2023/09/27

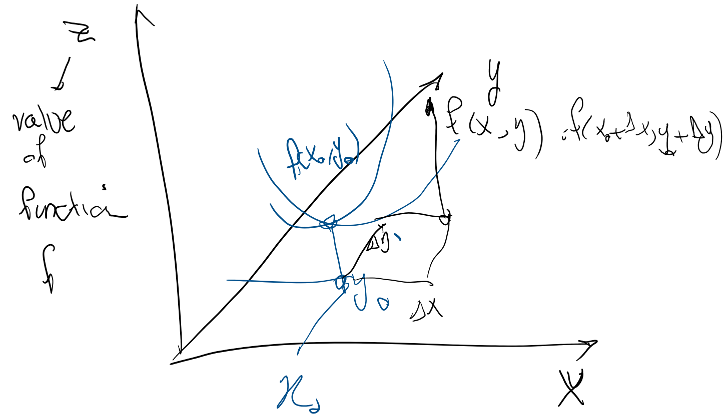
Wednesday, September 27, 2023 11:21 AM

Path to computing 1st increment automatically

Motivation from a function of two variables"

Statements for

minimum condition for f @ (x_0, y_0)



$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0, y_0) + \underbrace{\frac{\partial f}{\partial x}(x_0, y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\,\Delta y}_{\delta f \,=\, \text{1st increment}}$$

$(\delta f = 0)$

$$+ \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\,\Delta x^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\,\Delta x \Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\,\Delta y^2}_{\delta^2 f \;=\; \frac{1}{2}[\Delta x \;\; \Delta y] \underbrace{\begin{bmatrix} f_{,xx} & f_{,xy} \\ f_{,xy} & f_{,yy} \end{bmatrix}}_{\text{Hessian matrix}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}}$$

$f_{,xx} = \dfrac{\partial^2 f}{\partial x^2}$

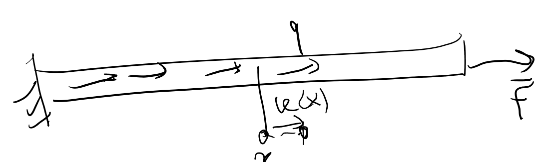
$(\delta^2 f > 0)$

$+$ HOT (higher order terms)

$$\boxed{\begin{array}{l} \text{Necessary condition for minimum value @ } (x_0, y_0) \\ f(x,y) \longrightarrow \delta f = f_{,x}\,\Delta x + f_{,y}\,\Delta y = 0 \quad @ \; (x_0, y_0) \end{array}} \qquad (1)$$

Can we do a similar thing for functionals?

u is the trial solution



$$\Pi(u, u') = \frac{1}{2}\int_0^L EA\, u'^2\, dx - \int_0^L q\, u\, dx - u(L)\, F$$

$$\Pi(\underset{\sim}{u},\underset{\sim}{u'}) = \frac{1}{2}\int_0^L EAu'^2\,dx - \int_0^L qu\,dx - \tilde{u}(L)\bar{F}$$

similar to x & y in eqn (1)

$$\delta\Pi = \frac{\partial\Pi}{\partial u}\delta u + \frac{\partial\Pi}{\partial u'}\delta(u')$$

$$= \frac{1}{2}\int_0^L\left(\underbrace{\frac{\partial EAu'^2}{\partial u}}_{0}\delta u + \underbrace{\frac{\partial EAu'^2}{\partial u'}}_{2EAu'}\delta u'\right)dx \quad \left(\text{as if } \frac{\partial EAy^2}{\partial y} = 2EAy\right)$$

$$-\int_0^L q\left[\underbrace{\frac{\partial u}{\partial u}}_{1}\delta u + \frac{\partial u}{\partial u'}\delta u'\right) + \underbrace{\frac{\partial u}{\partial u}}_{1}\delta u(L)\bar{F}$$

$$= \frac{1}{2}\int_0^L 2EAu'\delta u'\,dx - \int_0^L q\,\delta u\,dx - \delta u(L)\bar{F} = 0$$

$$\boxed{\delta\Pi = \int \delta u' EAu'\,dx - \int_0^L \delta u\, q\,dx - \delta u(L)\bar{F} = 0} \quad (2)$$

Equation (2) is identical to first increment statement we obtained last time by brute force (plugging $u+\delta u$) in the equation

Proof of how one can take derivative of functions:
https://rezaabedi.com/wp-content/uploads/Courses/FEM/FunctionalOptimum.pdf

$$\underset{\text{functional}}{\Pi}(y, y', y'', \ldots)$$

$$(3) \quad \boxed{\delta\Pi = \frac{\partial\Pi}{\partial y}\delta y + \frac{\partial\Pi}{\partial y'}\delta y' + \frac{\partial\Pi}{\partial y''}\delta y'' + \ldots}$$

$$\delta \Pi = \frac{\partial}{\partial y}\delta y + \frac{\partial}{\partial y'}\delta y' + \frac{\partial}{\partial y'} \dots$$

Functionals Optimality condition

- For a function $f:\mathbb{R}\to\mathbb{R}$ we observed that a necessary condition for optimality of f at x_0 was, $\delta f = \frac{\mathrm{d}f}{\mathrm{d}x}(x_0)\Delta x = 0$.
- What do we expect a necessary optimality condition for a functional Π be?
  a necessary extremum condition for Π at y is

$$\delta\Pi = 0, \text{ where } \delta\Pi \text{ is a shorthand for } \delta\Pi(y,\delta y) \tag{92}$$

- How to evaluate $\delta\Pi$?

$$\Pi = \Pi(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}) \quad \Rightarrow \quad \delta\Pi = \frac{\partial \Pi}{\partial y}\delta y + \frac{\partial \Pi}{\partial \frac{\mathrm{d}y}{\mathrm{d}x}}\delta(\frac{\mathrm{d}y}{\mathrm{d}x}) + \dots + \frac{\partial^n \Pi}{\partial {\frac{\mathrm{d}^n y}{\mathrm{d}x^n}}^n}\delta(\frac{\mathrm{d}^n y}{\mathrm{d}x^n}) \tag{93}$$

Note the similarity to the corresponding conditions for a function $f(x)$: $\delta f = \frac{\mathrm{d}f}{\mathrm{d}x}\Delta x = 0$.

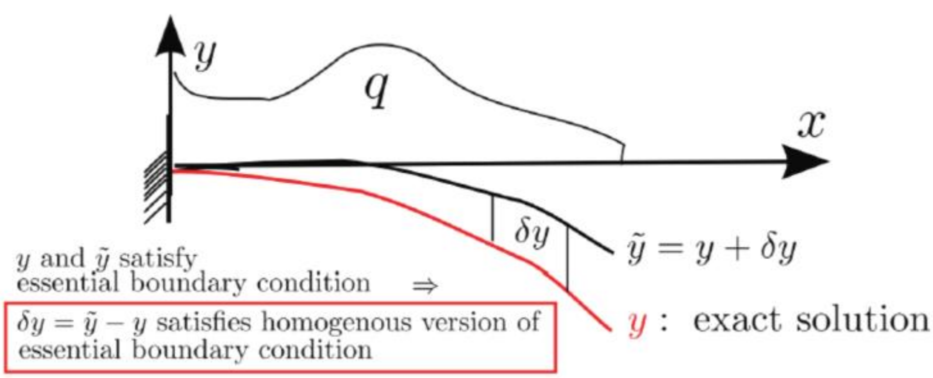
$$\text{Having a } \delta y \Rightarrow \tilde{y} = y + \delta y \Rightarrow \frac{\mathrm{d}^n \tilde{y}}{\mathrm{d}x^n} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \frac{\mathrm{d}^n \delta y}{\mathrm{d}x^n} \Rightarrow \boxed{\delta(\frac{\mathrm{d}^n y}{\mathrm{d}x^n}) = (\frac{\mathrm{d}^n \delta y}{\mathrm{d}x^n}) := \delta y^{(n)}} \tag{94}$$

Thus, noting that $y^{(n)} := \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$, (93) can be rewritten as,

$$\Pi = \Pi(y, y', \dots, y^{(n)}) \quad \Rightarrow \quad \boxed{\delta\Pi = \frac{\partial \Pi}{\partial y}\delta y + \frac{\partial \Pi}{\partial y'}\delta y' + \dots + \frac{\partial^n \Pi}{\partial {y^{(n)}}^n}\delta y^{(n)}.} \tag{95}$$

78 / 456

Summary: Functionals and function spaces



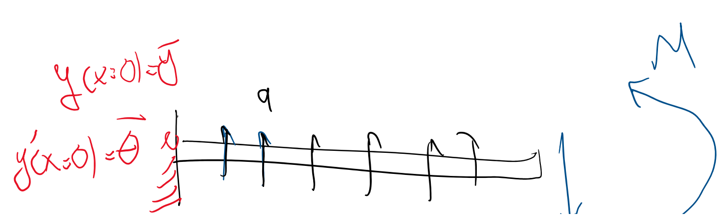
1. A necessary condition for the optimality of functional Π at y is (*cf.* (95)),

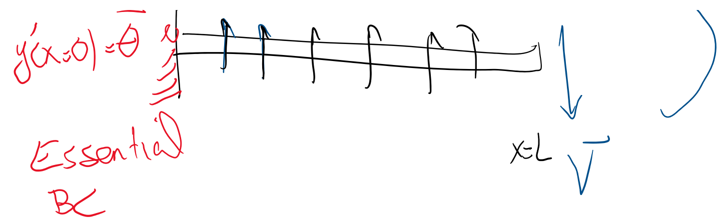
$$\Pi = \Pi(y, y', \dots, y^{(n)}) \quad \Rightarrow \quad \boxed{\delta\Pi = \frac{\partial \Pi}{\partial y}\delta y + \frac{\partial \Pi}{\partial y'}\delta y' + \dots + \frac{\partial^n \Pi}{\partial {y^{(n)}}^n}\delta y^{(n)}.} \tag{96}$$

2. Solution y satisfies all essential boundary conditions.
3. Increment δy satisfies the homogeneous version of all essential boundary conditions.

79 / 456

Beam problem





$$\Pi(y, y', y'') = \int_0^L \underline{\tfrac{1}{2} EI\, y''^2}\, dx - \int_0^L y\, q\, dx + \bar{V} y(L) - \bar{M} y'(L)$$

$$\delta\Pi = \int_0^L \frac{\partial \frac{1}{2} EI y''^2}{\partial y''}\, \delta y''\, dx - \int \frac{\partial y}{\partial y}\, \delta y\, q\, dx + \bar{V} \frac{\partial y}{\partial y}\, \delta y(L) - \bar{M} \frac{\partial y'}{\partial y'}\, \delta y'(L)$$

$$\delta\Pi = \int_0^L \delta y''\, EI\, y''\, dx - \int \delta y\, q\, dx + \underline{\delta y(L)}\, \bar{V} - \underline{\delta y'(L)}\, \bar{M} = 0$$

weak statement :)

$$\boxed{\delta y = w}$$

Compare this with Balance law -> Strong form -> WRS -> Weak statement on slides 51-57:

Find $y \in \mathcal{V} = \{u \in C^2(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta}\}$, such that, (62a)

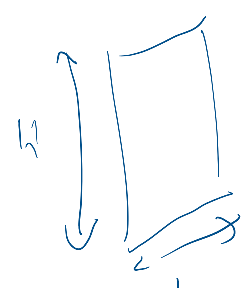
$\forall w \in \mathcal{W} = \{u \in C^2(\mathcal{D}) \mid u(0) = 0,\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0\}$ (62b)

$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right] \mathrm{d}x + \left\{-\underline{\frac{\mathrm{d}w}{\mathrm{d}x}} \bar{M} + \underline{w}\bar{V}\right\}_{x=L}$ (62c)

Summary

- Both $\mathcal{V}$ and $\mathcal{W}$ have the same regularity ($C^m(\mathcal{D})$): $m = M/2$, $M = 4$ is the order of the differential equation.
- The less demanding regularity conditions for the solution compared to the weighted residual statement ($C^M(\mathcal{D}) \rightarrow C^m(\mathcal{D})$) takes us to the same function space needed for the balance law (balance of linear and angular momentum for Euler Bernoulli beam.
- Both $\mathcal{V}$ and $\mathcal{W}$ exactly enforce the essential boundary conditions, with the difference that $\mathcal{W}$ satisfies the homogeneous version.

57 / 456

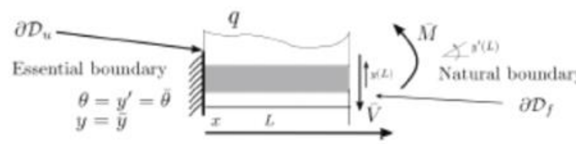


$I = \frac{1}{12} b H^3$

What else an energy method gives us?
What if we do integration by parts on an energy statement?

Energy method to Strong Form and Boundary Conditions

We realized the convenience of energy methods in deriving the weak form in one step. They can also be used to strong form <u>and</u> boundary conditions by the common approach of integration by part (divergence theorem in $D > 1$).



The weak form from (107) is:

$$\delta\Pi = \int_0^L \delta y''(x) EI y''(x)\,\mathrm{d}x - \int_0^L \delta y(x) q(x)\,\mathrm{d}x + \delta y(L)\bar{V} - \delta y'(L)\bar{M} = 0$$

Two consecutive integration by parts yield:

$$\int_0^L -\frac{\mathrm{d}\delta y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\,\mathrm{d}x - \int_0^L \delta y(x) q(x)\,\mathrm{d}x + \delta y(L)\bar{V} - \delta y'(L)\bar{M} + \delta y'(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\Big|_0^L = 0 \Rightarrow$$

$$\int_0^L \delta y\left(\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - q\right)\mathrm{d}x - \delta y'(L)\left(\bar{M} - EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(L)\right) - \delta y'(0) EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(0)$$

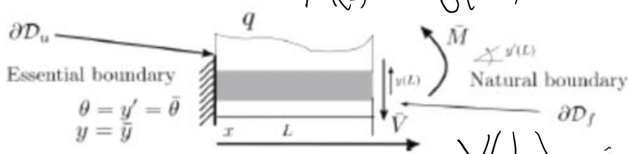
$$+ \delta y(L)\bar{V} - \{\delta y\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\}\Big|_0^L = 0$$

$$\int_0^L \delta y\left(\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - q\right)\mathrm{d}x - \delta y'(L)\left(\bar{M} - EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(L)\right) + \delta y(L)\left(\bar{V} - \frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\right)$$

$$+ \delta y(0)\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})(0) = 0$$

Energy method to Strong Form and Boundary Conditions

$M(L) = EIy''(L) = \bar{M}$



$V(L) = \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)(L) = \frac{dM}{dx}(L) = \bar{V}$

The terms in red in previous equation are zero because the function increment belongs to $\mathcal{W}$ given in (107):

$$\mathcal{W} = \{v \in C^2([0, L]) \mid v(0) = 0, \frac{\mathrm{d}v}{\mathrm{d}x}(0) = 0\}$$

Thus the last equation reduces to

Differential equation $(EIy'')'' - q = 0$

$$\int_0^L \delta y \left(\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - q \right) \mathrm{d}x \tag{113a}$$

$$- \delta y'(L) \left(\bar{M} - EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(L) \right) \tag{113b}$$

$$+ \delta y(L) \left(\bar{V} - \frac{\mathrm{d}}{\mathrm{d}x}(EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}) \right) = 0 \tag{113c}$$

By choosing δy such that $\delta y(L) = \frac{\mathrm{d}\delta y}{\mathrm{d}x}(L) = 0$ we reduce (113) to (113a) equal to zero for such δy increments:

Natural boundary conditions are "naturally" derived from an energy statement

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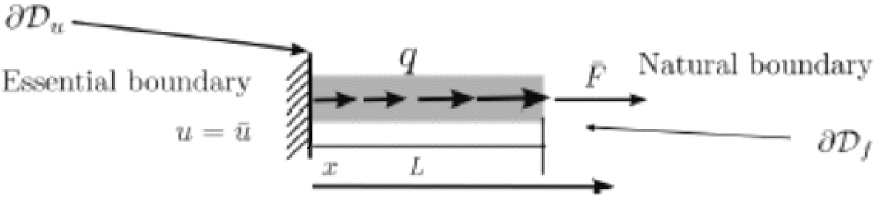
Please read virtual work slides

Energy Method vs. Principle of Virtual Work

- Principle of virtual work or virtual displacement in solid mechanics states that if $\mathbf{u}$ is the solution to a boundary value problem, the virtual internal and external works produces by admissible virtual displacements $\delta\mathbf{u}$ are equal.
- Virtual displacements $\delta\mathbf{u}$ refer to displacements that are zero at essential boundary values (so that solution displacement plus virtual displacement $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$ (*cf.* (79)) as another admissible trial function also satisfies essential boundary conditions).
- Virtual Displacement/Virtual work is basically the equation we obtain by minimizing the energy function $\delta\Pi = 0$.
- Similar principles (virtual temperature for heat flow in solids and virtual velocities for fluid flow) are also directly derived from $\delta\Pi = 0$.
- While principle of virtual work can be obtained from $\delta\Pi = 0$, it is often quite easy to directly write and equate internal and external works for a given problem.

I apply virtual displacement δu

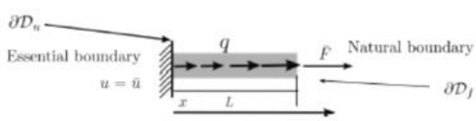
$u = \bar{u}$



Virtual internal work = virtual external work

$$\int \underbrace{\overset{\delta u' dx}{\delta \varepsilon\, dx}}_{\text{virtual displacement}} \underbrace{\sigma A}_{\text{force}} = \delta u(L)\,\bar{F} + \int_0^L \delta u(x)\, q(x)\, dx$$

Virtual work: 1D solid bar



Equation (98) can be written as,

Find $u \in \mathcal{V} = \{v \in C^1([0, L]) \mid v(0) = \bar{u}\}$, such that,
$\forall \delta u \in \mathcal{W} = \{v \in C^1([0, L]) \mid v(0) = 0\}$

$$\underbrace{\int_0^L \overbrace{\delta u'(x)}^{\frac{\mathrm{d}}{\mathrm{d}x}\delta u}\, \overbrace{EAu'(x)}^{F(u(x))}\, \mathrm{d}x}_{\text{Virtual Internal Work}} = \underbrace{\int_0^L \delta u(x) q(x)\, \mathrm{d}x + \delta u(L)\bar{F}}_{\text{Virtual External Work}} \tag{109}$$

Note that the internal work differential is:

$$\begin{aligned} \mathrm{d}V &= F(u(x)).(\delta u + \frac{\mathrm{d}}{\mathrm{d}x}\delta u \mathrm{d}x) - F(u(x)).\delta u \\ &= \frac{\mathrm{d}\delta u}{\mathrm{d}x}.F(u(x))\mathrm{d}x \end{aligned} \tag{110}$$