2023/10/16

Monday, October 16, 2023 11:15 AM

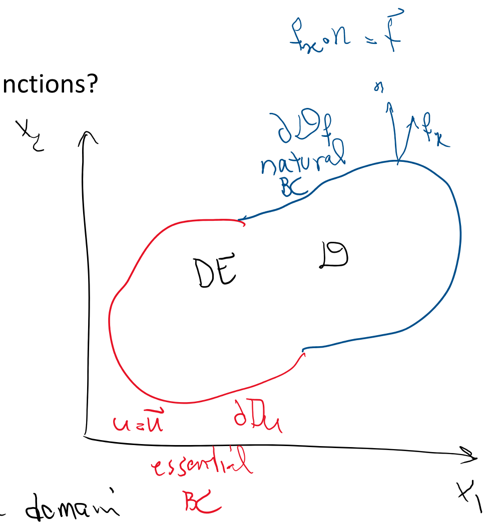
Why Least Square method is a Weighted Residual method with specific weight functions?

$R_i(u) = \mathcal{L}_M(u) - r \rightarrow$ source term

error in satisfying the differential equation

$R_f(u) = \bar{f} - f_x \cdot n = \bar{f} - \mathcal{L}_f(u)$

(1)



$\mathcal{L}_M$ & $\mathcal{L}_f$ are differential operators for inside the domain & natural boundary

Examples

bar problem

$R_i = (EAu')' + q$

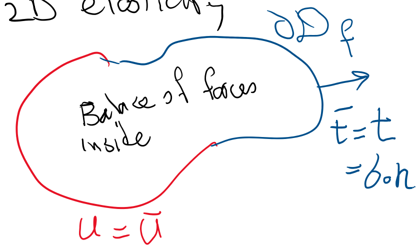
$= \mathcal{L}_M(u) - r$

$\mathcal{L}_M = (EA(\ )')'$, $r = -q$

$R_f = \bar{F} - F|_L = \bar{F} - (EAu')|_{x=L}$

$\mathcal{L}_f = EA(\ )'$

2D elasticity



$R_i(u) = \nabla \cdot \sigma + \rho b$

$= \nabla \cdot (C\varepsilon) + \rho b$

$= \underbrace{\nabla \cdot C\left(\frac{\nabla u + \nabla u^t}{2}\right)}\ + \rho b$

$= \mathcal{L}_M(u) - r$

$r = -\rho b$

$\mathcal{L}_M = \nabla \cdot C\left(\frac{\nabla + \nabla^t}{2}\right)$

$R_f = \bar{t} - t = \bar{t} - \sigma \cdot n$

$= \bar{t} - (C\varepsilon) \cdot n = \bar{t} - C\left(\frac{\nabla u + \nabla u^T}{2}\right) \cdot n$

$\mathcal{L}_f = C\left(\frac{\nabla + \nabla^T}{2}\right)$

2D heat conduction

$R_i(T) = \nabla \cdot q - Q$

$= \nabla \cdot -k\nabla T - Q$

$\mathcal{L}_M = \nabla \cdot -k\nabla$

$R_f = \bar{q} - q \cdot n$

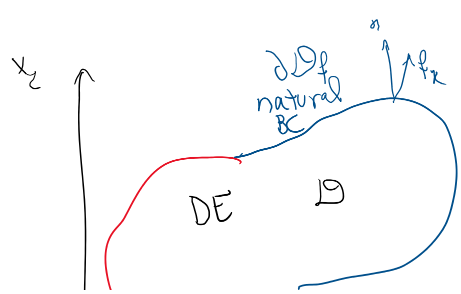
$= \bar{q} + (k\nabla T) \cdot n$

$\mathcal{L}_f = -k\nabla(\ )$

$R_i(u) = \mathcal{L}_M(u) - r \rightarrow$ source term

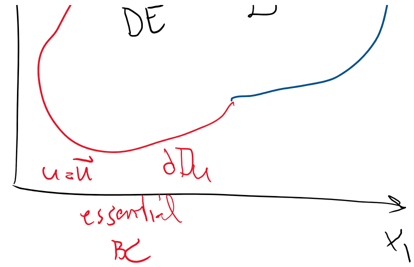
error in satisfying the differential equation

(1)



in satisfying the differential equation

$$R_f(u) = \bar{f} - f_{x \text{ on}} = \bar{f} - \mathcal{L}_f(u) \quad (1)$$



$$R^2 = \int_D R_i^2(u)\, dV + \int_{\partial D_f} R_f^2(u)\, dS \quad (2)$$

$$u = \phi_p + \sum_{i=1}^{n} a_i \phi_i \quad \text{discretization} \quad (3)$$

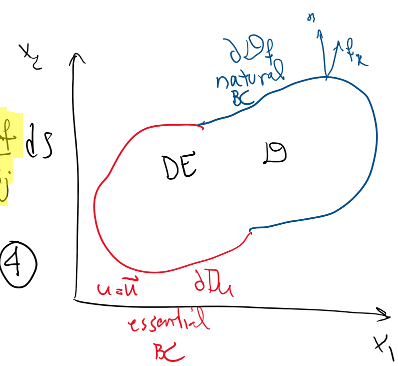
we have n unknowns $a_1, \dots, a_n$

$R^2(a_1, \dots, a_n)$, we want to minimize $R^2(a_1, \dots, a_n)$

$$\nabla R^2 = \begin{bmatrix} \frac{\partial R^2}{\partial a_1} \\ | \\ | \\ \frac{\partial R^2}{\partial a_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

$$\frac{\partial R^2}{\partial a_j} = \frac{\partial}{\partial a_j} \int_D \underbrace{R_i^2}_{\text{inside}}(a_1, \dots, a_n)\, dV + \int_{\partial D_f} R_f^2(a_1, \dots, a_n)\, dS$$

$$= \int_D \frac{\partial R_i^2}{\partial a_j} dV + \int_{\partial D_f} \frac{\partial R_f^2}{\partial a_j} dS = \int_D 2R_i \frac{\partial R_i}{\partial a_j} dV + \int_{\partial D_f} 2R_f \frac{\partial R_f}{\partial a_j} dS \quad (4)$$



we need to plug in values of R_i, $\frac{\partial R_i}{\partial a_j}$, R_f & $\frac{\partial R_f}{\partial a_j}$ in (4)

$$\frac{d\, y^2(x)}{dx} = 2y(x)\, y'(x)$$

(3) : $u = \phi_p + \sum_k a_k \phi_k$

$$\Rightarrow \quad R_i(a_1, \dots a_n) = R_i(u) = \underbrace{\mathcal{L}_M\left(\phi_p + \sum_{k=1}^{n} a_k \phi_k\right)} - r = \underbrace{\mathcal{L}_M(\phi_p) + \sum a_k \mathcal{L}_M(\phi_k)} - r$$

Example 1D for $\mathcal{L}_M = (EA(\,)')'$

$$\left(EA\left(\phi_p + \sum a_k \phi_k\right)'\right)' = (EA\phi_p')' + \left(EA(\phi_k)'\right)'$$

(because $\mathcal{L}_M$ is linear we can open it $\mathcal{L}_M(a_1 f_1 + a_2 f_2) = a_1 \mathcal{L}_M(f_1) + a_2 \mathcal{L}_M(f_2)$)

(because $\mathcal{L}_M$ is linear we can open it $\mathcal{L}_M(a_1 f_1 + a_2 f_2) = a_1 \mathcal{L}_M(f_1) + a_2 \mathcal{L}_M(f_2)$)

$\rightarrow R_i = \mathcal{L}_M(\phi_p) + a_1 \mathcal{L}_M(\phi_1) + \cdots + a_n \mathcal{L}_M(\phi_n) - r$ 5 i)

$$\frac{\partial R_i}{\partial a_j} = \mathcal{L}_M(\phi_j)$$ 5 ii

Similarly $R_f(u) = R_f(a_1, \dots, a_n) = \bar{F} - \mathcal{L}_f\left(\phi_p + \sum_k a_k \phi_k\right) \cdot n$

$= \bar{F} - \mathcal{L}_f(\phi_p) \cdot n - \sum_k a_k \mathcal{L}_f(\phi_k) \cdot n$ 5 iii) $\mathcal{L}_f$ is linear

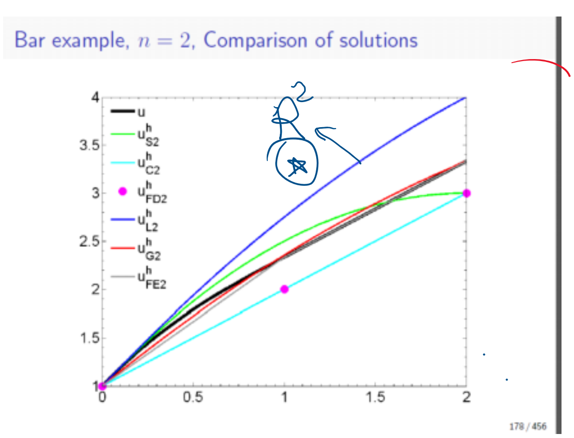
$$\frac{\partial R_f(u)}{\partial a_k} = -\mathcal{L}_f(\phi_k)$$ 5 iv)

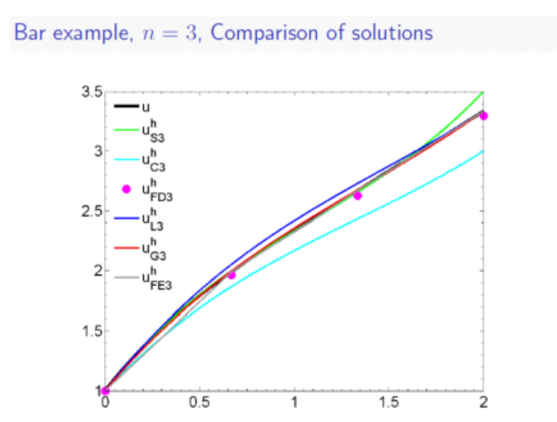
4 $$\int_D \frac{\partial R_i^2}{\partial a_j} dV + \int_{\partial D_f} \frac{\partial R_f^2}{\partial a_j} dS = \int_D 2R_i \underbrace{\frac{\partial R_i}{\partial a_j}}_{5ii} dV + \int_{\partial D_f} 2R_f \underbrace{\frac{\partial R_f}{\partial a_j}}_{(w_f)_j = -\mathcal{L}_f(\phi_j)} dS$$

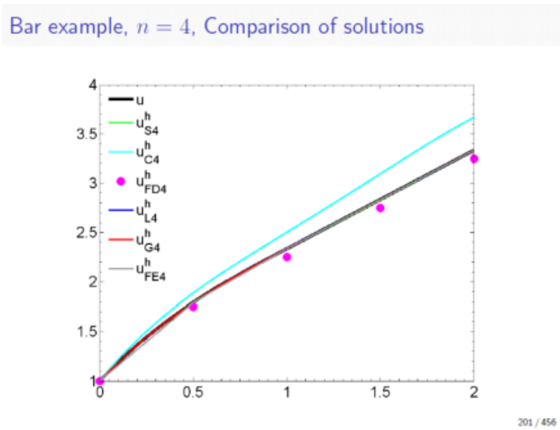
$w_j = \mathcal{L}_M(\phi_j)$

R2:
+: It can be added to a formulation to satisfy certain condition weakly, e.g. interpenetration.
+: Always get symmetric matrices

-: Minimizing the residual does not mean we have the best solution (uh closest to the exact solution)
-: Both solution and the weights get very high derivative orders (M)





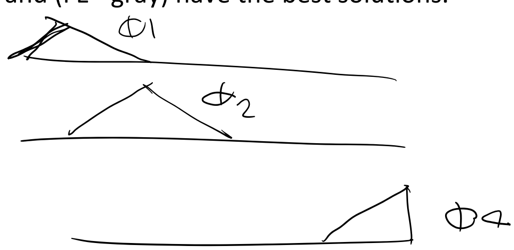


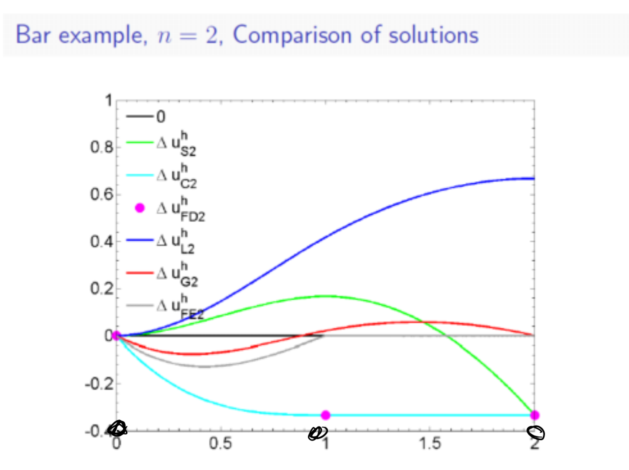
★ R^2 is not the closest to the exact solution $(u^h - u^{exact})$ because we minimize the error in

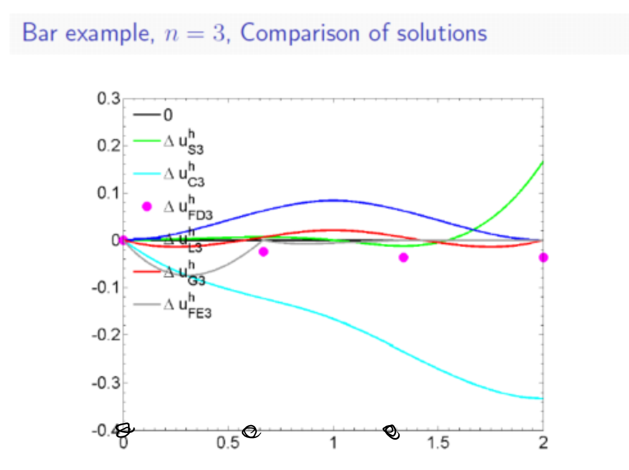
differential equation not w.r.t. the exact solution.

We see that the Galerkin methods (Spectral - red) and (FE - gray) have the best solutions.

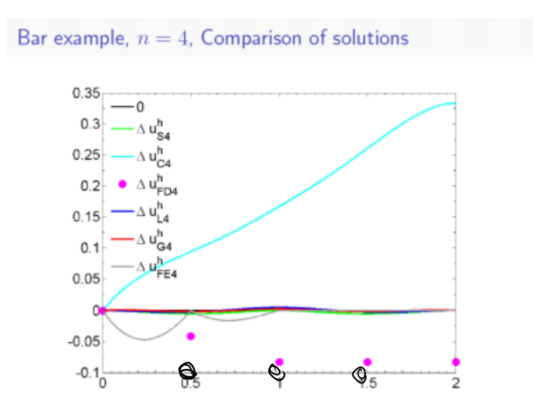
x, x^2, x^3







For FEM the errors are equal to zero ☺



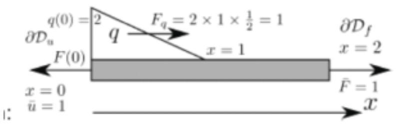
Unfortunately, this property is only for 1D and in 2D and 3D we don't have this very nice property

Why we have larger errors on the left side?

- The exact solution can be summarized as,

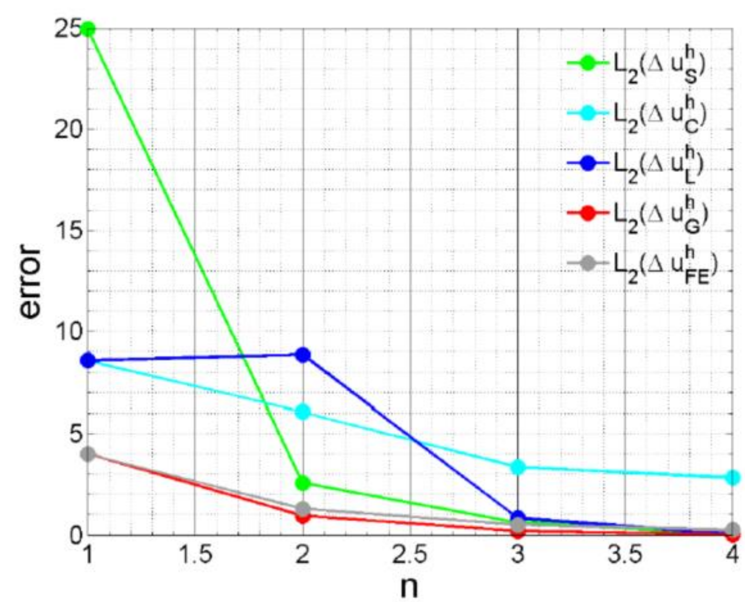
$$u(x) = \begin{cases} \frac{x^3}{3} - x^2 + 2x + 1 & 0 \le x \le 1 \\ x + \frac{4}{3} & 1 < x \le 2 \end{cases} \tag{179}$$

Because of the source term we have a higher order solution there and harder to capture that numerically



Error convergence:

Bar example, Error Convergence



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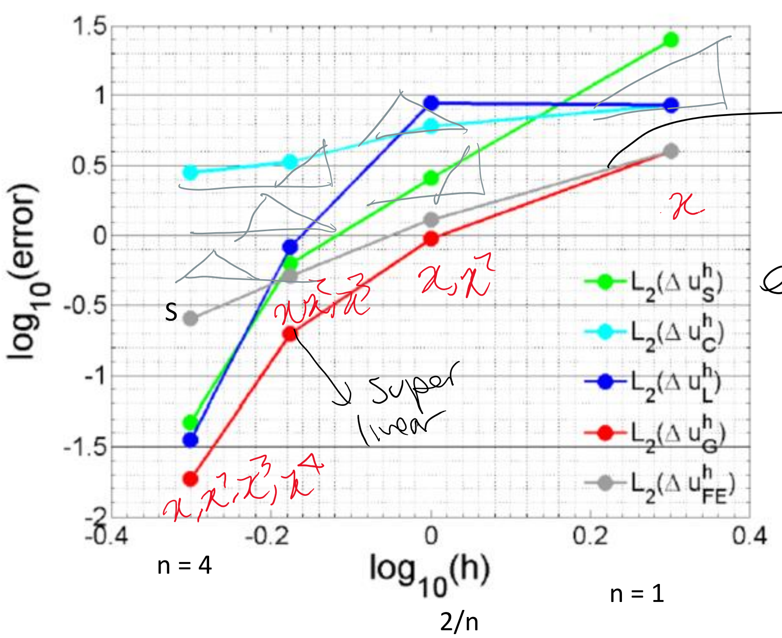
Bar example, Error Convergence



linear

$\left(\log(\text{error}) = C + \alpha \log(h)\right)$

e

$\text{error} = (e^C)\, h^{\alpha}$

$\text{error} = C\, h^{\overbrace{f(p)}} \approx 2 = p+1$

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n = 4

$\log_{10}(h)$

n = 1

2/n

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