundary condition, the weighted residual for this problem simplifies to:

$$\begin{aligned}
0 &= \int_{\mathcal{D}} w\mathcal{R}_i(y)\, dv + \int_{\partial\mathcal{D}_f} \mathbf{w}_f \mathcal{R}_f(y)\, ds \\
&= \int_0^L w\left(\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q\right) dx + \begin{bmatrix} -\frac{dw}{dx} \\ w \end{bmatrix} \cdot \begin{bmatrix} \bar{M} - M \\ \bar{V} - V \end{bmatrix} \Big|_{x=L} \\
&= \int_0^L w\left(\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - q\right) dx - \frac{dw}{dx}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L}
\end{aligned} \tag{55}$$

order 0   4 derivatives !

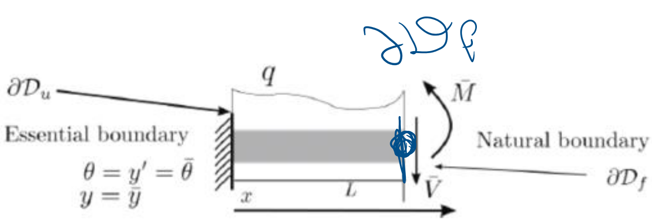
Next, we transfer derivatives from y to w (trial function to weight function). We note that

$$\begin{aligned}
\int_0^L \underbrace{w}_{0}\overbrace{\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)}^{4} dx &= \int_0^L \left[-\overbrace{\frac{dw}{dx}}^{1}\overbrace{\frac{d}{dx}EI\left(\frac{d^2y}{dx^2}\right)}^{3}\right] dx + \left[w\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)\right]\Big|_{x=0}^{x=L} \\
&= \int_0^L \left[\underbrace{\frac{d^2w}{dx^2}}_{2}EI\underbrace{\frac{d^2y}{dx^2}}_{2}\right] dx + [wV(y)]|_{x=0}^{x=L} - \left[\frac{dw}{dx}\underbrace{\left(EI\frac{d^2y}{dx^2}\right)}_{M}\right]\Big|_{x=0}^{x=L}
\end{aligned} \tag{56}$$

M   V   M

$M = EI\frac{d^2y}{dx^2}$

$V = \frac{dM}{dx} = \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)$



Plugging (55) in (56) yields,

0   4

Plugging (55) in (56) yields,

$$
\begin{aligned}
0 &= \int_0^L w\left(\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)-q\right)dx - \frac{dw}{dx}(\bar{M}-M(y))|_{x=L} + w(\bar{V}-V(y))|_{x=L} \\
&= \int_0^L \left[\frac{d^2w}{dx^2}EI\frac{d^2y}{dx^2} - wq\right]dx + \left[wV(y)\right] - \frac{dw}{dx}M(y)\Big]\Big|_{x=0}^{x=L} \\
&\quad - \frac{dw}{dx}(\bar{M}-M(y))|_{x=L} + w(\bar{V}-V(y))|_{x=L} \\
&= \int_0^L \left[\frac{d^2w}{dx^2}EI\frac{d^2y}{dx^2} - wq\right]dx \\
&\quad + \left\{wV(y) - \frac{dw}{dx}M(y) - \frac{dw}{dx}(\bar{M}-M(y)) + w(\bar{V}-V(y))\right\}_{x=L} \\
&\quad - \left\{wV(y) - \frac{dw}{dx}M(y)\right\}_{x=0}
\end{aligned}
\qquad (57)
$$

Jump terms from IBP

residuals on ∂D_f

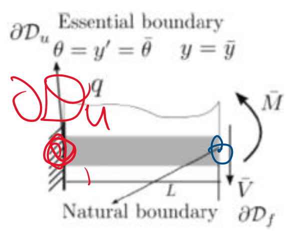
Weighted residual statement to Weak statement

This equation simplifies to

$$0 = \int_0^L \left[\frac{d^2w}{dx^2}EI\frac{d^2y}{dx^2} - wq\right]dx + \left\{-\frac{dw}{dx}\bar{M} + w\bar{V}\right\}_{x=L} \qquad (58a)$$

$$+\left\{w\,(V(y) - V(y)) - \frac{dw}{dx}(M(y) - M(y))\right\}_{x=L} \qquad (58b)$$

$$-\left\{wV(y) - \frac{dw}{dx}M(y)\right\}_{x=0} \qquad (58c)$$



∂D_f

Noting that $M(y) = EI\frac{d^2y}{dx^2}$ and $V(y) = \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)$ (second and third order derivatives in x):

chose your weight to be zero on essential BC

Essential boundary condition

We mentioned that the essential boundary condition is strongly enforced (That is, it is an "essential" condition). The essential conditions (54) require,

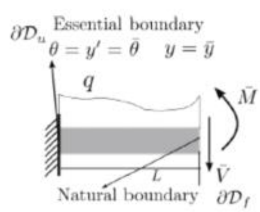
$$\mathcal{R}_u = \begin{bmatrix} \bar\theta - \theta \\ \bar y - y \end{bmatrix} = 0 \Rightarrow \boxed{\left\{ \begin{array}{c} \frac{dy}{dx} = \bar\theta \\ y = \bar y \end{array} \right\},\ \text{at } x = 0\ (\partial\mathcal{D}_u)} \qquad (59)$$

We discussed that to annihilate the high order derivatives of y in (58c):

$$-\left\{wV(y) - \frac{dw}{dx}M(y)\right\}_{x=0}$$

we set the corresponding weight functions identically zero:

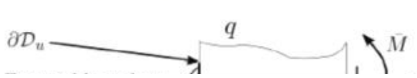
$$\left\{ \begin{array}{c} \frac{dw}{dx} = 0 \\ w = 0 \end{array} \right\},\quad \text{at } x = 0\ (\partial\mathcal{D}_u) \qquad (60)$$



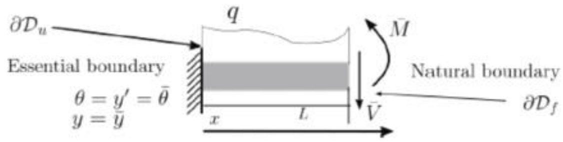
Summary

1. Trial, y, (solution) functions exactly satisfy all essential boundary conditions.
2. Weight, w, functions exactly satisfy the homogeneous essential boundary conditions.
3. If both conditions are satisfied we can form a *weak statement* that requires *only half* the highest derivative order. In fact, this enlarged space of functions is the same as the space of the original balance law.

Weak Statement (WS)



Weak statement



The weak statement for the Euler Bernoulli problem and the BCs in the figure are:

$$\text{Find } y \in \mathcal{V} = \{u \in C^2(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta}\},\ \text{such that,} \tag{62a}$$

$$\forall w \in \mathcal{W} = \{u \in C^2(\mathcal{D}) \mid u(0) = 0,\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0\} \tag{62b}$$

$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right] \mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L} \tag{62c}$$

Summary

- Both $\mathcal{V}$ and $\mathcal{W}$ have the same regularity ($C^m(\mathcal{D})$): $m = M/2$, $M = 4$ is the order of the differential equation.
- The less demanding regularity conditions for the solution compared to the weighted residual statement ($C^M(\mathcal{D}) \rightarrow C^m(\mathcal{D})$) takes us to the same function space needed for the balance law (balance of linear and angular momentum for Euler Bernoulli beam.
- Both $\mathcal{V}$ and $\mathcal{W}$ exactly enforce the essential boundary conditions, with the difference that $\mathcal{W}$ satisfies the homogeneous version.

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Weak statement

1. solution & weight have the same regularity

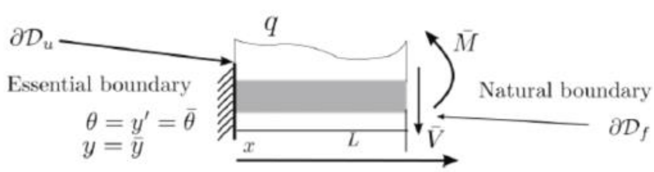
2. they satisfy essential BC

solution satisfies Essential BC
$y(0) = \bar{y}$
$y'(0) = \bar{\theta}$

weight satisfies homogeneous essential BC
$w(0) = 0$
$w'(0) = 0$

Compare this with WRS which is often not a good choice for solution

Weighted Residual Statement (WRS)



The weighted residual for the Euler Bernoulli problem and the boundary conditions in the figure are:

$$\text{Find } y \in \mathcal{V}^{\mathrm{WRS}} = \{u \in C^4(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta}\},\ \text{such that,} \tag{61a}$$

$\forall w \in \mathcal{W}^{\mathrm{WRS}} = C^0(\mathcal{D})$ no need to enforce the homogeneous essential BCs for WRS (61b)

$$0 = \int_{\mathcal{D}} w\mathcal{R}_i(y)\,\mathrm{d}v + \int_{\partial\mathcal{D}_f} \mathrm{w}_f \mathcal{R}_f(y)\,\mathrm{d}s$$

$$= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L} \tag{61c}$$

DE

natural BC error

So in this second version of the weighted residual statement, we no longer enforce essential boundary conditions weakly. The typical practice, like here, is to enforce the differential equation and natural boundary conditions weakly and the essential boundary conditions strongly.

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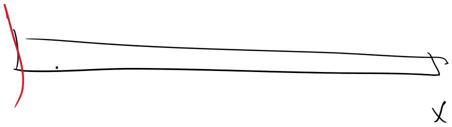
Discretization & solution space

$u = \bar{u} = 1$

Discretization & solution space

$u = \bar{u} = 1$



$x = L = 10$

for weighted residual statement (WRS) & weak form (WK) we need to satisfy essential BC exactly / a priori

numerical solution

$$u^h(x) = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \dots + a_n\phi_n(x)$$

$\phi_1 \dots \phi_n$ are basis functions

ϕ_p a particular solution

$$u^h(x=0) = \underline{\phi_p(0) + a_1\phi_1(0) + a_2\phi_2(0) \dots + a_n\phi_n(0)} = \bar{u}$$

essential BC

if we have $\phi_1(0) = 0$ $\phi_2(0) = 0$ $\phi_n(0) = 0$

then all we need is $\phi_p(0) = \bar{u}$

ϕ_p needs to satisfy essential BC

$\phi_1, \dots \phi_n$ " " (homog) essential BC

eg $\bar{u} = 1$

$u = \bar{u} = 1$

$x = 0$

ϕ_p : $\cos x$ ✓ / $\sin x$ ✗ / 1 ✓ / $1 + 15x$ ✓

choice of ϕ_i's ϕ_i / 1 ✗, $\underbrace{x, x^2, x^3, \sin\frac{\pi x}{L}}_{✓}$, $\cos\frac{\pi x}{2}$ ✗