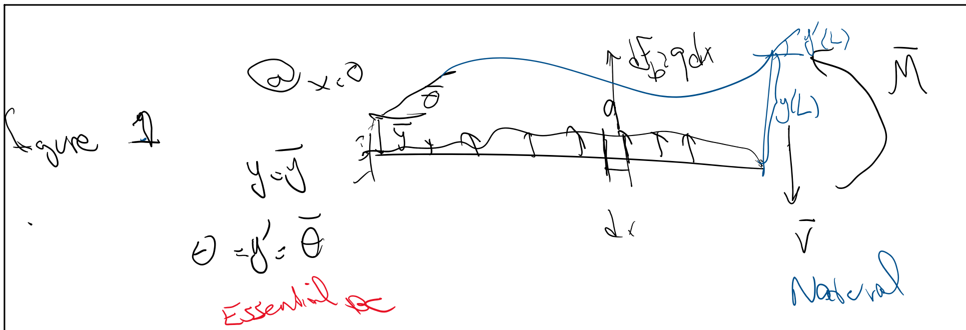
2023/09/25

Monday, September 25, 2023 11:19 AM

Energy methods:

- The correct (exact) solution has the minimum potential energy among all **trial solutions**.
  - **Trial solution**: Is a smooth enough function (so that the energy can be calculated) that satisfies **all ESSENTIAL BCs**.




Problem 1

natural BC

$$\Pi = \underset{\text{internal}}{V} - \underset{\text{external work}}{W} = \frac{1}{2}\int_0^L EI\,y''^2\,dx - (W_b + W_f)$$

(1)

$$W_b = \int_0^L y\,\underbrace{(q\,dx)}_{dF_b}$$

$$W_f = -\bar V\,y(L) + \bar M\,y'(L)$$

$$\Pi(y) = \frac{1}{2}\int_0^L EI\,y''^2\,dx - \int_0^L y\,q\,dx + \bar V\,y(L) - \bar M\,y'(L)$$

★

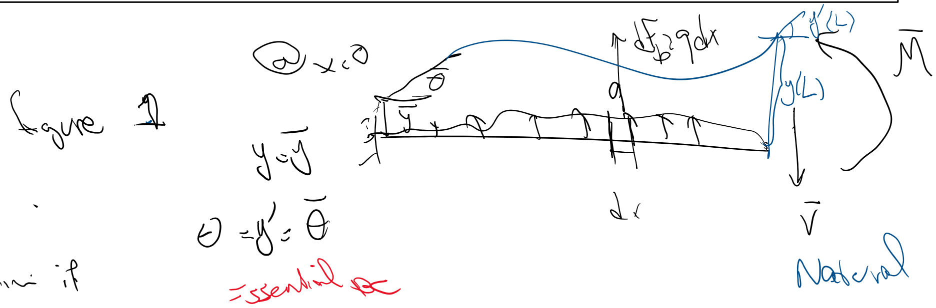


figure 1
@ x=0
$y=\bar y$
$\theta = y' = \bar\theta$
Essential BC
$dF_b = q\,dx$
$q$
$dx$
$y'(L)$
$y(L)$
$\bar M$
$\bar V$
Natural


$y$ is the correct (exact) solution if

$$\forall \tilde y \qquad \Pi(y) \le \Pi(\tilde y)$$

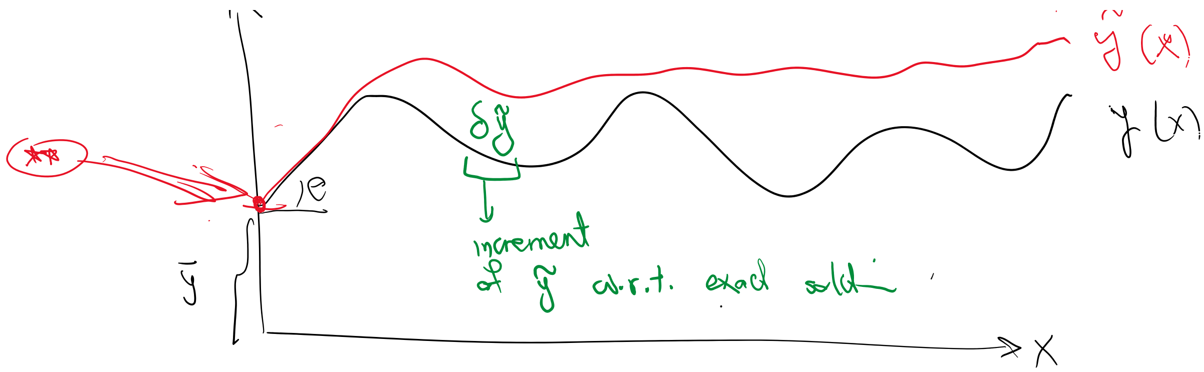
trial function

★ we can reduce this by 1 (see course notes) ★★

$$\underset{\text{space of solution}}{\tilde y \in \mathcal{V}} = \left\{ \tilde y \in C^{\cancel{2} \to 1} \;\middle|\; \tilde y(0) = \bar y \quad \tilde y'(0) = \bar\theta \right\}$$

the same as solution space

trial solution must satisfy in WK all essential BC




$$\delta\tilde{y} = \tilde{y} - y \in W = \{ f \in C^1([0,L]) \mid f(0) = 0 \;,\; f'(0) = 0 \}$$

the same as weight space in WK

$$\delta\tilde{y}(0) = \underbrace{\tilde{y}(0)}_{\bar{y}} - \underbrace{y(0)}_{\bar{y}} = \bar{y} - \bar{y} = 0$$

$$(\delta\tilde{y})'(0) = \tilde{y}'(0) - y'(0) = \bar{\theta} - \bar{\theta} = 0$$

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Similarly for the simpler bar problem we have the following statement

problem 2

Find $u \ni$ (such that) for all trial functions $\tilde{u}$ we have

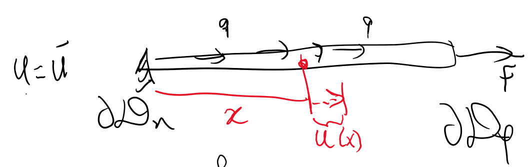



figure 2

$$\Pi(u) \le \Pi(\tilde{u})$$

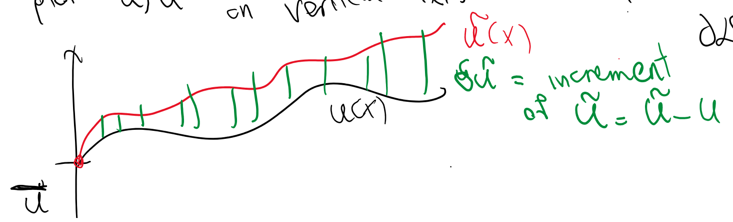
$$u \in V = \{ f \in C^{\cancel{1}\,0}([0,L]) \mid f(0) = \bar{u} \}$$

$$\delta\tilde{u} = \tilde{u} - u \in W = \{ f \in C^{\cancel{1}\,0}([0,L]) \mid f(0) = 0 \}$$

(2)

$$\Pi(u) = V - N = \underbrace{\frac{1}{2}\int_0^L EA\,u'^2\,dx}_{\circledast} - \int_0^L u\,q\,dx - \bar{F}\,u(L)$$

If I plot $u, \tilde{u}$ on vertical axis



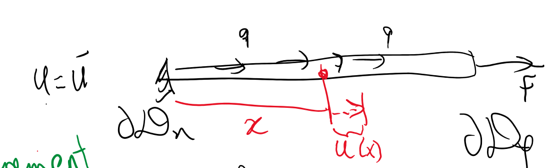
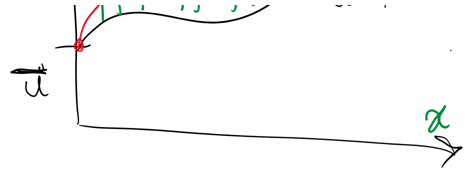




figure 2




$$\Pi(\tilde{u}) = \Pi(\underbrace{u + \delta\tilde{u}}_{\tilde{u}}) = \int_0^L \frac{1}{2} EA\,\tilde{u}'^2\,dx - \int_0^L q\tilde{u}\,dx - F\tilde{u}(L)$$

$$= \int_0^L \frac{1}{2} EA\left[(u+\delta\tilde{u})'\right]^2 - \int_0^L q\,(u+\delta\tilde{u})\,dx - F\,(u+\delta\tilde{u})(L)$$

$$= \int_0^L \frac{1}{2} EA\,(u'^2 + 2u'\delta\tilde{u}' + \delta\tilde{u}'^2) - \int_0^L q\,u\,dx - \int_0^L q\,\delta\tilde{u}\,dx - Fu(L) - F\delta\tilde{u}(L)$$

$$\Pi(\tilde{u}) = \underbrace{\int_0^L \frac{1}{2} EA\,u'^2\,dx - \int_0^L q\,u\,dx - Fu(L)}_{\Pi(u)}$$ "—" terms

$$+ \underbrace{\int_0^L \delta\tilde{u}' EA\,u'\,dx - \int \delta\tilde{u}\,q\,dx - \delta\tilde{u}(L)F}_{\delta\Pi \text{ "1st increment"}}$$ "=" terms

$$+ \underbrace{\int_0^L \frac{1}{2} EA\,(\delta\tilde{u}')^2\,dx}_{\delta^2\Pi \text{ "2nd increment"}}$$ "≡" term

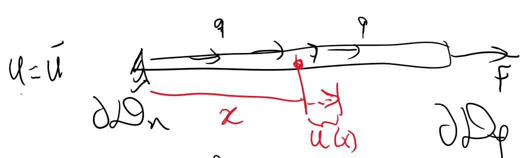
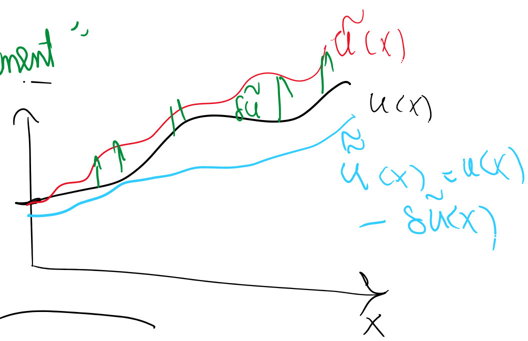



figure 2



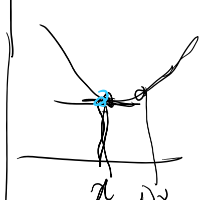

$$\underbrace{0 \le}_{\substack{\text{because} \\ u \text{ is the} \\ \text{exact solution}}} \Pi(\tilde{u}) - \Pi(u) = \delta\Pi + \delta^2\Pi \qquad (3)$$

$$\delta\Pi = \int \delta\tilde{u}'\,EAu'\,dx - \int_0^L \delta\tilde{u}\,q\,dx - \delta\tilde{u}(L)F$$

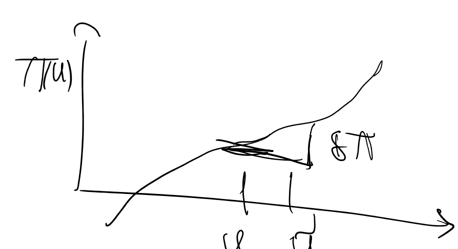
$$\delta^2\Pi = \int \frac{1}{2} EA\,((\delta\tilde{u})')^2$$

& $\Pi(u) \le \Pi(\tilde{u})$

similar to
$f(x+\Delta x) - f(x)$
$= f'(x)\Delta x$
$+ \frac{1}{2} f''(x)\Delta x^2$
$+ \dots$




$f'(x) \neq 0$ eg $f(x) \dots$




For minimum condition

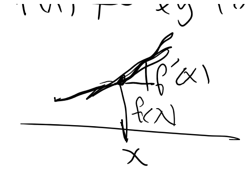
$\delta\Pi$ must be zero so if $\delta\Pi = 0$ eqn (3)

$\delta\Pi$ must be zero so if $\delta\Pi = 0$ eqn (3) simplifies to

$$\Pi(\tilde{u}) - \Pi(u) = \delta^2\Pi \geq 0$$

$$\delta^2\Pi = \int_0^L \underbrace{\tfrac{1}{2} EA(\delta\tilde{u}')^2}_{\geq 0} dx \geq 0$$

$\Pi(u) \leq \Pi(\tilde{u})$




$\Rightarrow$ All we need to satisfy for the exact solution $u$ is

$$\boxed{\delta\Pi = 0} \quad (4)$$

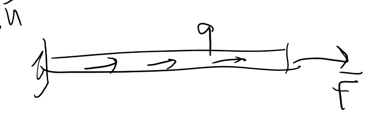
Energy minimization gives $(\delta\Pi = 0)$. That is

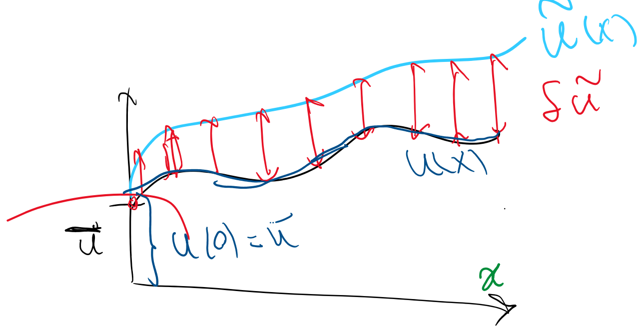
Find $u \in \mathcal{V} = \{ f \in C^{\cancel{*}0}([0,L]) \;\; f(0) = \bar{u} \}$

$\ni \forall \underbrace{\delta\tilde{u}}_{w(x)} \in \mathcal{W} \{ f \in C^{\cancel{*}0}([0,L]) \;\; f(0) = 0 \}$

$$\delta\Pi = \int_0^L \underbrace{\delta\tilde{u}'}_{w'(x)} EA\, u' dx - \int_0^L \underbrace{\delta\tilde{u}}_{w(x)} q\, dx - \underbrace{\delta\tilde{u}(L)}_{w(L)} F = 0$$

$u = \bar{u}$




This is identical to the weak statement that we derived on 9/13

Find $u \in \mathcal{V} = \{ f \in C^0(0,L) \;\; f(0) = \bar{u} \} \ni$

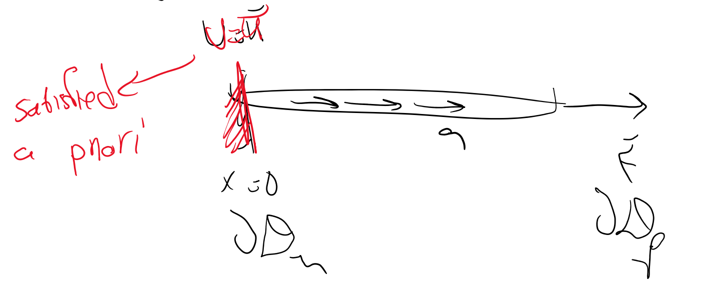
$\forall w \in \mathcal{W} = \{ f \in C^0(0,L) \;\; f(0) = 0 \}$

& : orders are balanced
single weight, satisfy essential

$\forall \omega \in \mathcal{W} = \{ f \in C^1(0,L) \quad f(0) = 0 \}$ & $\therefore$ orders are balanced & slnd weight satisfy essential BC (for weight the homogeneous version of $\bar{u}$)

$$\int_0^L \underline{\omega'} EA\, u'\, dx = \int_0^L q\, dx + \omega(L)\bar{F}$$

the weak statement

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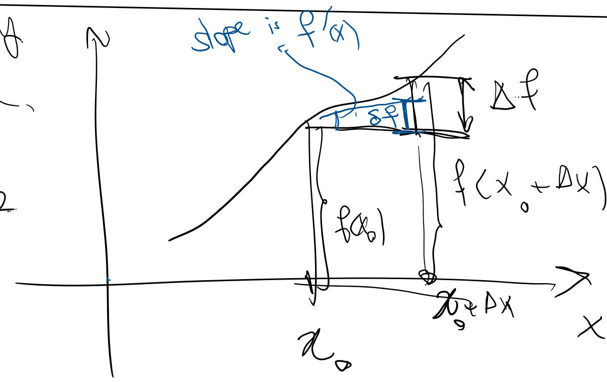
$\Delta f = \delta f + \delta^2 f + \delta^3 f \ldots$

$\delta f = f'(x_0)\Delta x$

$\delta^2 f = \frac{1}{2} f''(x_0)\Delta x^2$




$$\underbrace{\Delta f}_{\text{total variation}} = f(x_0 + \Delta x) - f(x_0) = \underbrace{f(x_0) + f'(x_0)\Delta x + \frac{1}{2} f''(x_0)\Delta x^2 + \ldots}_{\text{Taylor's series of } f(x_0+\Delta x)} - f(x_0)$$

$$= \underbrace{f'(x_0)\Delta x}_{\delta f} + \underbrace{\frac{1}{2} f''(x_0)\Delta x^2}_{+\delta^2 f} + \frac{1}{6} f'''(x_0)\Delta x^3 + \ldots + \delta^3 f$$

1st increment — second increment — 3rd increment ..

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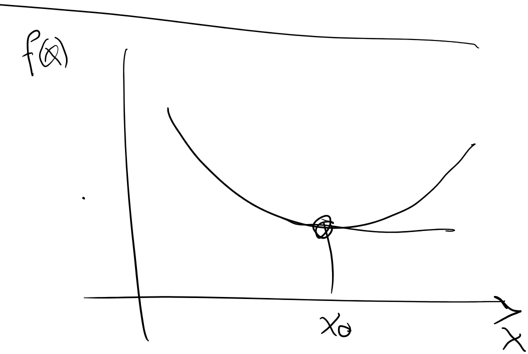
Minimum value

$\Delta f = \delta f + \delta^2 f + \delta^3 f \ldots$

$\delta f = f'(x_0)\Delta x = 0$

$\delta^2 f = \frac{1}{2} f''(x_0)\Delta x^2$



$\delta^2 f = \frac{1}{2} f''(x_0) \Delta x$ | $x_0$ $\bar{x}$

$$\Delta f = \cancel{\delta f} + \frac{1}{2} \underbrace{f''(x)}\ \underbrace{\Delta x^2}_{>0} > 0$$

$$f''(x_0) > 0$$

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Same for functionals

$$\Delta \Pi = \underbrace{\delta \Pi}_{=0} + \underbrace{\delta^2 \Pi}_{>0} + - -$$

for a min condition