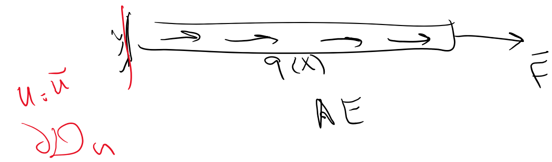
2023/10/04

Wednesday, October 4, 2023 11:20 AM

## 2) Collocation method

fig 2 last time




more general

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fig 2

$$R_i = (EAu')' + q =$$

inside

$$\Big(\underbrace{\mathcal{L}_M}_{\text{Differential operator for inside the domain}}(u) - \underbrace{r}_{\text{source term}}\Big)$$

general expression

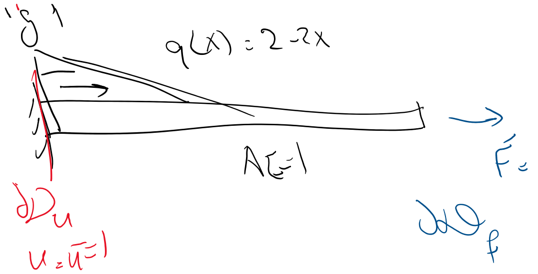


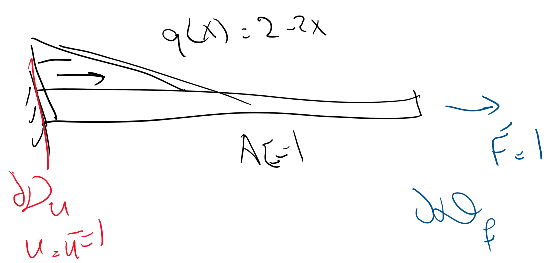

fig 1 from last time

Differential operator for inside the domain WRS

bar: $\mathcal{L}_M(\cdot) = (EA(\cdot)')'$ $\qquad r = -q$

$$R_f = \bar{F} - F\Big|_{x=L} = \bar{F} - EAu'\Big|_{x=L}$$

for numerical example (fig 1)




$$(1)\ \begin{cases} R_i = u'' + q(x) \\ R_f = 1 - u' \end{cases}$$

We discretized it for n = 2

$$u = \phi_p + a_1\phi_1 + a_2\phi_2$$

I chose $\phi_p = 1$, $\phi_1 = x$, $\phi_2 = x^2$

$$(2)\ u = 1 + a_1 x + a_2 x^2 = 1 + [x \quad x^2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow u' = [1 \quad 2x]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, u'' = [0 \quad 2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

plug (2) in (1)

$$R_i = u'' + q(x) = \underbrace{[0 \quad 2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{u''} + q(x) \qquad 0 \le x \le 2$$

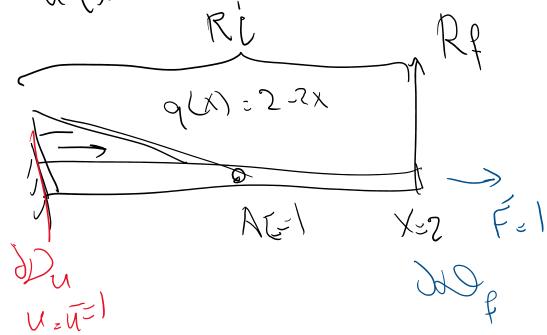
$$R_f = 1 - u'(x=2) = 1 - \underbrace{[1 \quad 2\times 2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} = 1 - [1 \quad 4]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$(x = 2)$

$(x=2)$ $\underbrace{\qquad\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{u'(x=2)}$

I need 2 eqns to solve $a_1$ & $a_2$:

Let's choose $R_f(x=2)=0$ as one of them




$$\text{Eq 1)}\quad R_i(x=1)=0 \rightarrow \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \overbrace{q(x=1)}^{0} = 0$$

$$\text{Eq 2)}\quad R_f = 0 \qquad 1 - \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

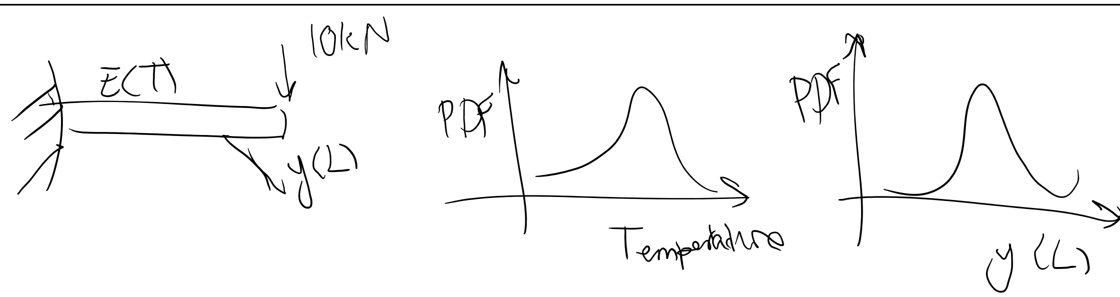
only applies to $x=2$

1

$$\boxed{Ka = F \qquad K = \begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ -1 \end{bmatrix}}$$

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u^h_{C2}(x) = 1 + a_1 x + 0x^2 = 1 + x$$

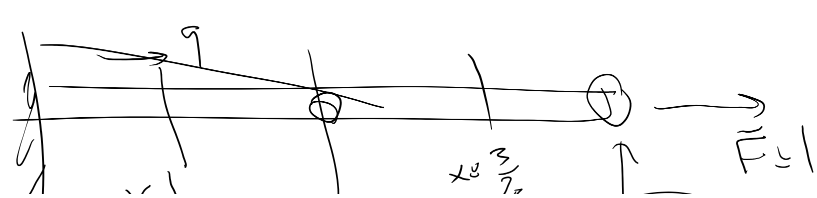
C2 → Collocation, $h=2$

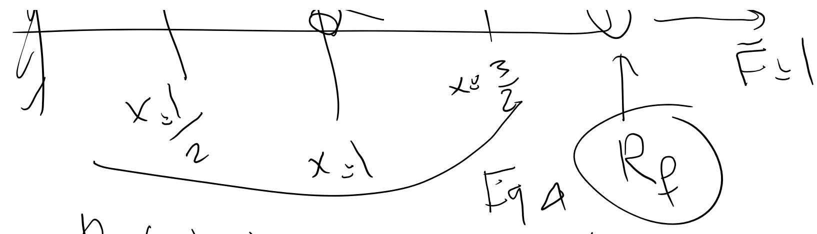



FYI: Collocation method finds application in solving stochastic differential equations.

HW

$n=4$






$R_i(x)=0$ Eqns 1 to 3

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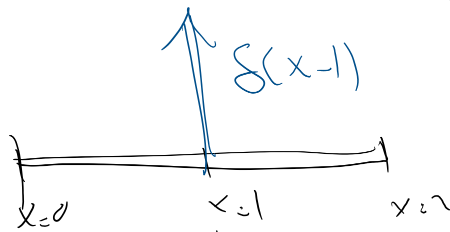
Collocation method can be cast as a WR method with appropriate weights:

WRS from last time

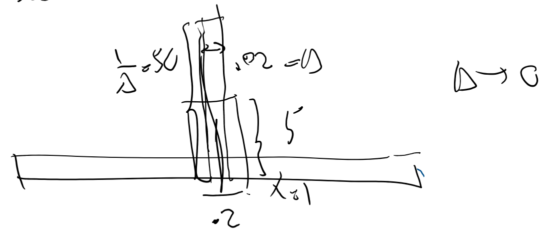
$$\text{(2)} \quad \int_0^2 \omega \underbrace{\left( [0\ \ 2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q(x) \right)}_{R_i(x)} dx + \omega(2)\underbrace{\left(1 - [1\ \ 4]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right)}_{R_f} = 0$$

$\delta$ delta Dirac functi




$$\text{(3)} \quad \int_0^L f(x)\,\delta(x - x_0)\,dx = f(L_0)$$

$$0 < x_0 < L$$




Plug $\omega = \delta(x-1)$ in (2)

$$\int_0^2 \underbrace{\delta(x-1)}_{\omega} \underbrace{\left( R_i(x) \right)}_{\text{like } f \text{ in eqn 3}} dx + \delta(2-1)(R_f) = 0$$

$$= R_i(x=1) + \cancel{\delta(1)}^{\,0} R_f = 0$$

$$\Longrightarrow R_i(x=1) = [0\ \ 2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \cancel{q(1)}^{\,0} = 0$$

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$R_f=0$

$$\int^2 \omega R_i(x)\,dx + \omega(x=2)R_f = 0$$

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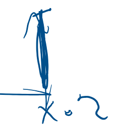


$\omega = \mathcal{V}_{\{2\}}$

$$\omega = \mathcal{V}_{\{2\}} \Longrightarrow 0 + 1 R_f = 0$$

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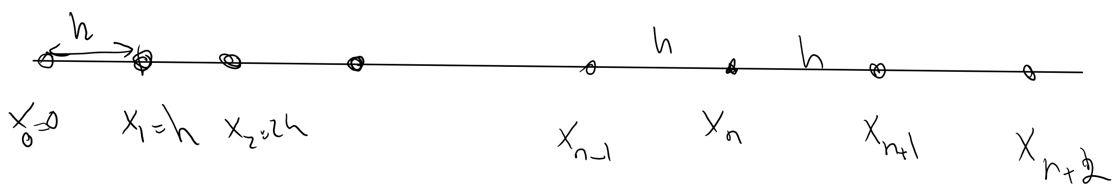
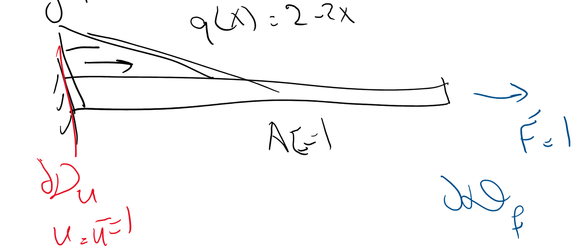
Finite Difference


fig 3




For our example we have

fig 1




$$R_i = \underbrace{u''}_{\text{approximated by Finite Differences (FD)}} + q(x)$$

Examples from fig 3

$$u'(x_n) := u_n' \approx \frac{u_{n+1} - u_n}{h} = \frac{f(x_n+h) - f(x_n)}{h}$$

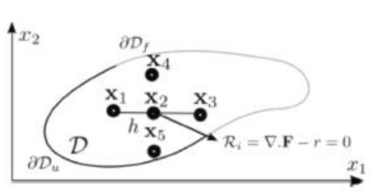
if h is small

$$\approx \frac{u_n - u_{n-1}}{h}$$

$$u''(x_n) = u_n'' = \frac{u_{n+1} + u_{n-1} - 2u_n}{h^2}$$

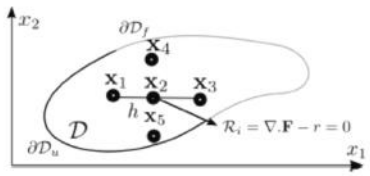
## Collocation method versus Finite Difference

n = 2




- Both Collocation and Finite Difference methods directly work with the strong form and boundary conditions.
- Collocation method is a particular class of weighted residual method where the solution is interpolated as $\mathbf{u}^h = a_j\phi_j + \phi_p$.
- Finite Difference does not interpolate the solution with trial function. Rather, it uses discrete values of the function on often regular grids to approximate differential operators.
- Differential operators in Finite Difference method are approximate, where as in collocation

## Collocation method versus Finite Difference

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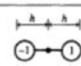
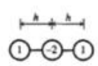
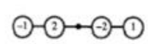
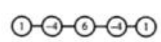
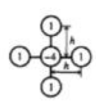
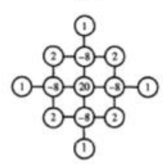
- Collocation method is a particular class of weighted residual method where the solution is interpolated as $\mathbf{u}^h = a_j\phi_j + \phi_p$.
- Finite Difference does not interpolate the solution with trial function. Rather, it uses discrete values of the function on often regular grids to approximate differential operators.
- Differential operators in Finite Difference method are approximate, where as in collocation method the solution $\mathbf{u}^h$ exactly satisfies the strong form at $\mathbf{x}_i$.
- As an example, let us assume the differential operator $L_M$ in $\mathcal{R}_i$ includes a Laplacian operator $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$. The finite difference approximation of Laplacian on a uniform grid with size $h$ would be,

$$\Delta u(\mathbf{x}_2) = \frac{1}{h^2}\left(u(\mathbf{x}_1) + u(\mathbf{x}_3) + u(\mathbf{x}_4) + u(\mathbf{x}_5) - 4u(\mathbf{x}_2)\right) \tag{150}$$



## Finite Difference Stencils

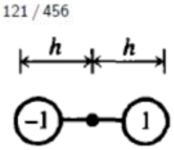
**TABLE 3.1** *Finite difference approximations for various differentiations*

| Differentiation | Finite difference approximation | Molecules |
|---|---|---|
| $\left.\frac{dw}{dx}\right\vert_i$ | $\frac{w_{i+1} - w_{i-1}}{2h}$ |  |
| $\left.\frac{d^2w}{dx^2}\right\vert_i$ | $\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$ |  |
| $\left.\frac{d^3w}{dx^3}\right\vert_i$ | $\frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2h^3}$ |  |
| $\left.\frac{d^4w}{dx^4}\right\vert_i$ | $\frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4}$ |  |
| $\nabla^2 w\vert_{i,j}$ | $\frac{-4w_{i,j} + w_{i+1,j} + w_{i,j+1} + w_{i-1,j} + w_{i,j-1}}{h^2}$ |  |
| $\nabla^4 w\vert_{i,j}$ | $[20w_{i,j} - 8(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) + 2(w_{i+1,j+1} + w_{i-1,j+1} + w_{i-1,j-1} + w_{i+1,j-1}) + w_{i+2,j} + w_{i-2,j} + w_{i,j+2}$ $w_{i,j-2}]/h^4$ |  |

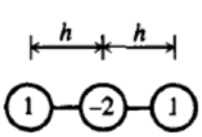
Source:Bathe's book, section 3.3.5.



$\left.\frac{dw}{dx}\right|_i$ $\qquad \frac{w_{i+1} - w_{i-1}}{2h}$



$\left.\frac{d^2w}{dx^2}\right|_i$ $\qquad \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$



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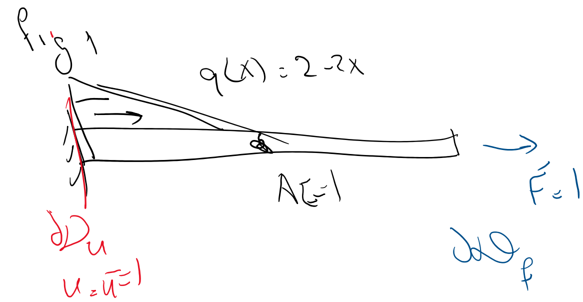
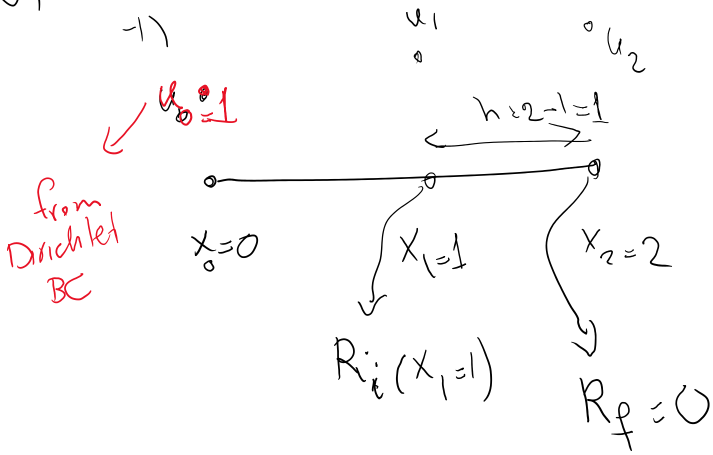
Solve our sample problem with FD

Solve our sample problem with FD




$n=2$ $(u_1=?,\ u_2=?)$

Similar to FD method we satisfy Diff eqn ($R_i=0$) or BC ($R_f=0$) at certain points (FD certain points are $x_i$'s)

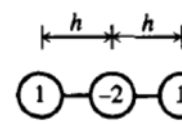



$$R_i = u'' + q(x)$$

$$R_f = 1 - u'$$

@ $x_1=1$ $R_i=0$ $\quad u_1'' \approx \dfrac{1\times u_2 - 2\times u_1 + 1\times u_0}{h^2}$, $q(x=1) = 2-2\times1 = 0$

$$\left.\frac{d^2w}{dx^2}\right|_i \quad \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$$



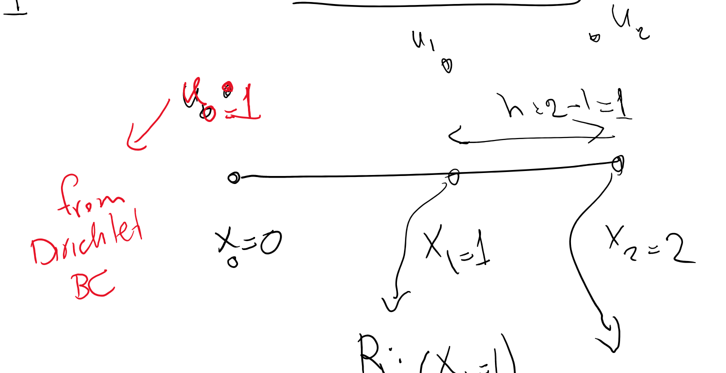

$R_i(x_1)=0 \implies u_2 - 2u_1 + u_0 = 0 \implies \boxed{-2u_1 + u_2 = -1}$ eq1

$u_0 = 1$

eq2 $R_f(x_2)=0 = 1 - u' = 0$

$$1 - \frac{u_2 - u_1}{\underset{1}{h}} = 0$$

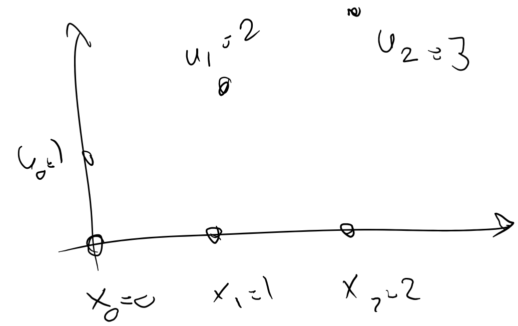
$\boxed{u_1 - u_2 = -1}$ eq 2



$$\boxed{u_1 - u_2 = -1} \quad \text{eq 2}$$

eq 1 & 2 : $Ku = F$ $\qquad K = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \quad F = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = K^{-1} F = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$




- Collocation and FD are similar in "satisfying" the DE and BC at certain points.
- Difference: FD no function discretization of the solution and we only have point values.

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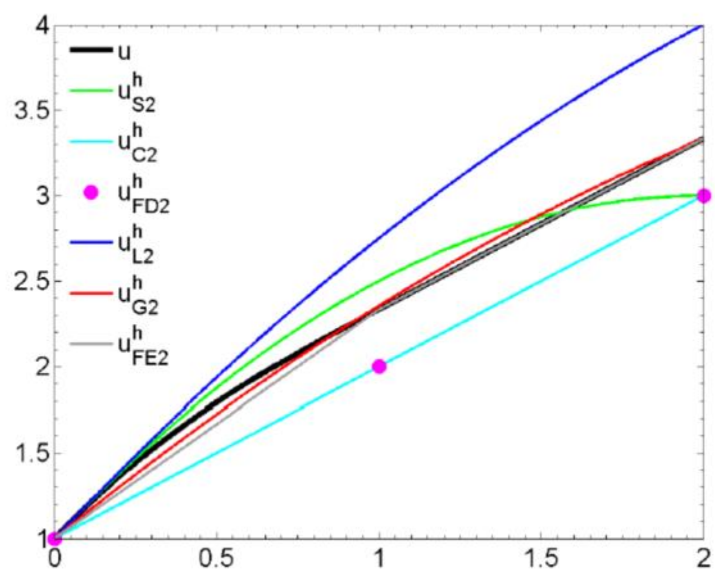
Bar example, $n = 2$, Comparison of solutions

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Galerkin method:

$$\boxed{\text{Galerkin} \quad w = \phi}$$

From last time we have

$$\int_0^L \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} R_i \, dx + \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}\Bigg|_{x=2} R_f = 0$$

$$\int_0^2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \left( \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q(x) \right) dx + \begin{bmatrix} w\phi_1 \\ w\phi_2 \end{bmatrix}_{x=2} \left( 1 - \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\phi_1 = x$ $\quad \phi_2 = x^2$

$$\Rightarrow \quad K = \int \begin{bmatrix} x \\ x^2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} - \begin{bmatrix} x \\ x^2 \end{bmatrix}_{x=2} \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -\frac{32}{3} \end{bmatrix}$$

$$F = -\int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} (2-2x)\, dx - \begin{bmatrix} x \\ x^2 \end{bmatrix}_{x=2} \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} -7/3 \\ -25/6 \end{bmatrix}$$

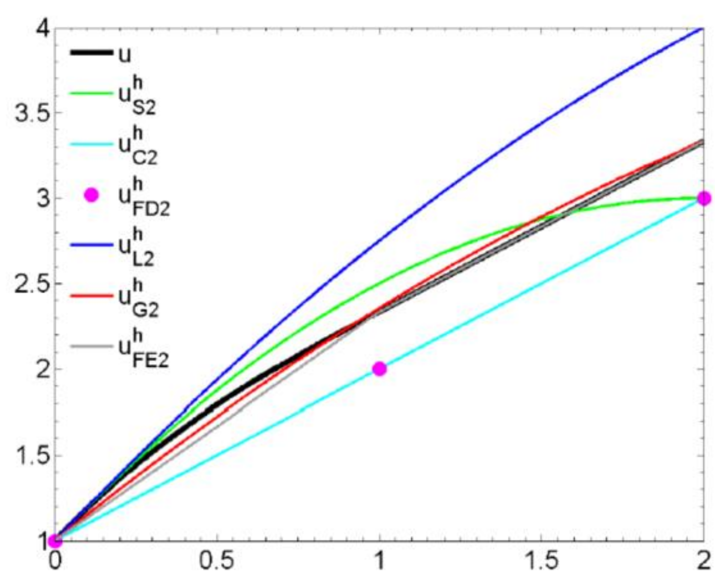
$$Ka = F \Rightarrow a = K^{-1} F = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix}$$

$$\boxed{u^h_{G2} = \phi_p + a_1 \phi_1 + a_2 \phi_2 = 1 + \frac{37}{24} x - \frac{3}{16} x^2}$$

Galerkin

## Bar example, $n = 2$, Comparison of solutions

n = 2

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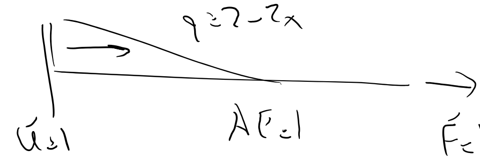
Can we solve Galerkin method with weak form (WK)?


$$\int_0^L w' EA u' \, dx = \int_0^L w q \, dx + w(L) F$$

$$\int_0^2 w' u' \, dx = \int_0^{\cancel{2}\,1} w (2-2x) \, dx + w(L) \times 1$$

fig 1




$$u = \phi_p + [\phi_1 \cdots \phi_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

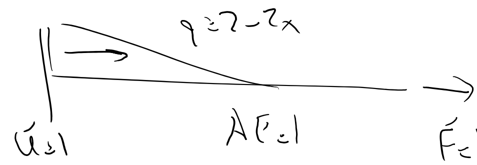
$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \underset{Glk}{=} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}$$

$$\boxed{\int_0^2 \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}' \underbrace{\left( \phi_p' + [\phi_1' \cdots \phi_n'] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)}_{u'} dx = \int_0^1 \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} (2-2x) \, dx + \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} (2)}$$

For Galerkin method with

n unknowns

fig 1



$n = 2 \quad \phi_1 = x \quad \phi_2 = x^2$ for Galerkin

$\phi_p = 1$

$$\underbrace{\left(\int_0^2 \begin{bmatrix} x \\ x^2 \end{bmatrix}' \begin{bmatrix} x & x^2 \end{bmatrix}' dx\right)}_{K} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix} (2-2x)\,dx + \begin{bmatrix} x \\ x^2 \end{bmatrix}\Bigg|_2}_{F}$$

$$\begin{bmatrix} 1 \\ 2x \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix} \qquad F = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix}$$

$$a = \begin{bmatrix} \frac{37}{24} \\ -3/16 \end{bmatrix}$$

$$\boxed{u^h_{(2)} = 1 + \frac{37}{24}x - \frac{3}{16}x^2}$$