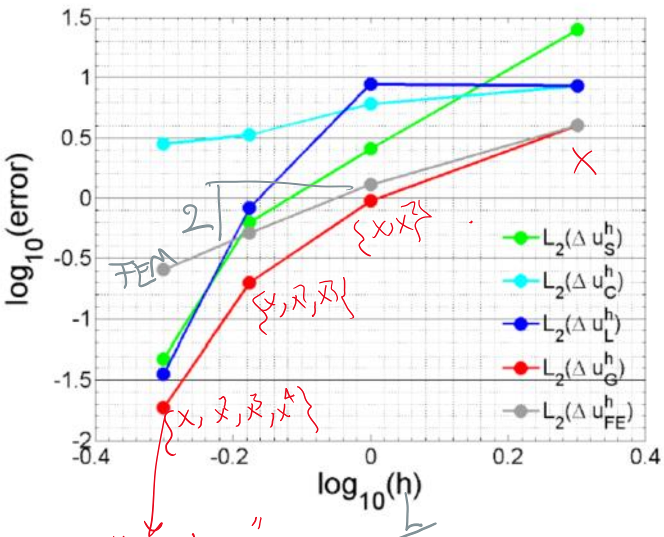
Bar example, Error Convergence

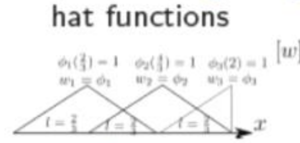
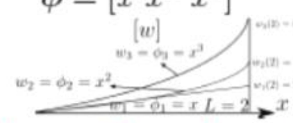
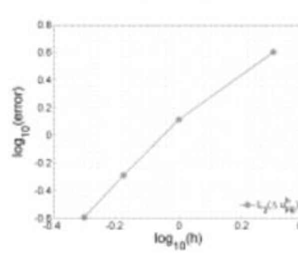
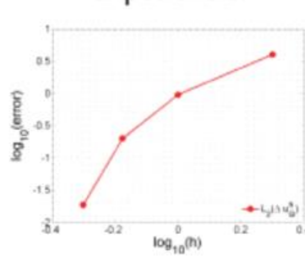


"everytime"
the slope of convergence increases by adding one more higher order term

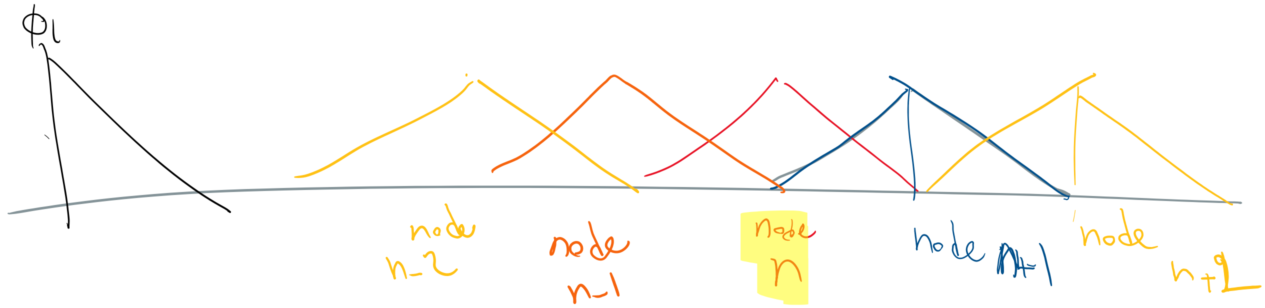
FEM
slope $\alpha = p+1$
$\log(\text{error}) = C + \alpha \log h$

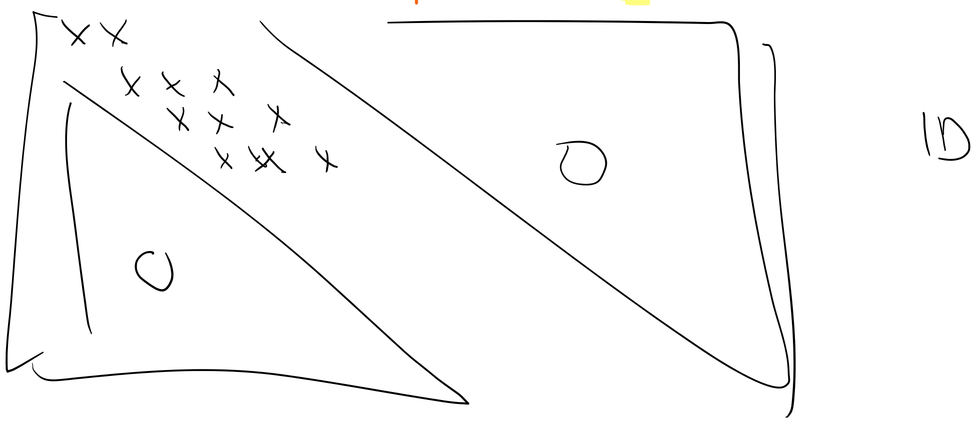
$$\boxed{\text{error} = C h^{\alpha} \quad \text{here } \alpha = p+1}$$

Observations: FE versus spectral methods

Feature	Finite Element	Spectral Methods
Trial Functions Example	Local / Finite Regularity hat functions 	Globally Smooth $\phi = [x\ x^2\ x^3]$ 
Matrix $\mathbf{K}$ Example	Sparse $\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$	Full (diagonal for orthogonal ϕ) $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & \frac{32}{3} & 24 & \frac{256}{5} \\ 8 & 24 & \frac{288}{5} & 128 \\ 16 & \frac{256}{5} & 128 & \frac{2048}{7} \end{bmatrix}$
order of accuracy of u^h (p)	fixed ($e.g.$, $p = 1$)	$\nearrow$ vs. n ($e.g.$, $p = n$)
Convergence Example	Linear: $e = Ch^{\alpha}$ $\alpha = 2$ 	higher than linear exponential 
Geometry	Very general geometries	simple ($e.g.$, rectangular) in practice to get diagonal $\mathbf{K}$

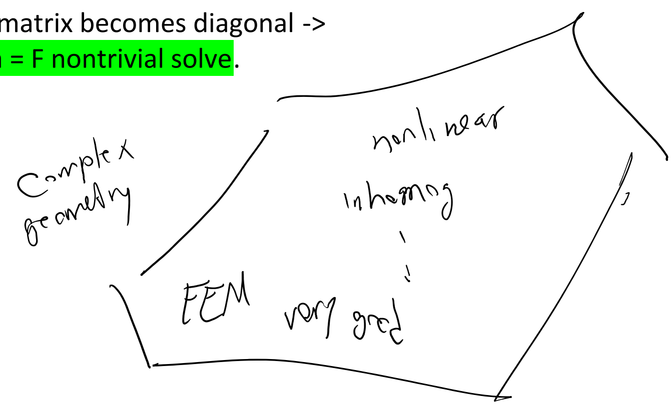
ϕ_1





If the problem is:
- On simple geometries (rectangle, circle, sphere, etc)
- Linear PDE
- Homogeneous (material) properties

We can use basis functions that for spectral method the K matrix becomes diagonal ->
We would have super exponential convergence without Ka = F nontrivial solve.



FYI:

Diagonal matrix for spectral methods

- The global nature of trial functions ϕ in spectral method results in full $\mathbf{K}$ matrices that are expensive to solve.
- To circumvent this problem we employ trial functions that make $\mathbf{K}$ diagonal.
- In weak statement $K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_{\mathcal{D}} L_m^w(\phi_i) L_m(\phi_j)\,\mathrm{dv}$.
- If the problem is self-adjoint $\mathcal{A}(.,.)$ is an inner product and we can construct an orthogonal trial function basis ϕ_i for example using Gram Schmidt method.
- Given the particular form of $\mathcal{A}$ (from L_m^w and L_m) and domain of integration $\mathcal{D}$ ($[0\ 1]$, $[-1\ 1]$, semi-infinite, infinite, *etc.*) we employ various trigonometric and orthogonal polynomial spaces. Some examples are:
    - $\phi_k(x) = e^{ikx}$ Fourier spectral method.

$[-1\ 1]$, semi-infinite, infinite, *etc.*) we employ various trigonometric and orthogonal polynomial spaces. Some examples are:

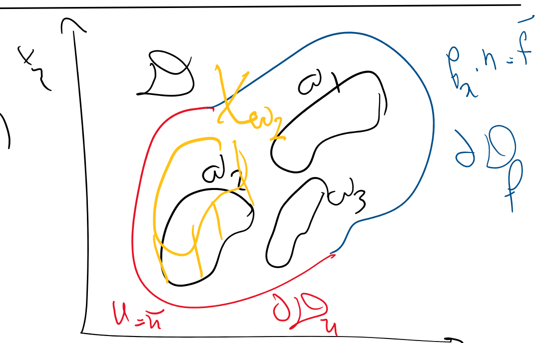
- $\phi_k(x) = e^{ikx}$ Fourier spectral method.
- $\phi_k(x) = T_k(x)$ Chebyshev spectral method.
- $\phi_k(x) = L_k(x)$ or $P_k(x)$ Legendre spectral method. → HW extra credit
- $\phi_k(x) = \mathcal{L}_k(x)$ Laguerre spectral method.
- $\phi_k(x) = H_k(x)$ Hermite spectral method.

where $T_k(x)$, $L_k(x)(P_k(x))$, $\mathcal{L}_k(x)$, and $H_k(x)$ are the Chebyshev, Legendre, Laguerre, and Hermite polynomials of degree k, respectively.

- The orthogonal property of these functions is for simple geometries. That is why spectral methods are more popular for simple geometries where we can take advantage of their exponential convergence property while keeping computational costs low by using orthogonal trial functions.

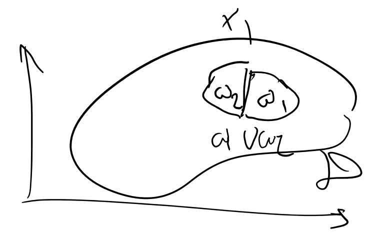
Imagine we have discretized the solution as follows (n unknowns)

$$u^h(x_1,x_2) = \phi_p(x_1,x_2) + \sum_{i=1}^{n} a_i \phi_i(x_1,x_2)$$



— Balance law

$$\underline{\forall\, \omega \subseteq \mathcal{D}} \quad \int_{\partial\omega} f_x \cdot n\, ds = \int_{\omega} r\, dV$$



to solve $a_1, \ldots, a_n$ we need n ω's

this is equivalent to subdomain with

$$w = \chi_{\omega_1}, \ldots \quad \omega_n = \chi_{\omega_n}$$

$$\boxed{\text{Subdomain : satisfying balance law on } n \text{ sets}}$$

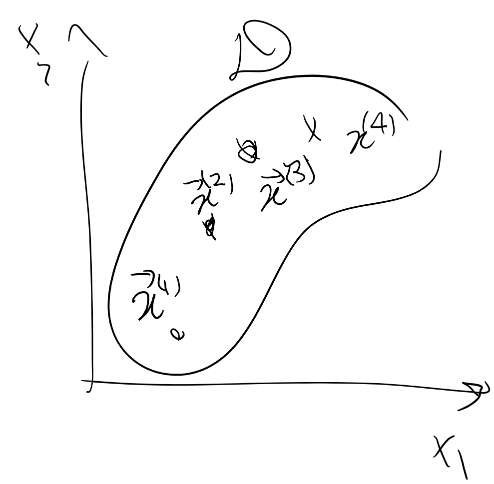
Balance law

⇓

Divergence $\forall\omega \int_{\omega} (\nabla \cdot f_x - r)\, dV = 0$

⇓ ⇓

localization $\nabla \cdot f_x - r = 0$ PDE



n unknowns $a_1, \ldots, a_n$

n unknowns $a_1, \ldots, a_n$

x_1

Satisfy PDE and Natural BC @ $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ locations $\Rightarrow$

$a_1, \ldots, a_n$ obtained $\Longleftrightarrow$ Collocation

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$\Downarrow$

WRS

$$\int_{D} \omega (\underbrace{\nabla \cdot f_x - r}_{R_i}) dV + \int_{\partial D_f} (\bar{f} - f_x \cdot n) ds = 0$$

IBP
Gauss $\Downarrow$

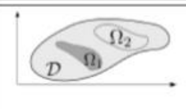
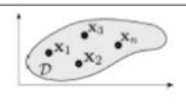
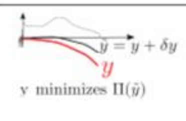
weak statement

$$\int_{D} L_m(\omega) \underbrace{D}_{\text{material property}} L_m(u)\, dV = \int_{D} \omega\, r\, dV + \int_{\partial D_f} \omega \bar{F}\, ds$$

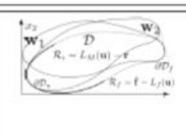
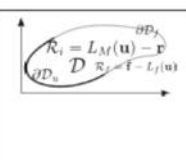
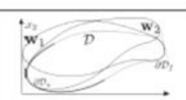
Galerkin $W = \phi \Longleftrightarrow$ Best job done in terms of energy

Equations for discrete systems

- After discretizing the solution with n unknowns we need n equations to solve the discrete problem.
- The n equations are derived from different interpretation of the equations we derived so far. All these equations have a "for all" condition. In discrete systems the "for all" condition is replaced by a finite set.

Approach	Equation	Figure	Discretization	Discretization method
Balance Law (20)	$\forall \Omega \subset \mathcal{D}: \int_{\partial\Omega} (\mathbf{f}.\mathbf{n}) \mathrm{ds} - \int_{\Omega} \mathbf{r}\, \mathrm{dv} = \mathbf{0}$		Change $\forall \Omega$ to $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$	Similar to subdomain method in WRM
Strong Form (23)	$\forall \mathbf{x} \in \mathcal{D}: \nabla.\mathbf{f} - \mathbf{r} = \mathbf{0}$		Change $\forall \mathbf{x}$ to $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$	Collocation method in WRM. Also FD & FV.
Energy Method (80)	$\forall \tilde{y} \in \mathcal{V}:\ \Pi(y) \le \Pi(\tilde{y})$		$\forall\{\tilde{a}_1, \ldots, \tilde{a}_n\}$: $\Pi(a_1, \ldots, a_n) \le \Pi(\tilde{a}_1, \ldots, \tilde{a}_n) \Rightarrow \frac{\partial \Pi}{\partial a_1} = \cdots = \frac{\partial \Pi}{\partial a_n} = 0$	Ritz Energy Method. Also yields Weak Form.

Equations for discrete systems

Approach	Equation	Figure	Discretization	Discretization method
Weighted Residual Method (45)	$\forall \mathbf{w} \in \mathcal{W}$: $\int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i\ \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f\ \mathrm{ds} = 0$		Change $\forall \mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$	Weighted Residual Method (WRM)
Least Square (51)	$R^2 = \int_{\mathcal{D}} \mathcal{R}_i^2\ \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2\ \mathrm{ds} = 0$		Change $R^2 = 0$ to $\forall\{\tilde{a}_1, \ldots, \tilde{a}_n\}$: $R^2(a_1, \ldots, a_n) \le R^2(\tilde{a}_1, \ldots, \tilde{a}_n) \Rightarrow \frac{\partial R^2}{\partial a_1} = \cdots = \frac{\partial R^2}{\partial a_n} = 0$	Least Square method, a WRM for linear L_M (& L_f).
Weak Form (74)	$\forall \mathbf{w} \in \mathcal{W}$ $\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u})\ \mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r} \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}} \mathrm{ds}$		Change $\forall \mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$	Weak Formulation

Appendix: Function spaces (optional)

Function space

Appendix: Function spaces (optional)

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Function space

Galerkin Weak Statement Function spaces

1. We first reduce the highest derivative order $M = 2m$ in the strong form (and weighted residual statement) to m in the weak statement.
2. Next, we observe that the functions should only be in $H^m(\mathcal{D})$. We observed that $H^m(\mathcal{D}) \subset C^{m-1}(\mathcal{D})$. In practice, the finite element trial functions that are in $C^{m-1}(\mathcal{D})$ are also $H^m(\mathcal{D})$.

Conventional (continuus) finite element methods:

Strong Form order $M = \underline{2m}$ $\Rightarrow$

Trial functions are $\underline{C^{m-1}}$

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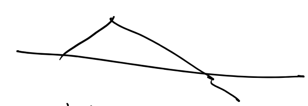
$u'' + q = 0 \quad M = 2$
$m = 1$

$C^{1-1} = C^0$ functions



$EIy^{(4)} + q = 0 \quad M = 4$
$m = \frac{M}{2} = 2$

$C^{2-1} = C^1$ functions are needed

1D elements

Element types:

1. 1D solid bar element.
2. Truss element.

Concepts:

1. Global (weighted residual) vs local (element level) perspectives.
2. Stiffness matrix.
3. Forces: 1.Source term; 2.Natural BC; 3.Essential BC, 4.Nodal.
4. Nodes, elements, shape function, dof.
5. Nodes with more than one dof (truss).
6. Element local coordinate system ξ (bar).
7. Rotation of element local coordinate system (truss).
8. Full stiffness $\mathbf{K}$ (free + prescribed dofs) vs (free only dofs) $\mathbf{K}_{ff}$.
9. High order differential equations (*e.g.*, C^1 beam elements).
10. Multiphysics coupling (beams: axial, bending, & torsional coupling).

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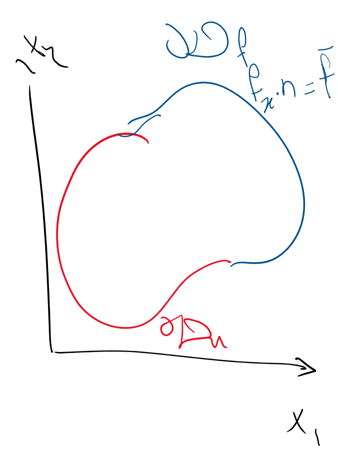
From this point, we'll only work with the weak statement and employ the Galerkin method

$w = \phi$

General weak statement for a self adjoint problem

$$\int_{\mathcal{D}} L_m(w)\, D\, L_m(u)\, dV = \int_{\mathcal{D}} w \cdot r\, dV + \int_{\partial\mathcal{D}_f} w \bar{F}\, ds$$



L_m

D material section property

1D bar $\int w' \underline{EA} u' \, dx = \int_{\mathcal{D}} w \underset{\sim}{q}^{r} \, dV$

$L_m = (\;)' \quad D = EA$

1D beam $\int_{\mathcal{D}} w'' EI\, y'' \, dx = \int_{\mathcal{D}} w \underset{\sim}{q}^{r} \, dV$

$L_m = (\;)'' \quad D = EI$

Heat conduction in 2D

$$\int_{\mathcal{D}} (\nabla w)\, k\, \nabla T\, dV = \int_{\mathcal{D}} w \underset{\sim}{Q}^{r} \, dV$$

$$\int_{\mathcal{D}} (\nabla w) k \nabla T \, dV = \int_{\mathcal{D}} w Q \, dV$$

$\mathcal{L}_m = \nabla$, $D = k$

2D elasticity

$$\int \varepsilon(w) \, \mathbb{C} \, \varepsilon(u) \, dV = \int w \overbrace{(\rho b)}^{r} dV$$

$$\mathcal{L}_m = \frac{\nabla + \nabla^t}{2} \quad \left(\varepsilon(u) = \frac{\nabla u + (\nabla u)^t}{2} \right)$$

$D = \ddot{\mathbb{C}}$ elasticity tensor

For all these linear self-adjoint problems, we want to obtain a formula for the stiffness matrix and various forms of force vectors

$$u^h(\vec{x}) = \phi_p(\vec{x}) + \sum_{i=1}^{n} a_i \phi_i(\vec{x}) = \phi_p + \underbrace{[\phi_1 \cdots \phi_n]}_{\phi} \underbrace{\begin{bmatrix} a_1 \\ | \\ | \\ a_n \end{bmatrix}}_{a}$$

$$= \phi_p + \phi a$$

n: # unknowns

ϕ_p satisfies essential BC $\forall x \in \partial \mathcal{D}_u \quad \phi_p(x) = \bar{u}(x)$

ϕ_i satisfy homog " " $\forall x \in \partial \mathcal{D}_u \quad \phi_i(x) = 0$

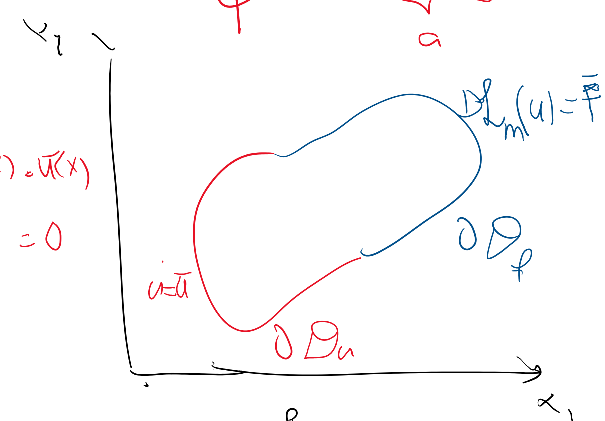


fig 1

$$\underbrace{\int_{\mathcal{D}} \mathcal{L}_m(w) D \mathcal{L}_m(u) dV}_{} = \int_{\mathcal{D}} w \, r \, dV + \int_{\partial \mathcal{D}_f} w F \, ds$$

$$\boxed{\begin{array}{l} \mathcal{A}(w,u) = (w,r) + (w,\bar{F})_N \\ \text{where } \mathcal{A}(w,u) = \int_{\mathcal{D}} \mathcal{L}_m(w)\, D\, \mathcal{L}_m(u)\, dV \\ (w,r) = \int_{\mathcal{D}} w\, r\, dV \qquad (f,g) = \int_{\mathcal{D}} f\, g\, dV \\ (w,\bar{F})_N = \int_{\partial\mathcal{D}_f} w\, \bar{F}\, ds \qquad (f,g)_N = \int_{\partial\mathcal{D}_f} f \cdot g\, ds \end{array}} \quad (1)$$

A is called the bilinear form

$$\boxed{\begin{array}{ll} \text{i)} & \mathcal{A}(w_1+w_2,u) = \mathcal{A}(w_1,u) + \mathcal{A}(w_2,u) \\ \text{ii)} & \mathcal{A}(w,u_1+u_2) = \mathcal{A}(w,u_1) + \mathcal{A}(w,u_2) \end{array}} \quad (2)$$

i) $\mathcal{A}(w_1+w_2,u) = \int_{\mathcal{D}} \mathcal{L}_m(w_1+w_2)\, D\, \mathcal{L}_m(u)\, dV = \int \underbrace{\left(\mathcal{L}_m(w_1) + \mathcal{L}_m(w_2)\right)}_{\substack{\mathcal{L}_m(w_1+w_2) \\ \text{linear}}} D\, \mathcal{L}_m(u)\, dV$

$= \int_{\mathcal{D}} \mathcal{L}_m(w_1)\, D\, \mathcal{L}_m(u)\, dV + \int \mathcal{L}_m(w_2)\, D\, \mathcal{L}_m(u)\, dV$

$= \mathcal{A}(w_1,u) + \mathcal{A}(w_2,u)$

Similar process for ii)

Side note: If we solve a nonlinear PDE, linearity w.r.t. weight persists, but not w.r.t. solution u

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Use bilinear property (2), plug

$U^h = \phi_p + \sum_{i=1}^{n} a_i \phi_i$ into eqn (1) (weak statement)

$$A(\overset{\phi_i}{\widetilde{w_i}}, \underbrace{\phi_p + \sum_{j=1}^{n} a_j \phi_j}_{\text{use linearity}}) = (\overset{\phi_i}{\widetilde{w_i}}, r) + (\overset{\phi_i}{\widetilde{w_i}}, \bar{F})_N$$

w_i = weight function # $i = \phi_i$ Galerkin

$$\sum_{j=1}^{n}\left(\mathcal{A}(\phi_i,\phi_j)\,a_j\right)+\mathcal{A}(\phi_i,\phi_p) = (\phi_i,r)+(\phi_i,\bar{F})_N$$

$$\underset{\forall i\in\{1,\dots,n\}}{\Longrightarrow}\ (4)\ \sum_{j=1}^{n}\mathcal{A}(\phi_i,\phi_j)\,a_j = (\phi_i,r)+(\phi_i,\bar{F})_N-\mathcal{A}(\phi_i,\phi_p)$$

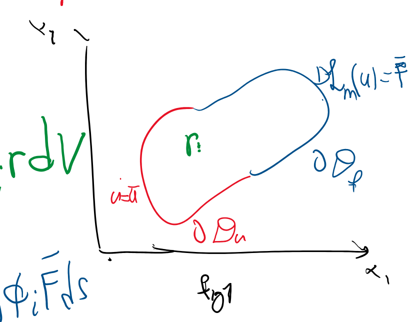
$$\underbrace{\begin{bmatrix}\mathcal{A}(\phi_1,\phi_1) & \mathcal{A}(\phi_1,\phi_2) & \cdots & \mathcal{A}(\phi_1,\phi_n)\\ \mathcal{A}(\phi_2,\phi_1) & \cdots & & \\ & & & \\ & & & \mathcal{A}(\phi_n,\phi_n)\end{bmatrix}}_{K}\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}=\begin{bmatrix}(\phi_1,r)\\(\phi_2,r)\\ \vdots\\(\phi_n,r)\end{bmatrix}+\begin{bmatrix}(\phi_1,\bar{F})_N\\(\phi_2,\bar{F})_N\\ \vdots\\(\phi_n,\bar{F})_N\end{bmatrix}-\begin{bmatrix}\mathcal{A}(\phi_1,\phi_p)\\ \mathcal{A}(\phi_2,\phi_p)\\ \vdots\\ \mathcal{A}(\phi_n,\phi_p)\end{bmatrix}$$

$$K a = F_r + F_N - F_b$$

(5)

$$(\phi_i,r)=\int_{\mathcal{D}}\phi_i\, r\, dV$$

$$(\phi_i,\bar{F})_N=\int_{\partial\mathcal{D}_f}\phi_i\,\bar{F}\,ds$$



Discrete Galerkin formulation for solid bar

- Using the bilinearity of $\mathcal{A}$ we obtain,

$$\begin{aligned}&\text{Find } \mathbf{a} = \{a_1, a_2, \ldots, a_{n_\mathrm{f}}\} \text{ such that } \forall I \in \{1, 2, \ldots, n_\mathrm{f}\}\\ &\mathcal{A}(\phi_I, \phi_J)\, a_J = (\phi_I, q)_r + \left(\phi_I, \bar{F}\right)_N - \mathcal{A}(\phi_I, \phi_p)\end{aligned} \tag{297}$$

- It is customary to denote the RHS components as,

$$F_{rI} := (\phi_I, q)_r \qquad \text{Force from Source terms} \tag{298a}$$

$$F_{NI} := \left(\phi_I, \bar{F}\right)_N \qquad \text{Force from Natural BCs} \tag{298b}$$

$$F_{DI} := \mathcal{A}(\phi_I, \phi_p) \qquad \text{Force from Essential BCs} \tag{298c}$$

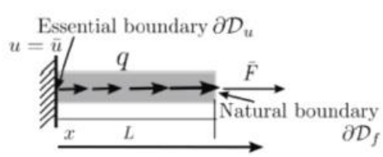
- That is, $\mathbf{a}$ is obtained from the system,

$$K_{IJ}\, a_J = F_I \text{ for} \tag{299a}$$

$$K_{IJ} = \mathcal{A}(\phi_I, \phi_J) \tag{299b}$$

$$\begin{aligned}F_I &= F_{rI} + F_{NI} - F_{DI}\\ &= (\phi_I, q)_r + \left(\phi_I, \bar{F}\right)_N - \mathcal{A}(\phi_I, \phi_p)\end{aligned} \tag{299c}$$

Discrete Galerkin formulation for solid bar



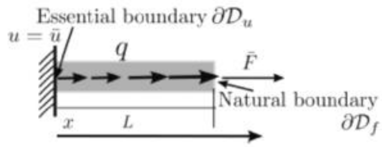
- Using the definitions and vector interpretation of the linear and bilinear forms the stiffness and force vectors are,

$$\begin{aligned}\mathbf{K} = \mathcal{A}\left(\phi^{\mathrm{T}}, \phi\right) &= \int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}x}[\phi]^{\mathrm{T}}\, EA \frac{\mathrm{d}}{\mathrm{d}x}[\phi]\ \mathrm{dv}\\ &= \int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_{n_\mathrm{f}}\end{bmatrix} EA \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_{n_\mathrm{f}}\end{bmatrix}\ \mathrm{d}x\end{aligned} \tag{300a}$$

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N - \mathbf{F}_D \tag{300b}$$

- Force $\mathbf{F}$ is the sum of forces induced from source term, $\mathbf{F}_r$, natural boundary conditions, $\mathbf{F}_N$, and essential boundary conditions, $\mathbf{F}_D$.

Discrete Galerkin formulation for solid bar



- $\mathbf{F}_r$, $\mathbf{F}_N$, and $\mathbf{F}_D$ are given by,

$$\mathbf{F}_r = \left(\phi^{\mathrm{T}}, q\right)_r = \int_{\mathcal{D}} [\phi]^{\mathrm{T}} q \,\mathrm{dv} = \int_0^L \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n_f} \end{bmatrix} q \,\mathrm{d}x \tag{301a}$$

$$\mathbf{F}_N = \left(\phi^{\mathrm{T}}, \bar{F}\right)_N = \int_{\partial\mathcal{D}_f} [\phi]^{\mathrm{T}} \bar{\mathbf{F}}.\mathbf{N} \,\mathrm{ds} = \left(\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n_f} \end{bmatrix} \bar{F} \right)_{x=L} \tag{301b}$$

$$\mathbf{F}_D = \mathcal{A}\left(\phi^{\mathrm{T}}, \phi_p\right) = \int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}x}[\phi]^{\mathrm{T}} EA \frac{\mathrm{d}}{\mathrm{d}x}\phi_p \,\mathrm{dv} = \int_0^L \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n_f} \end{bmatrix} EA \frac{\mathrm{d}}{\mathrm{d}x}\phi_p \,\mathrm{d}x \tag{301c}$$