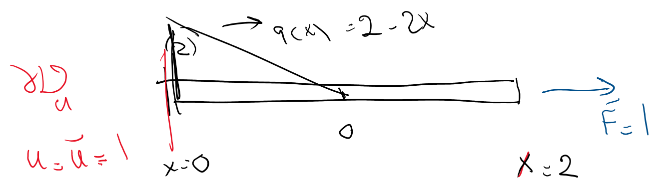
2023/10/02

Monday, October 2, 2023 11:18 AM

In previous sessions we used different approaches to obtain the WRS or the Weak statement (WK)

Now we want to solve the problem numerically

fig 1



numerical solution

$$u^h = \sum_{i=1}^{n} a_i \phi_i(x) + \phi_p(x)$$

n = # of unknowns

(i) ϕ_i = Basis functions : Sastify (homogeneous) version of all essential BC's

ii) ϕ_p = Particular solution → satisfies all essential BC

a_i's are unknowns

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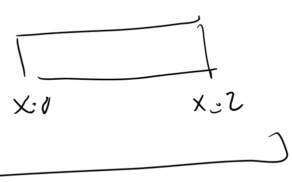
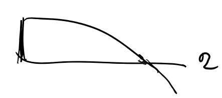
ii) $\phi_p(0) = 1$ e.g 1



$\cos x$



i) $\phi_i(0) = 0$ $n=2$ x , x^2 $\sin x$, $\sin 2x$

$u^h(x) = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x) = 1 + a_1 x + a_2 x^2$ $\cos x + a_1 \sin x + a_2 \sin 2x$

$u^h(0) = 1 + a_1 \cdot 0 + a_2 0 = 1$ 1 ✓ 1 ✓

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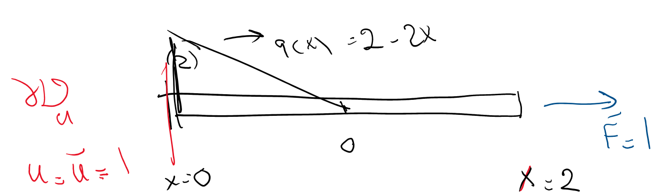
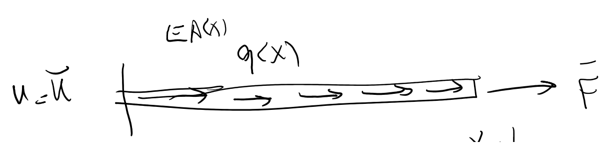
fig 1 $E=1, A=1$

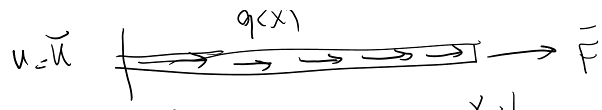


Solving the problem by WRS (and later WK):

WRS



WRS



$$\int_0^L \omega R_i\,dx + \underset{f}{\underbrace{\omega_f}}(L)\,R_f(L) = 0$$

we'll see that for the least square method $\omega_f \neq \omega$

$$\forall\omega: \int_0^L \omega\,\big(\underbrace{(EAu')' + q}_{R_i}\big)\,dx + \omega_f(L)\big(\bar{F} - \underbrace{EAu'(L)}_{F(L)}\big) = 0 \quad (1)$$

Since eventually for figure (1) I'll use E=1, A=1, let's do this substitution now

$$(2)\; u^h = \phi_p + \sum_{i=1}^{n} a_i\phi_i = \phi_p + [\phi_1 \cdots \phi_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \phi_p + \phi a$$

(1) for E=1, A=1

$$= \int_0^L \omega\,(u^{h''} + q)\,dx + \omega_f(L)\big(\bar{F} - \underbrace{EA}_{1}\,u^{h'}(L)\big) = 0 \quad (3)$$

plug (2) in (3)

$$\int_0^L \omega\Bigg(\underbrace{\phi_p''(x) + [\phi_1''(x) \cdots \phi_n''(x)]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{u^{h''}} + q(x)\Bigg)dx$$

$$+\;\omega_f(L)\Bigg(\bar{F} - \underbrace{\Bigg(\Big[\phi_p'(x) + [\phi_1'(x) \cdots \phi_n'(x)]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\Big]\Bigg)\Bigg|_{x=L}}_{u^{h'}(L)}\Bigg) = 0 \quad (4)$$

We need to satisfy equation (4) for n equations (so ending up with n equations, n unknowns a1, ..., an)
Let's call the weight functions w1(x), ..., wn(x)

$$\omega = \begin{bmatrix} \omega_1(x) \\ \omega_2(x) \\ \vdots \\ \omega_n(x) \end{bmatrix} \quad (5)$$

plug (5) in (4):

$$\int_0^L \begin{bmatrix} \omega_1(x) \\ \vdots \\ \omega_n(x) \end{bmatrix}\Bigg(\phi_p''(x) + [\phi_1'' \cdots \phi_n''](x)\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + q(x)\Bigg)dx$$

$[a_1 \;|$

$$\int_0 \begin{bmatrix} \vdots \\ w_n(x) \end{bmatrix} \left(\phi_p'' \ldots \begin{bmatrix} \vdots \\ a_n \end{bmatrix} \right)$$

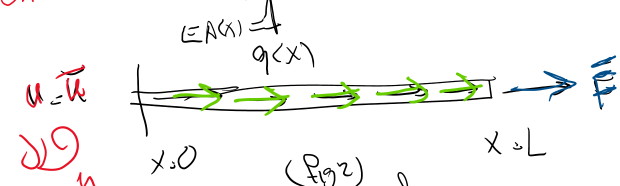
$$+ \begin{bmatrix} w\phi_{f_1}(L) \\ \vdots \\ w\phi_{f_n}(L) \end{bmatrix} \left(\bar{F} - \left[\phi_p'(L) + [\phi_1'(L) \ \cdots \ \phi_n'(L)] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right] \right) = 0$$

$$\Rightarrow \underbrace{\overbrace{\left(\int_0^L \underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}}_{n\times 1} (x) \, \underbrace{EA}_{\text{for constant } EA} \, \underbrace{[\phi_1'' \ \cdots \ \phi_n''](x)}_{1\times n} dx - \underbrace{\begin{bmatrix} w\phi_{f_1}(L) \\ \vdots \\ w\phi_{f_n}(L) \end{bmatrix}}_{n\times 1} EA \, \underbrace{[\phi_1'(L) \ \cdots \ \phi_n'(L)]}_{1\times n} \right)}^{n\times n \text{ matrix}}}_{K} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \underbrace{- \int_0^L \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} (x) \, \underbrace{EA}_{\text{for constant } EA} \, \phi_p''(x) \, dx + \begin{bmatrix} w\phi_{f_1}(L) \\ \vdots \\ w\phi_{f_n}(L) \end{bmatrix} EA \, \phi_p'(L)}_{F_D \text{ (Dirichlet BC force)}}$$

$$\underbrace{- \int_0^L \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} (x) q(x) \, dx}_{F_r \text{ (Force from source term } r=-q)}$$

(6) $$\underbrace{- \bar{F} \begin{bmatrix} w\phi_{f_1}(L) \\ \vdots \\ w\phi_{f_n}(L) \end{bmatrix}}_{F_N \text{ (Force from Neuman BC)}}$$



(fig 2) general

In general $Ka = F$

K: stiffness matrix

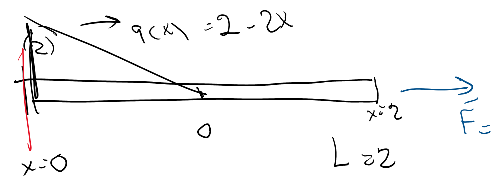
F: force vector

$$F = \underbrace{F_r}_{\text{source term}} + \underbrace{F_N}_{\text{Ne...}} - F_D$$

$$1 = \underbrace{I_r}_{\text{source term force}} + \underbrace{N}_{\text{Neumann BC force}} - \underbrace{D}_{\text{Dirichlet BC force}}$$

Let's solve problem 1 with n = 2 and $\phi_1 = x$, $\phi_2 = x^2$, $\phi_p = 1$

$$\phi = [x, x^2]$$
$$\phi' = [1, 2x]$$
from eq 6 $\quad \phi'' = [0, 2]$



$$K = \left(\int_0^L \underbrace{\begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}}_{n \times 1} (x) \underbrace{\left[\phi_1'' \cdots \phi_n''\right](x)}_{1 \times n} \underbrace{EA}_{\text{for constant } EA} dx - \overbrace{\underbrace{\begin{bmatrix} \omega_{f_1}(L) \\ \vdots \\ \omega_{f_n}(L) \end{bmatrix}}_{n \times 1} EA \underbrace{[\phi_1'(L) \cdots \phi_n'(L)]}_{1 \times n}}^{n \times n \text{ matrix}} \right)$$

$$= \int_0^2 \begin{bmatrix} \omega_1(x) \\ \omega_2(x) \end{bmatrix} [0 \quad 2] \, dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} [1 \quad 2x] \Big|_{x=2}$$

$$\Longrightarrow \text{(7a)} \quad \boxed{K = 2\int_0^2 \begin{bmatrix} 0 & \omega_1(x) \\ 0 & \omega_2(x) \end{bmatrix} dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} [1 \quad 4]}$$

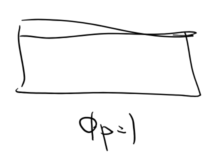
Dirichlet BC force , from eq 6

$$F_D = \int_0^L \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} (x) \overset{0}{\phi_p''}(x) \underbrace{EA}_{\text{for constant } EA} dx + \begin{bmatrix} \omega_{f_1}(L) \\ \vdots \\ \omega_{f_n}(L) \end{bmatrix} \underbrace{\overset{0}{\phi_p'}(L)}_{EA} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(7b)}$$

$\phi_p = 1$

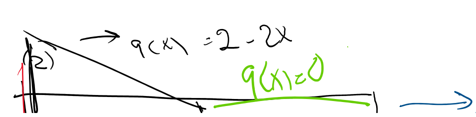
$\phi_p' = 0$

$\phi_p'' = 0$

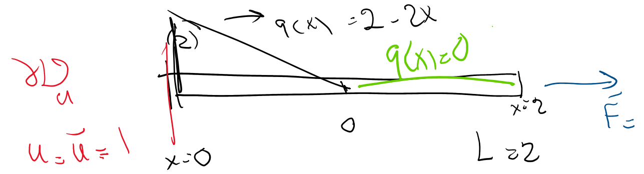


Neumann BC force , from eq 6

$$F_N = - \bar{F} \begin{bmatrix} \omega_{f_1}(L) \\ \vdots \\ \omega_{f_n}(L) \end{bmatrix}$$



$$F_N = - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ w_{f_n}(L) \end{bmatrix}$$



$$F_N = - \begin{bmatrix} w_{f_1}(2) \\ w_{f_2}(2) \end{bmatrix} \quad (7c)$$

Source term q: from eqn 6

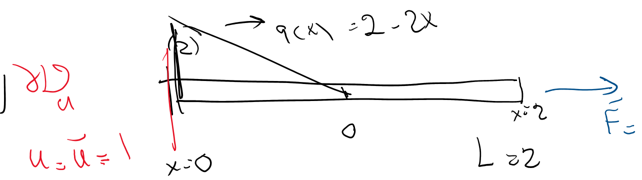
$$F_r = - \int_0^L \begin{bmatrix} w_1(x) \\ w_2(x) \end{bmatrix} q(x)\, dx = \int_0^1 \begin{bmatrix} w_1(x) \\ w_2(x) \end{bmatrix} (2 - 2x)\, dx \quad (7d)$$

7a, – – 7d are

(7)

EA = 1

$$K = \int_0^2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} [0 \quad 2]\, dx - \begin{bmatrix} w_{f_1}(2) \\ w_{f_2}(2) \end{bmatrix} [1 \quad 4]\, \delta w_u$$



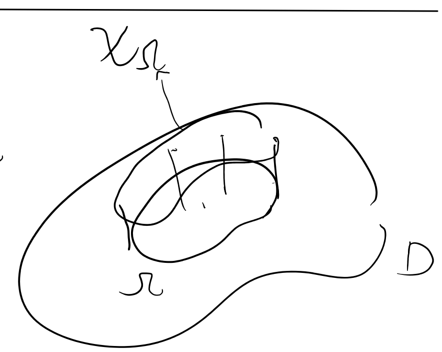
$$F = \underbrace{- \int_0^1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} (2 - 2x)\, dx}_{F_r} \underbrace{- \begin{bmatrix} w_{f_1}(2) \\ w_{f_2}(2) \end{bmatrix}}_{F_N}$$

We'll make difference choices for weights to get different numerical methods

Method 1) Subdomain method

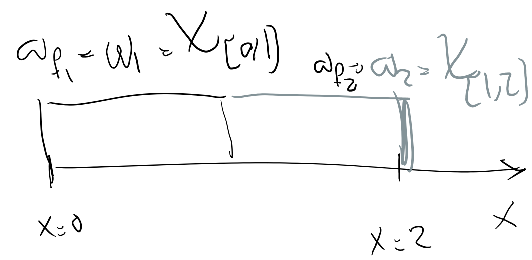
Background

$$\underbrace{\chi_\Omega}_{\text{characteristic function of } \Omega}(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$



characteristic function of Ω

In subdomain method, the weights are characteristic functions



Equation (7):

$$K = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} [0 \;\; 2]\, dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} [1 \;\; 4]$$

$$\underbrace{K_{i\cdot}}_{\text{row } i \text{ of } K} = \int_0^2 \omega_i [0 \;\; 2] - \omega_{f_i}(2)\, [1 \;\; 4]$$

$$\underbrace{K_{1,\cdot}}_{\text{row 1 of } K} = \int_0^2 \chi_{[0,1]} [0,2] - \chi_{[0,1]}(2)\, [1,4]$$

$x=0$ $x=1$ $x=2$

$$= \int_0^1 [0 \;\; 2]\, dx - 0 \times [1 \;\; 4] = [0 \;\; 2]$$

$$\underbrace{K_{2,\cdot}}_{\text{row 2 of } K} = \int_0^2 \chi_{[1,2]} [0,2] - \underbrace{\chi_{[1,2]}(2)}\, [1,4]$$

$\omega_2 = \chi_{[1,2]}$

$$= \int_1^2 (0,2) - (1)\, [1 \;\; 4] = [0 \;\; 2] - [1 \;\; 4]$$
$$= [-1, -2]$$

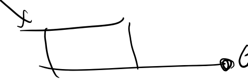
$$K = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix} \begin{array}{l} \rightarrow \text{row 1} \\ \rightarrow \text{row 2} \end{array}$$

From equation (7):

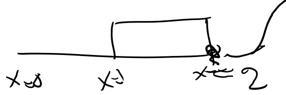
$$F = \underbrace{-\int_0^1 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} (2-2x)\, dx}_{F} - \underbrace{\begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix}}_{f}$$

$$F \quad \int^1 \quad (x)(2-2x)\, dx \quad \overbrace{\qquad}^{0} (2) = \int^1$$

$$F_1 = -\int_0^1 \chi_{[0,1]}(x)\,(2-2x)\,dx - \overbrace{\chi_{[0,1]}(2)}^{0} = -\int_0^1 (2-2x)\,dx - 0 = -1$$



$$F_2 = -\int_0^1 \chi_{[1,2]}(x)(2-2x)\,dx - \chi_{[1,2]}(2) = 0 - 1 = -1$$



$$F = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix} \qquad Ka = F$$

$$\Longrightarrow \quad a = \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix}$$

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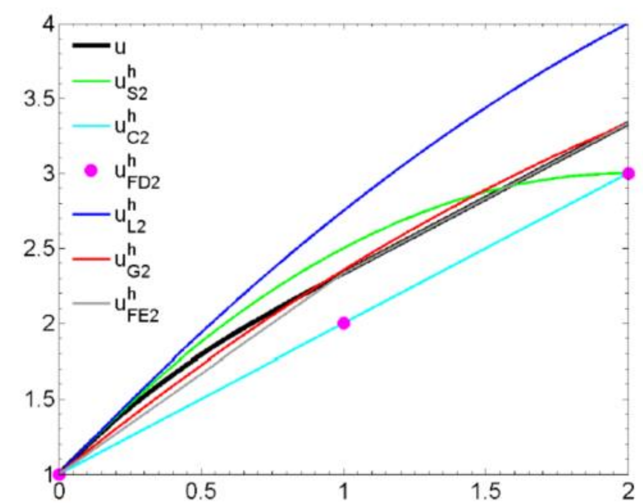
numerical

$$u^h_{S2}(x) = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x) = 1 + 2x - \frac{1}{2}x^2$$

subdomain $\quad$ $n=2$

Bar example, $n = 2$, Comparison of solutions



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