2023/09/20

Wednesday, September 20, 2023 10:59 AM

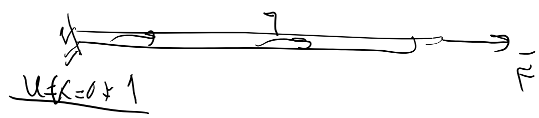
From last time we observed that the essential (Dirichlet) BC can be satisfied for the discrete solution uh by writing it as a summation of a particular solution ϕ_p that satisfies the essential BC and n (number of unknowns) basis functions ϕ_i that satisfy the HOMOGENEOUS essential BC

$$u^h = \phi_p + \underbrace{\sum_{i=1}^{n}}_{\text{we can drop this (summation convention)}} a_i \phi_i = \phi_p + a_i \phi_i = \underbrace{\phi_p(x)}_{\text{particular function}} + \underbrace{[\phi_1(x)\ \phi_2(x) \dots \phi_n(x)]}_{\text{basis functions},\ 1\times n} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\text{unknowns},\ n\times 1}$$

ϕ_p : satisfies essential BC

ϕ_i : satisfy homog. " "

Example 1



ϕ's

$\phi_i(x=0) = 0$ some choices for ϕ_i $\{x, x^2, \sin x, x(1-x)\}$

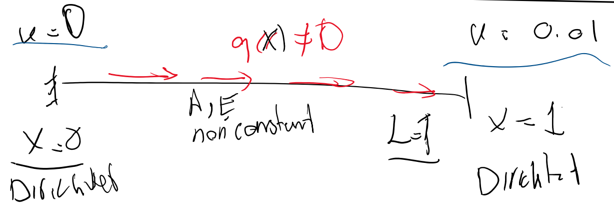
bad choices $\{\cos x\}$

$\phi_p(x=0) = 1$ some choices $\{1, 1-x, \cos x\}$

we'll later solve a problem for $n=2$

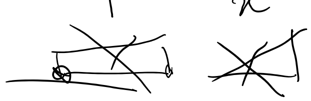
$$u^h = 1 + a_1 x + a_2 x^2 \qquad (\phi_p = 1,\ \phi_1 = x,\ \phi_2 = x^2)$$

Example 2



which ones are good choices for ϕ_i's

(1)	(2)	(3)	(4)	(5)	(6)
$\cancel{1}$	$\cancel{x}$	x^2	$1-x$	$x(1-x)$	$\sin(\pi x)$



$$\left.\begin{aligned} \phi_i(0) &= 0 \\ \phi_i(1) &= 0 \end{aligned}\right\}$$ basis function must be zero @ all essential BC points

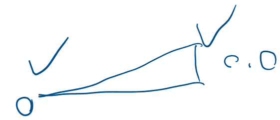
$n = 2$ good choices

$n=2$ good choice

(2a) $u^h(x) = a_1\, x(1-x) + a_2\, x^2(1-x)$

or (2b) $u^h(x) = a_1 \sin \pi x + a_2 \sin 2\pi x$

ϕ_p must satisfy all essential BCs $\quad \phi_p = 0.01x$



we need to add ϕ_p to 2 to have the complete form of u^h

$u^h(x) = a_1\, x(1-x) + a_2\, x^2(1-x) + 0.01x$ $\quad$ good choice for $n=2$

or $u^h(x) = a_1 \sin \pi x + a_2 \sin 2\pi x + 0.01x$

verify $u^h(0)=0 \quad u^h(1) = 0.01$ $\rightarrow$ ϕ_p satisfies essential BC

$u^h(0) = a_1(0) + a_2(0) + (0.01)0 = 0$

$u^h(1) = a_1(0) + a_2(0) + (0.01)1 = 0.01$

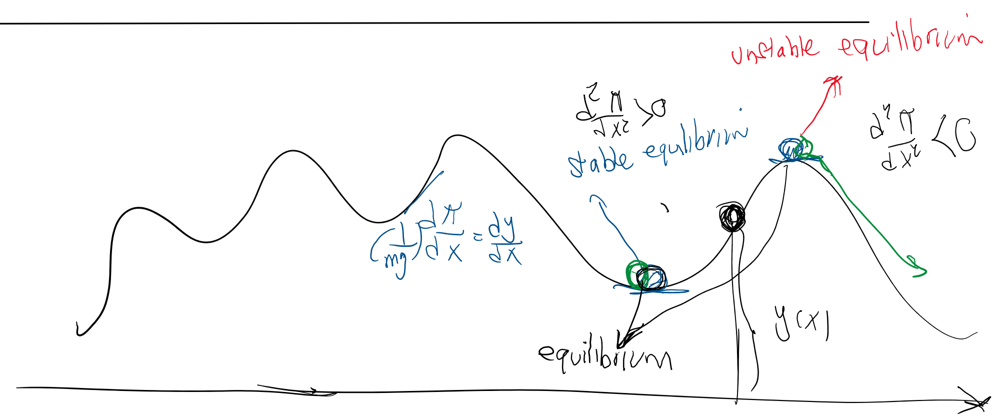
all ϕ_i's are zero @ all essential BC's

Energy methods
Motivation

y

$\Pi(x) = mg\,y(x)$

equilibrium ($\sum F_y = 0$)



$\Longleftrightarrow \quad \frac{d\Pi(x)}{dx} = 0$

in addition to this

$\frac{d^2\Pi}{dx^2} < 0$ $\quad$ unstable

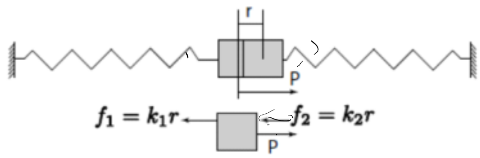
$\frac{d^2\Pi}{dx^2} > 0$ $\quad$ stable

Reason: Stable equilibrium $\Longleftrightarrow$ minimum potential energy

③

Example:

Find r for equilibrium state



Approach 1: Balance of forces $\quad -k_1 r - k_2 r + P = 0 \Rightarrow \boxed{r = \dfrac{P}{k_1+k_2}}$ (4)

Approach 2 Energy method

$$\Pi = \underset{\text{internal energy}}{V} - \underset{\text{external work}}{W} = \underbrace{\frac{1}{2}k_1(r)^2 + \frac{1}{2}k_2(-r)^2}_{V} - \underbrace{(Pr)}_{\text{external work}}$$

$$= \frac{1}{2}k_1 r^2 + \frac{1}{2}k_2 r^2 - Pr$$

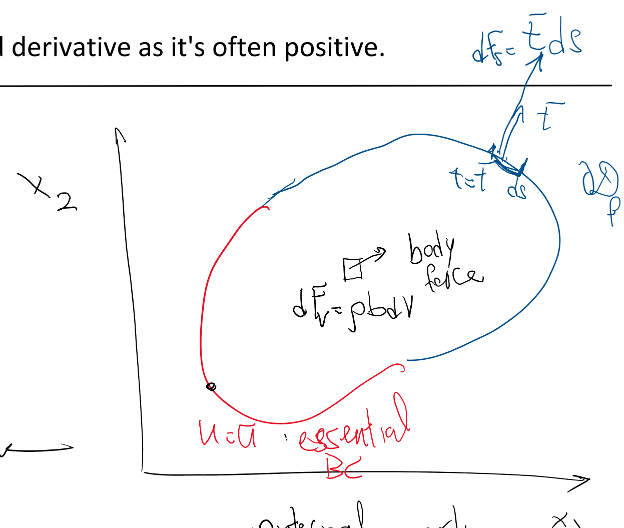
equilibrium ; $\dfrac{d\Pi}{dx} = k_1 r + k_2 r - P = (k_1+k_2)r - P = 0$

$\Rightarrow \quad r = \dfrac{P}{k_1+k_2} \quad$ eq4

stable or not? $\qquad \dfrac{d^2\Pi}{dx^2} = k_1 + k_2 > 0$

In many practical case (not all) we don't need to calculate the second derivative as it's often positive.

Hot to calculate the potential energy for a continuum system



$$\Pi = V - W - T$$

$$V = \int_{\Omega} e(\varepsilon)\,dV \qquad \text{internal energy}$$

$e(\varepsilon)$: internal energy density

similar to energy stored in spring in example above

external work

similar to Pr in example above

$$W = W_b + W_f$$

$$W_b = \int_{\Omega} \underset{\text{displacement}}{u}\,(\underset{\text{force}}{dF_v}) = \int_{\Omega} u\,\rho b\,dV \qquad \text{body external work}$$

$$W_f = \int u\,(df_s) = \int_{\partial\Omega} u\,\underbrace{\bar{t}\,ds}_{df_s} \qquad \text{natural BC}$$

$$W_f = \int_{\partial\Omega_f} u\,(df_s) = \int_{\partial\Omega_f} u \underbrace{t\,ds}_{df_s}$$

natural BC

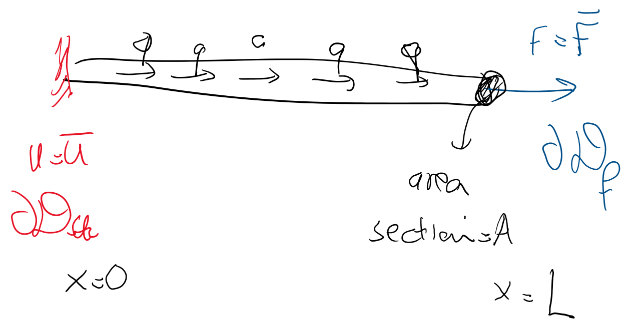
$$T = \int_{\Omega} \underbrace{\frac{1}{2}\dot{u}^2}_{\text{kinetic energy density}} \overbrace{\rho\, dV}^{dm}$$

Kinetic energy

in this course we don't have the dynamic terms ($T = 0$ in ME/AE 517)

Example for Π: Bar problem

$$\Pi = V - W - \cancel{T}^{0} \qquad (5a)$$

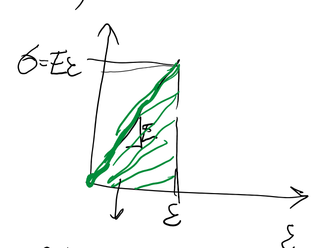


$$V = \int_{\Omega} e(\varepsilon)\, dV = \int_{x=0}^{x=L} e(\varepsilon)\, \underbrace{(A\,dx)}_{dV}$$

$$= \int_{x=0}^{L} \frac{1}{2} E \varepsilon^2 A\, dx$$

note $\varepsilon = u' \left(= \frac{du}{dx}\right)$

1D elasticity



$$e(\varepsilon) = \frac{\varepsilon \cdot \sigma}{2} = \frac{1}{2} E \varepsilon^2$$

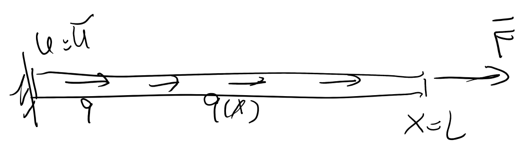
$$\rightarrow \boxed{V = \int_0^L \frac{1}{2} E A u'^2\, dx} \qquad (5b)$$

$$W = \underbrace{\int_0^L u\, (\underbrace{q\, dx}_{dF_V})}_{\text{external work from } q} + u(L)\, \bar{F} \qquad (5c)$$

Plug 5b & 5c in 5a to get

unknown is u

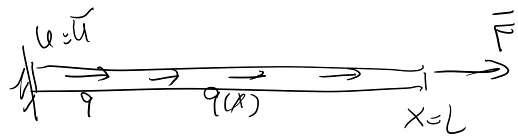
$$\Pi(u(x)) = \int_0^L \frac{1}{2} E A\, u'^2\, dx - \int_0^L u\, q\, dx - u(L)\, \bar{F} \qquad (6)$$



unknown is u

$$\underline{\Pi(u(x))} = \int_0^L \frac{1}{2} EA u'^2 dx - \int_0^L u q\, dx - u(L)\bar{F}$$

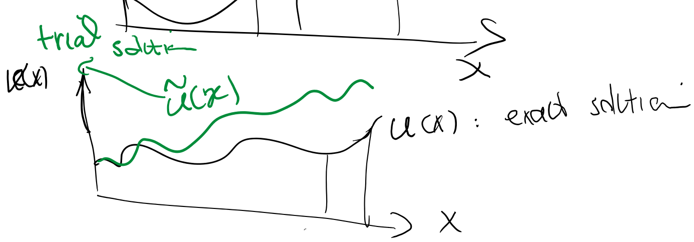
6



A scalar function of a function is called a functional

eq 6

$\Pi \longrightarrow$ potential $\underline{\text{energy}}$ scalar value



$$\Pi(u(x)) \le \Pi(\tilde{u}(x))$$

$u(x)$: exact solution; $\tilde{u}(x)$: a "trial" solution

$\boxed{\text{the energy of the exact solution is minimum}}$

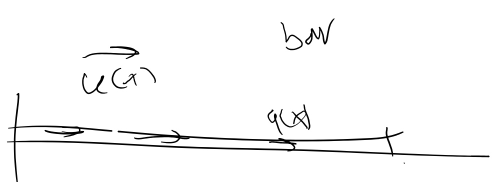
We basically need to calculate the increment (similar to derivative of functions of real number) of potential energy Π to obtain the exact solution.

1. Useful links for energy method (not necessary to apply energy approach in the derivation of weak statement)– link Functional optimization: How an equation for first variation of a functional (e.g. equations 93, 95 on slide 78) can be derived. You clearly do not need to read this document for this course and this is only provided as a related material for students that want to understand the logic behind the derivation of equations 93, 95. – link Exact calculation of total, first, and second variations for a simple example: In this document the total variation of the energy functional for the bar problem is directly calculated. The first and second variations are directly obtained and higher variations are zero for this simple functional. It is observed that the first variation is exactly the same as what we would have obtained by equation 96 on slide 78.

From <http://rezaabedi.com/teaching/me-517-finite-elements/>

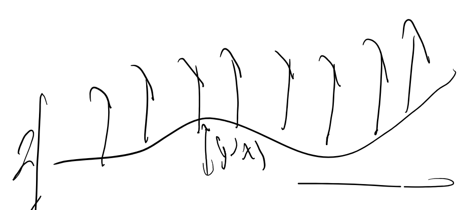
For a beam problem we can show that V is

bar



$$V = \int_0^L \frac{1}{2} EA u'^2 dx$$

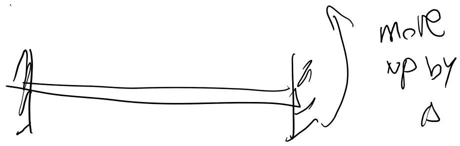
beam

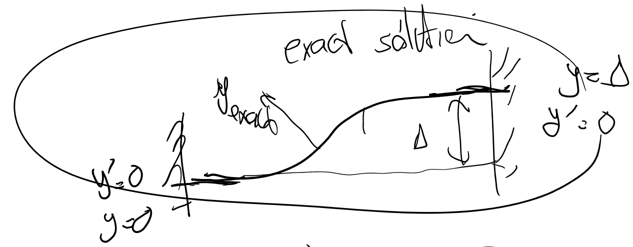


$$V = \int_0^L \frac{1}{2} EI y''^2 dx$$

$\int_0$ $\qquad V = \int_0 \frac{}{2} \ldots y$

beam example





$$\Pi = V - W^{0} = V = \underline{\frac{1}{2} EI\, y''^{2}_{exact}\, dx} > 0$$

trial solution $\qquad \tilde{y}_1 = 0 \qquad V(\tilde{y}_1) = \int \frac{1}{2} EI\, \tilde{y}_1''^{2}\, dx = 0$

$$\text{Why } \Pi(y_{exact}) > 0 = \Pi(\tilde{y}_1) \;?$$

$$\tilde{y}_1' = 0$$

Whereas I had mentioned that the potential energy of the exact solution is less than equal to the energy of ALL trial functions?

$$\Pi(y_{exact}) \le \Pi(\tilde{y})$$

All $\tilde{y}$