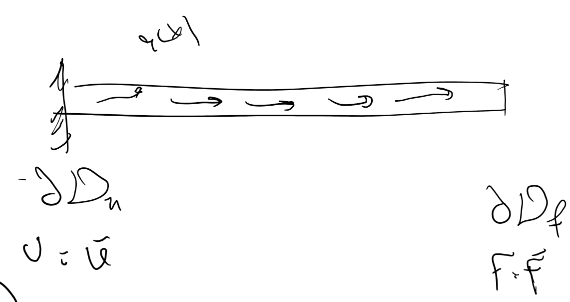
Energy methods:

Ritz method

The idea here is to compute the discrete form of potential energy and then minimize it:

fig 1)

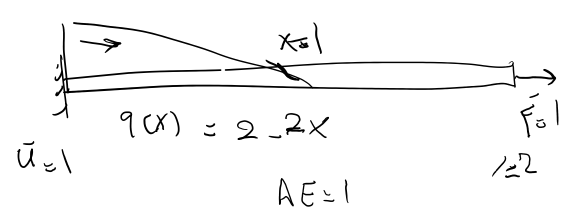


$$\Pi(u) = \underbrace{V(u)}_{\text{internal energy}} - \underbrace{W(u)}_{\text{external energy}}$$

$$= \underbrace{\int_0^L \frac{1}{2} EA u'^2 dx}_{V} - \underbrace{\left(\int_0^L u q \, dx + \bar{F} u(L)\right)}_{W} \quad (1)$$

Apply eqn (1) to our numerical example

fig 2)



$$\boxed{\Pi(u) = \int_0^2 \frac{1}{2} u'^2 dx - \int_0^1 u(2-2x)dx - u(2)} \quad (2)$$

FIRST: discretize u: $u \to u^h$

$$u^h(x) = \phi_p(x) + a_1 \phi_1(x) + a_2 \phi_2(x) \qquad \text{for } n = 2$$

our choices where $\phi_p = 1$ $\quad \phi_1(x) = x$, $\quad \phi_2(x) = x^2$

$$\boxed{u^h(x) = 1 + a_1 x + a_2 x^2} \quad (3)$$

Plug (3) in (2):

$$\Pi(u^h) = \int_0^2 \frac{1}{2}\left[\left(1 + a_1 x + a_2 x^2\right)'\right]^2 dx - \int_0^1 \left(1 + a_1 x + a_2 x^2\right)(2-2x)\,dx$$

$$- \left(1 + a_1 x + a_2 x^2\right)\Big|_{x=2}$$

$$-(1 + a_1 x + a_2 x^2)\Big|_{x=2}$$

$$\boxed{\Pi(a_1, a_2) = \underbrace{\int_0^2 \frac{1}{2}(a_1 + 2a_2 x)^2 dx}_{(i)} - \underbrace{\int_0^1 (1 + a_1 x + a_2 x^2)(2-2x)dx}_{ii} \underbrace{- (1 + 2a_1 + 4a_2)}_{iii}} \quad (4)$$

Second operation: Minimize potential energy (4)

$$\Pi(a_1, a_2) = \underbrace{\left(a_1^2 + 4a_1 a_2 + \frac{16}{3} a_2^2\right)}_{(i)} - \underbrace{\left(\frac{7}{3} a_1 + \frac{25}{6} a_2 + 2\right)}_{ii \,\&\, iii}$$

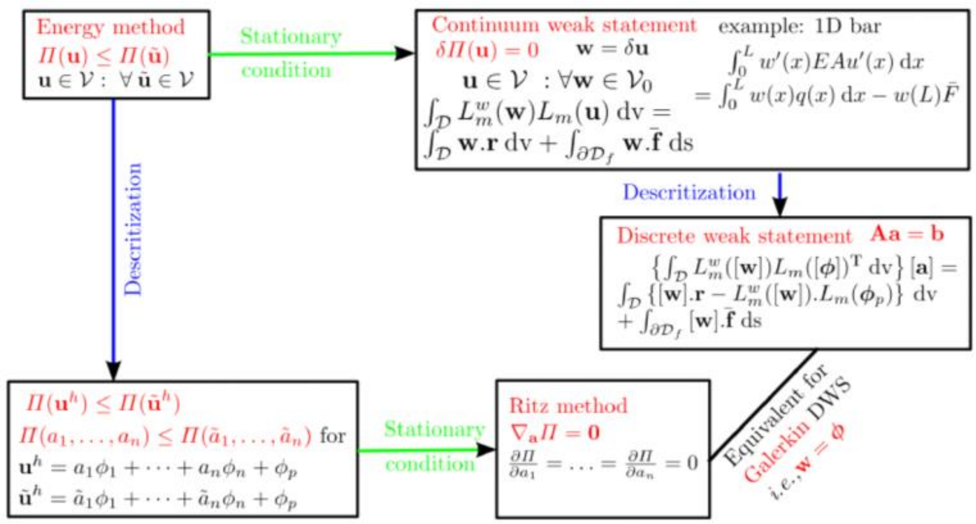
$$\nabla \Pi = \begin{bmatrix} \frac{\partial \Pi}{\partial a_1} \\ \frac{\partial \Pi}{\partial a_2} \end{bmatrix} = \begin{bmatrix} 2a_1 + 4a_2 - \frac{7}{3} \\ 4a_1 + \frac{32}{3} a_2 - \frac{25}{6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Ka = F \qquad K = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix} \qquad F = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix}$$

$$a = K^{-1} F = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix} \longrightarrow u^h(x) = \underset{\phi_p}{1} + \underset{a_1}{\frac{37}{24}} \underset{\phi_1}{x} - \underset{a_2}{\frac{3}{16}} \underset{\phi_2}{x^2} \quad (5)$$

This solution matches the solution of Galerkin method using WRS and Weak Statement from the last time!

Relation between Energy Method and Weak Statement



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Ritz method (first discretize energy, then minimize) is equivalent to weak statement (first minimize) applied to discrete Galerkin method.

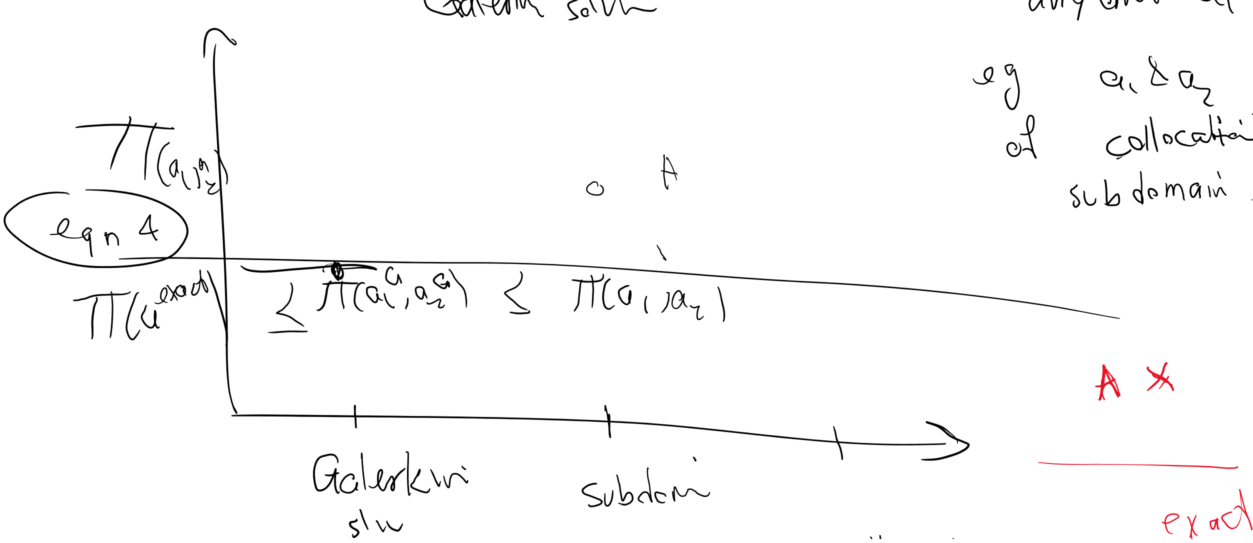
If fact, for Galerkin method, we not only obtain the minimum energy within the space of solution, the energy error is also minimum (we'll skip the proof)

$$u^h(x) = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x)$$

Galerkin method

$$\underbrace{\Pi(a_1, a_2)}_{\text{Galerkin solution}} \le \underbrace{\Pi(1 + a_1\phi_1 + a_2\phi_2)}_{\text{any other } a_1 \,\&\, a_2}$$

eg a_1 & a_2 of collocation subdomain etc



Galerkin s/n

$a_1 = 37/24$

$a_2 = -\frac{3}{16}$

Subdomain

$a_1 = 2$

$a_2 = -\frac{1}{2}$

Collocation

exact s/n

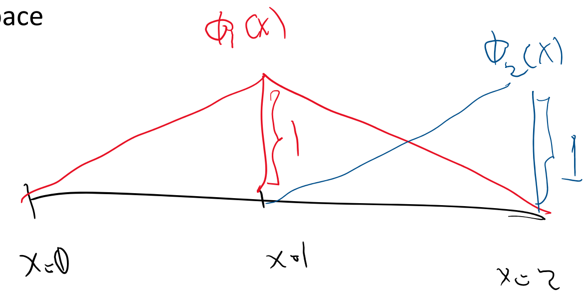
$$\Pi(\text{error}) = \Pi(u^h - u^{exact})$$

this is also the smallest for Galerkin method

Galerkin solution: Not only has the smallest energy within the space of discretization (e.g. for example uh = phi_p + a1 phi1 + a2 phi2, it also has the lowest energy error.

Let's still work with the Galerkin method, but with a different solution space

Let's use Finite Element basis functions



$$u^h(x) = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x) \qquad \phi_p = 1$$

Galerkin solution:

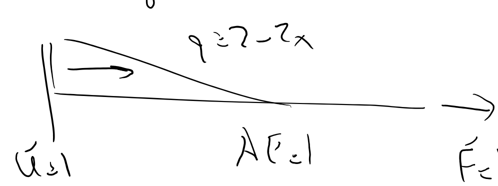
$w_1 = \phi_1 \qquad w_2 = \phi_2$, refer to last class

Equation (*)

$$\int_0^2 \begin{bmatrix}\phi_1 \\ \vdots \\ \phi_n\end{bmatrix}' \underbrace{\left(\phi_p' + [\phi_1' \cdots \phi_n'] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)}_{u'} dx = \int_0^1 \begin{bmatrix}\phi_1 \\ \vdots \\ \phi_n\end{bmatrix} (2-2x)\,dx + \begin{bmatrix}\phi_1 \\ \vdots \\ \phi_n\end{bmatrix} (2)$$

For Galerkin method with n unknowns

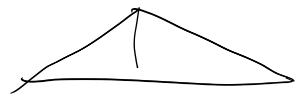
fig 1

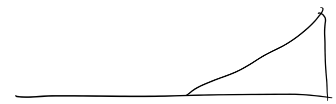


$u=1$ $A=1$ $F=1$

$n=2$

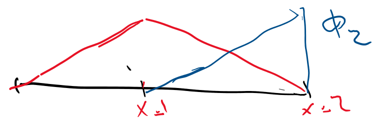
ϕ_1



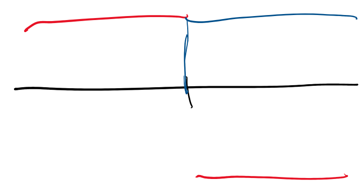


ϕ_2

ϕ's



ϕ's



$$\phi_1(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases} \qquad \phi_2(x) = \begin{cases} 0 & 0<x<1 \\ x-1 & 1<x<2 \end{cases}$$

$$\phi_1'(x) = \begin{cases} 1 & 0<x<1 \\ -1 & 1<x<2 \end{cases} \qquad \phi_2'(x) = \begin{cases} 0 & 0<x<1 \\ 1 & 1<x<2 \end{cases}$$

eqn $\star \longrightarrow$

$$K = \int_0^2 \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} [\phi_1' \quad \phi_2'] \, dx = \int_0^1 \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} [\phi_1' \quad \phi_2'] \, dx + \int_1^2 \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} [\phi_1' \quad \phi_2'] \, dx$$

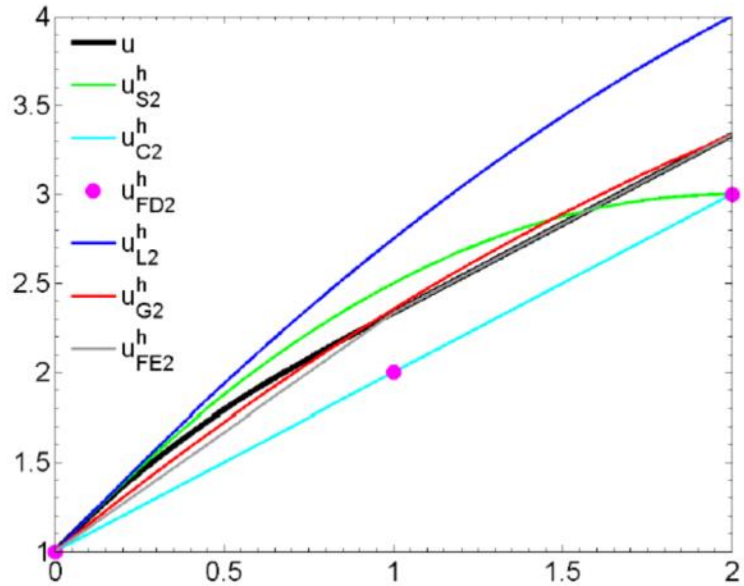
$$= \int_0^1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0] \, dx + \int_1^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1 \quad 1] \, dx = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F = \int_0^1 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} (2-2x) \, dx + \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{x=2} = \int_0^1 \begin{bmatrix} x \\ 0 \end{bmatrix} (2-2x) \, dx + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$a = K^{-1} F \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 7/3 \end{bmatrix} \Longrightarrow \cancel{u^h(x) = 1 + \frac{4}{3} x + \frac{7}{3} x^2}$$

$$\boxed{u^h_{FEM_2}(x) = 1 + \frac{4}{3} \phi_1(x) + \frac{7}{3} \phi_2(x)}$$

Bar example, $n = 2$, Comparison of solutions



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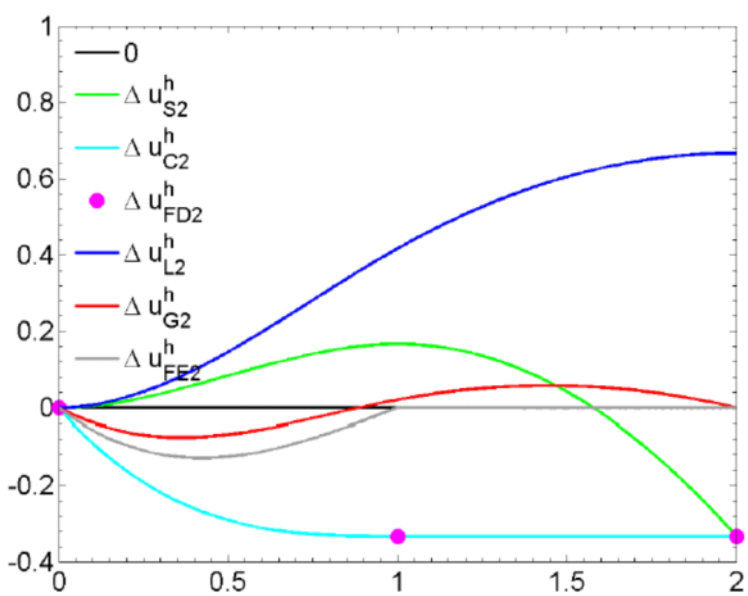
Bar example, $n = 2$, Comparison of solutions



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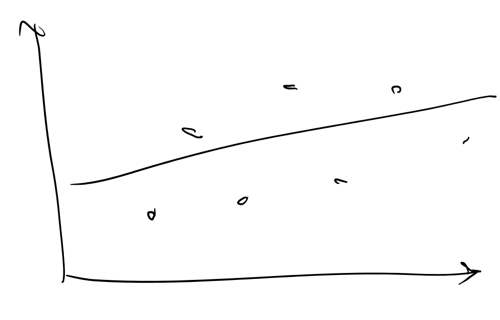
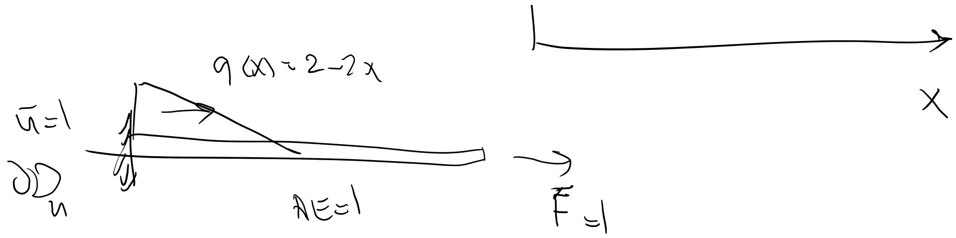
Least square method:
Least square is used in different settings, e.g. in linear regression:



$q(x) = 2-2x$



$$u^h(x) = 1 + a_1\phi_1 + a_2\phi_2 \quad , \text{ use } \quad \phi_1 = x, \quad \phi_2 = x^2$$

$$u^h(x) = 1 + a_1 x + a_2 x^2$$ this satisfies essential BC $R_u(x=0) = \bar{u} - u = 1 - 1 = 0$

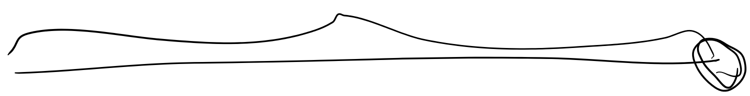
$$R_i(x) = u^{h''} + q(x) = 2a_2 + q(x) \qquad 0 < x < 2$$

note $u^{h'} = a_1 + 2a_2 x$

$u^{h''} = 2a_2$

$$R_f(x) = \bar{F} - F = \bar{F} - EAu^{h'}(x=L) = 1 - u^{h'}(x=2) = 1 - [a_1 + 4a_2)$$

$$R_i = 2a_2 + q(x)$$



$$R_f = 1 - (a_1 + 4a_2)$$

$$R^2 = \int_0^2 R_i^2(x)\,dx + R_f^2$$

$$= \int_0^2 \left(2a_2 + q(x)\right)^2 dx + \left(1 - (a_1 + 4a_2)\right)^2 =$$

$$\int_0^1 \left(2a_2 + \overbrace{2-2x}^{q}\right)^2 dx + \int_1^2 (2a_2)^2 dx + \left(1 - (a_1 + 4a_2)\right)^2$$

$$\boxed{R^2(a_1, a_2) = 1 + 20a_2^2 - 2a_1 - 8a_2 + 8a_1 a_2} \qquad (6)$$

Minimize R^2

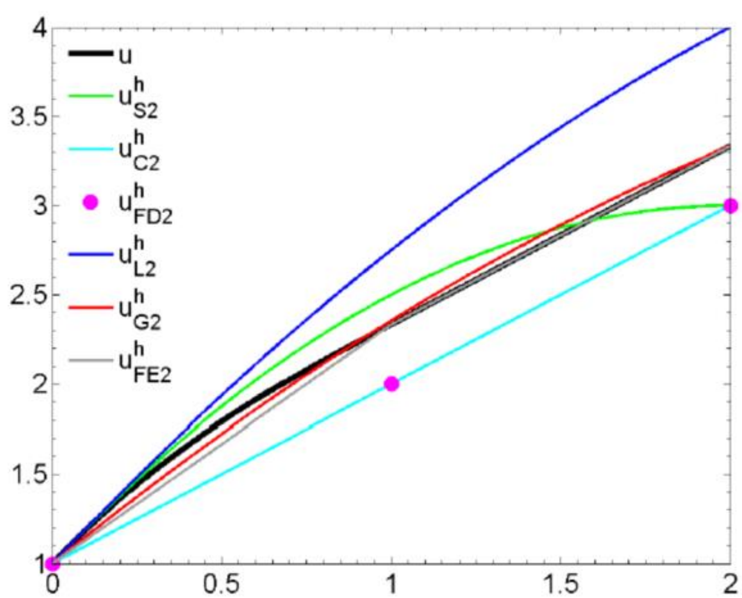
Minimize R^2

$$\nabla R^2 = \begin{bmatrix} \partial R^2 / \partial a_1 \\ \partial R^2 / \partial a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2a_1 + 8a_2 - 2 \\ 8a_1 + 4a_2 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow Ka = F \qquad K = \begin{bmatrix} 2 & 8 \\ 8 & 48 \end{bmatrix}, \quad F = \begin{bmatrix} 2 \\ -1/4 \end{bmatrix}$$

$$a = K^{-1}F = \begin{bmatrix} 2 \\ -1/4 \end{bmatrix} \longrightarrow \boxed{u^h_{L2}(x) = 1 + 2x - \frac{1}{4}x^2} \quad (7)$$

Bar example, $n = 2$, Comparison of solutions



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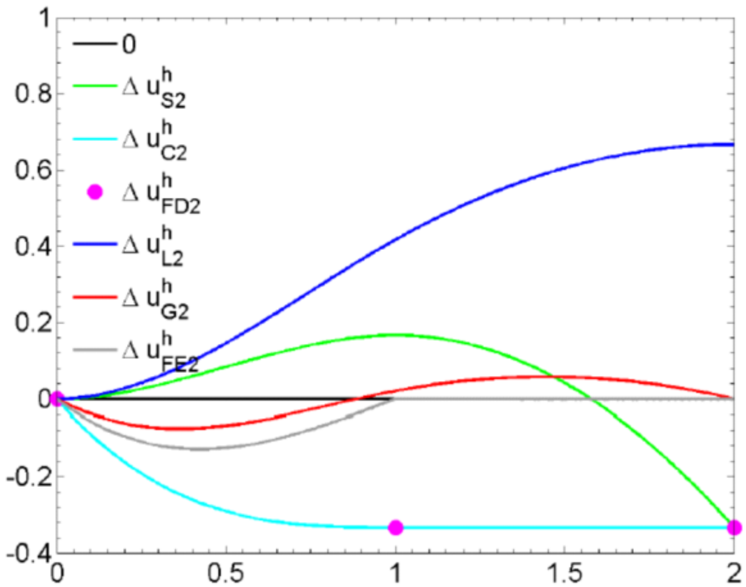
Bar example, $n = 2$, Comparison of solutions



Least square is a weighted residual with specific weights

$$R_i(u^h) = u^{h''} + q(x) = \underbrace{L_M(u^h)}_{L_M = (\;)''} - \underbrace{r}_{r = -q}$$

$$R_f = \underbrace{\bar{F}}_{1} - \underbrace{EA\,u^{h'}}_{EA=1} = 1 - \underbrace{u^{h'}}_{L_f(u^h) = (\;)'}$$

$$\boxed{\begin{aligned} w &= L_M(\phi) \\ w_f &= -L_f(\phi) \end{aligned}}$$

Least square

$$\int_0^2 w \left([\phi_1'' \quad \phi_2''] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q(x) \right) dx + w_f(x=2) = 0$$

$$w = L(\phi) = \begin{bmatrix} \phi_1'' \\ \phi_2'' \end{bmatrix}, \qquad w_f = -\begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}$$

$$K = \int_0^2 \begin{bmatrix} \phi_1'' \\ \phi_2'' \end{bmatrix} [\phi_1'' \quad \phi_2''] \, dx + \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} [\phi_1' \quad \phi_2'] \Big|_{x=L}$$

$$= \int_0^2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} [0 \quad 2] \, dx + \begin{bmatrix} 1 \\ 2x \end{bmatrix} [1 \quad 2x] \Big|_2 = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 4 & 24 \end{bmatrix}$$

$$F = -\int_0^1 \begin{bmatrix} \phi_1'' \\ \phi_2'' \end{bmatrix} (2-2x) - \begin{bmatrix} -\phi_1' \\ -\phi_2' \end{bmatrix} \Big|_1 = -\int_0^1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} [2-2x] \, dx - \begin{bmatrix} -1 \\ -2x \end{bmatrix} \Big|_2$$

$$= -\begin{bmatrix} 0 \\ 4(x - \frac{x^2}{2}) \end{bmatrix} \Big|_0^1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = -\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= -\begin{bmatrix} \\ 4(x - \frac{x^2}{2}) \end{bmatrix}\Bigg|_0^1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = -\begin{bmatrix} \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \quad a = K^{-1} F = \begin{bmatrix} 2 \\ -\frac{1}{4} \end{bmatrix}$$

matches the solution above