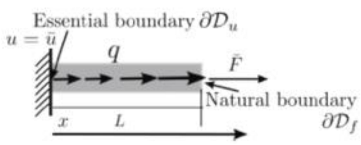
2023/10/23

Monday, October 23, 2023 11:17 AM

Discrete Galerkin formulation for solid bar



- $\mathbf{F}_r$, $\mathbf{F}_N$, and $\mathbf{F}_D$ are given by,

$$\mathbf{F}_r = \left(\phi^{\mathrm{T}}, q\right)_r = \int_{\mathcal{D}} [\phi]^{\mathrm{T}} q \, \mathrm{d}v = \int_0^L \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n_f} \end{bmatrix} q \, \mathrm{d}x \tag{301a}$$

$$\mathbf{F}_N = \left(\phi^{\mathrm{T}}, \bar{F}\right)_N = \int_{\partial\mathcal{D}_f} [\phi]^{\mathrm{T}} \bar{\mathbf{F}}.\mathbf{N} \, \mathrm{d}s = \left(\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n_f} \end{bmatrix} \bar{F} \right)_{x=L} \tag{301b}$$

$$\mathbf{F}_D = \mathcal{A}\left(\phi^{\mathrm{T}}, \phi_p\right) = \int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}x}[\phi]^{\mathrm{T}} EA \frac{\mathrm{d}}{\mathrm{d}x}\phi_p \, \mathrm{d}v = \int_0^L \frac{\mathrm{d}}{\mathrm{d}x} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n_f} \end{bmatrix} EA \frac{\mathrm{d}}{\mathrm{d}x}\phi_p \, \mathrm{d}x \tag{301c}$$

$$\left(\mathcal{L}_m(\Phi)\right)^t D \, \mathcal{L}_m(\Phi_p)$$

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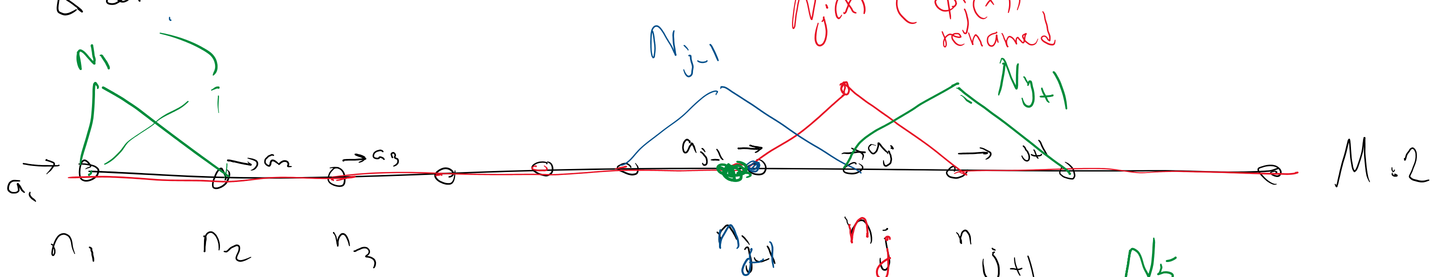
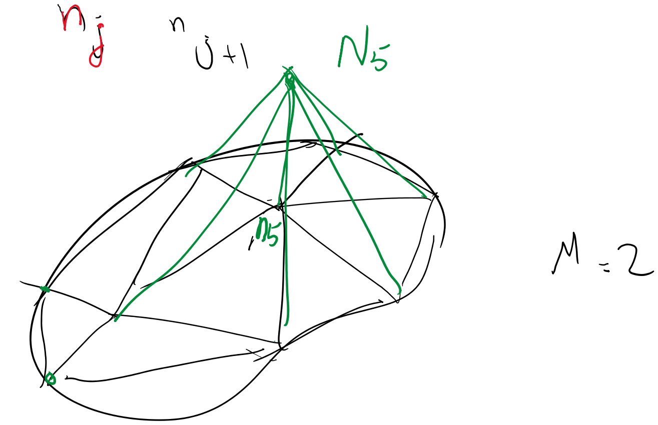
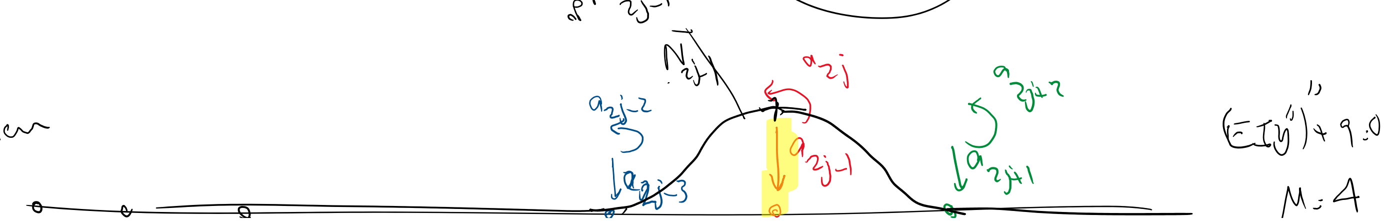
In finite element method (FEM) we denote basis functions (phis) by N and call them shape functions.

Shape functions take the value of 1 @ one degree of freedom (dof) & zero else where

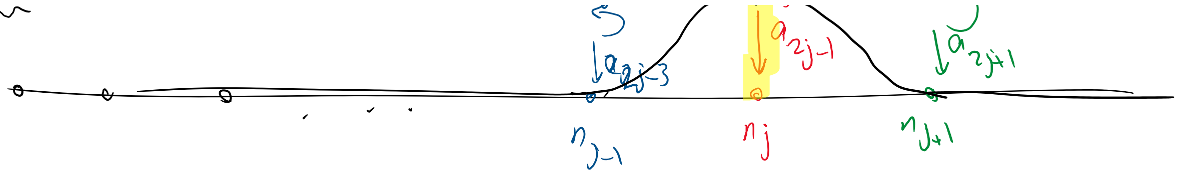


2D heat conduction





problem



$EIy'''' + q = 0$

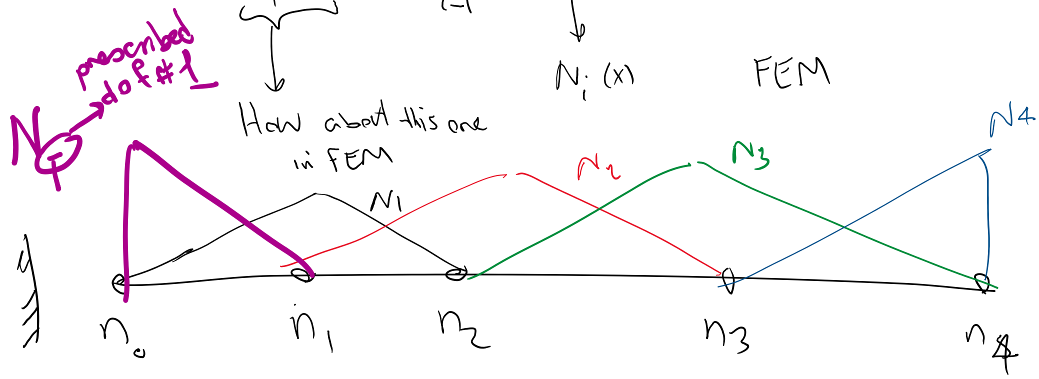
$M = 4$

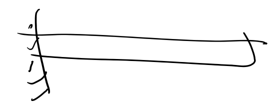
$$u^h(x) = \underbrace{\phi_p(x)}_{\text{How about this one in FEM}} + \sum_{i=1}^{n} a_i \phi_i(x)$$

$\phi_i(x) \to N_i(x)$ FEM

$n_f = \#\ \underline{\text{free}}$ dofs $= 4$

$n_p = \#$ prescribed dofs $= 1$





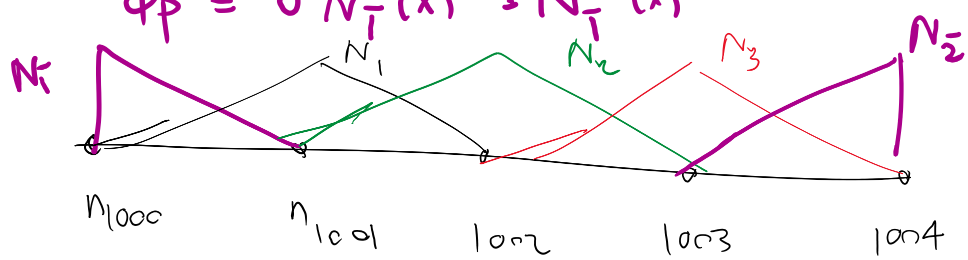
$u = \bar{u} = 1$

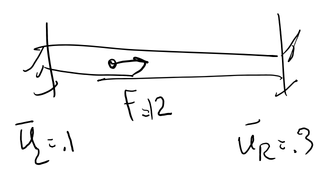
we used $\phi_p = \bar{U} = 1$ before how about now?

$$\phi_p = \bar{U} N_{\bar{1}}(x) = N_{\bar{1}}(x)$$

$n_f = 3$

$n_p = 2$





$N_{(\bar{i})}$ or $N_{\bar{i}}$: i refers to dof #

$$u^h(x) = \phi_p(x) + \sum_{i=1}^{n_f} a_i N_i(x) = \underbrace{\left(\sum_{\bar{i}=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(x) \right)}_{\phi_p} + \sum_{i=1}^{n_f} a_i N_i(x)$$

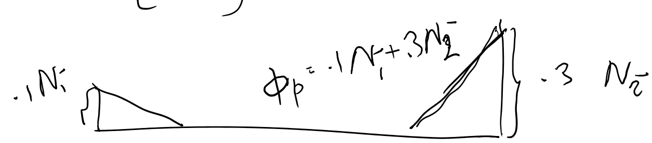
same as

$$u^h(x) = \begin{bmatrix} N_{\bar{1}}(x) & N_{\bar{2}}(x) \end{bmatrix} \begin{bmatrix} a_{\bar{1}} \\ a_{\bar{2}} \end{bmatrix} + \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

this example

$a_{\bar{1}} = 0.1$

$a_{\bar{2}} = .3$



In terms of notation we write

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$$u^h = \sum_{i=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(x) + \sum_{i=1}^{n} a_i N_i(x)$$

$$= \bar{N}\bar{a} + N a$$

$$\bar{N} = [N_{\bar{1}} \; \cdots \; N_{\bar{n}_p}] \qquad \bar{a} = \begin{bmatrix} a_{\bar{1}} \\ \vdots \\ \bar{a}_{n_p} \end{bmatrix}$$

$$N = [N_1 \; \cdots \; N_{n_f}] \qquad a = \begin{bmatrix} a_1 \\ \vdots \\ a_{n_f} \end{bmatrix}$$

n_f = # free dofs

n_p = # prescribed dofs

another notation

$$u^h = \underbrace{\sum_{i=1}^{n_p} a_i^p N_i^p(x)}_{\phi_p} + \sum_{i=1}^{n_f} a_i^f N_i^f(x)$$

$$= N^p a^p + N^f a^f$$

$$N^p = [N_1^p \cdots N_{n_p}^p] \quad a^p = \begin{bmatrix} a_1^p \\ \vdots \\ a_{n_p}^p \end{bmatrix}$$

$$N^f = [N_1^f \cdots N_{n_f}^f] \quad a^f = \begin{bmatrix} a_1^f \\ \vdots \\ a_{n_f}^f \end{bmatrix}$$

often a^f & N^f are simply written as a & N

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2D example

Heat conduction

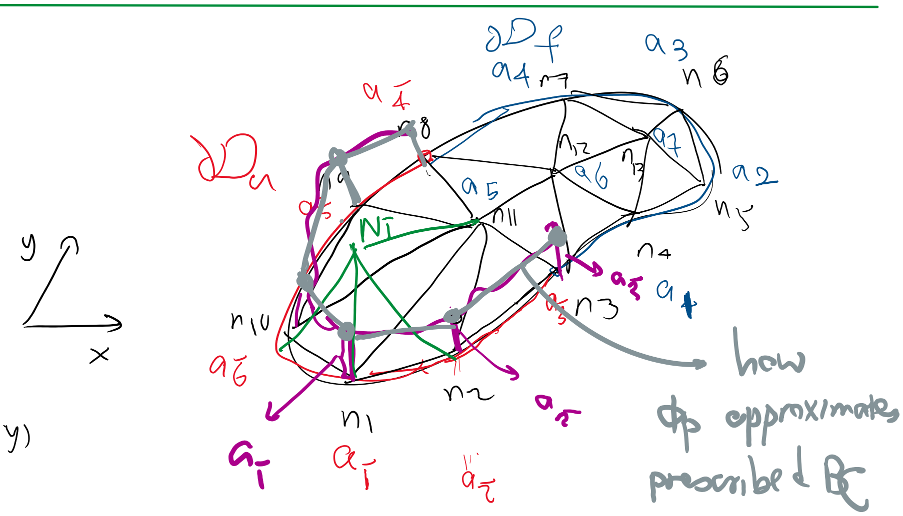
each node has 1 dof

(Temperature)

$n = 13$

$n_f = 7$

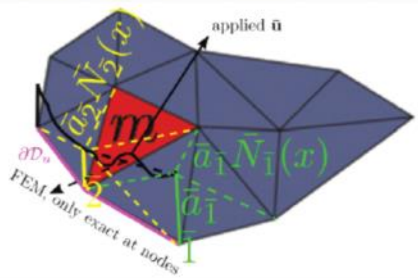
$n_{\bar{p}} = 6$



$$\phi_p = a_{\bar{1}} N_{\bar{1}}(\vec{x}) + a_{\bar{2}} N_{\bar{2}}(x) + \cdots + a_{\bar{6}} N_{\bar{6}}(x)$$

$$\phi_p(n_2) = 0 \quad a_{\bar{2}}\, 1 \quad + \cdots \quad 0 \quad = a_{\bar{2}}$$

B. Essential Boundary Conditions



$$
\begin{array}{rl}
n_{\mathrm{p}} & := \text{number of (prescribed) dof on } \partial\mathcal{D}_u \\
\bar{(.)} & = \text{decoration for prescribed dofs} \\
\{\bar{1}, \ldots, \bar{n_{\mathrm{p}}}\} & = \text{ global dofs on } \partial\mathcal{D}_u \\
\bar{\mathbf{a}} = \left[\bar{a}_{\bar{1}}, \ldots, \bar{a}_{\bar{n_{\mathrm{p}}}}\right]^{\mathrm{T}} & = \text{vector of prescribed values for these dofs} \\
\bar{\mathbf{N}} = \left[\bar{N}_{\bar{1}}, \ldots, \bar{N}_{\bar{n_{\mathrm{p}}}}\right] & = \text{(row) vector of shape functions for these dofs} \\
\bar{\mathbf{B}} = \left[\bar{B}_{\bar{1}}, \ldots, \bar{B}_{\bar{n_{\mathrm{p}}}}\right] & = \text{(row) vector of ``displacement to strain'' for these dofs}
\end{array}
\quad (308)
$$

- ϕ_p is formed by,

$$\boxed{\phi_p = \bar{\mathbf{N}}\bar{\mathbf{a}} = \sum_{i=1}^{n_{\mathrm{p}}} \bar{a}_{\bar{i}}\bar{N}_{\bar{i}} \;\Rightarrow\; \frac{\mathrm{d}\phi_p}{\mathrm{d}x} = \bar{\mathbf{B}}\bar{\mathbf{a}} = \sum_{i=1}^{n_{\mathrm{p}}} \bar{a}_{\bar{i}}\bar{B}_{\bar{i}}} \quad (309)$$

- While this construction guarantees satisfaction of essential boundary conditions at nodes $\bar{1}, \ldots, \bar{n_{\mathrm{p}}}$, as shown in the figure the intermediate values may not match $\bar{u}$.

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Now that we have established how the solution is interpolated, we want to form K, FD, Fr, FN (and a new force Fn)

1) K matrix

$n =$ # unknowns

$$K = \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} \underbrace{D}_{\text{material section property}} \mathcal{L}_m [\phi_1 \cdots \phi_n] \, dV$$

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in FEM language $\quad \phi_1 = N_1 \cdots \quad \phi_{n_f} = N_{n_f}$

$$K = \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{n_f} \end{bmatrix} D \underbrace{\mathcal{L}_m \overbrace{[N_1 \cdots N_{n_f}]}^{N^f \text{ (or } N)}}_{B^f \text{ (or } B)} dV$$

$$\boxed{K = K^{ff} = \int_{\mathcal{D}} (B^f)^t D B^f \, dV}$$ (2)

$B^f = \mathcal{L}_m N^f$ B "displacement to strain map"

What about F_D Dirichlet BC force vector

Recall

$$F_D = \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} D \mathcal{L}_m (\phi_p) \, dV$$

compare this with

$$K = \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} D \mathcal{L}_m (\phi_1, \ldots \phi_n) \, dV$$

In FEM language

$$F_D = \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} D \mathcal{L}_m \left(\sum_{i=1}^{np} a_i^p N_i^p \right) dV$$

$$= \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} D \mathcal{L}_m \left([N_1^p \cdots N_{np}^p] \begin{bmatrix} a_1^p \\ \vdots \\ a_{np}^p \end{bmatrix} \right) dV$$

$$\widehat{F_D} = \left(\int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} D \mathcal{L}_m ([N_1^p \cdots N_{np}^p]) \right) \begin{bmatrix} a_1^p \\ \vdots \\ a_{np}^p \end{bmatrix}$$

prescribed

both f

recall

$$K^{ff} = K = \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} D \mathcal{L}_m [N_1 \cdots N_{nf}] \, dV$$

f

f $\qquad \mathcal{B} \quad \lfloor N_{nf} \rfloor$

free

Summary of K & F_D

$$K = K^{ff} = \int_{\mathcal{D}} (B^f)^t D \underbrace{B^f}_{\text{a.k.a } B} dV \qquad B^f = \mathcal{L}_m(\lfloor N_1^f \cdots N_{n_f}^f \rfloor)$$

$$B^p = \mathcal{L}_m(\lfloor N_1^p, \cdots N_{n_p}^p \rfloor)$$

$$F_D = K^{fp} a^p$$

$$K^{fp} = \int_{\mathcal{D}} (B^f)^t D B^p dV$$

(2)

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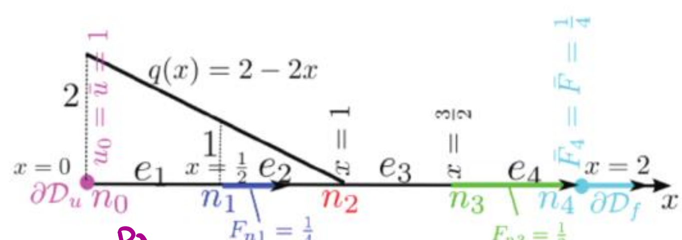
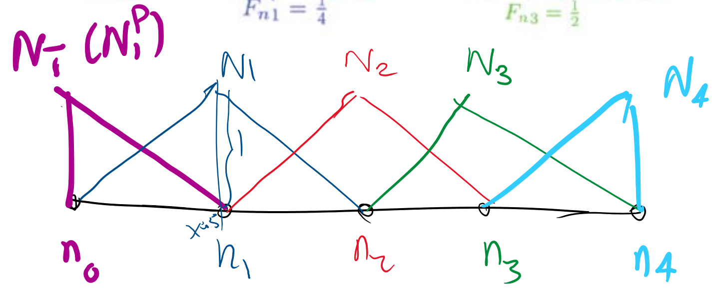
Numerical example



Bar problem

$$\int \underbrace{W'}_{} \overset{D}{\widetilde{EA}}\, u' - - -$$

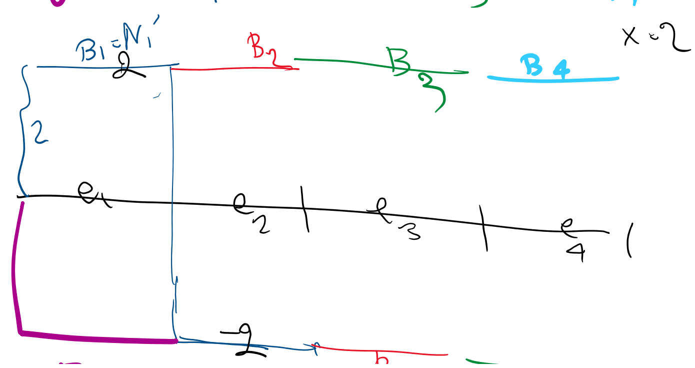
$$\mathcal{L}_m(W) \rightarrow \mathcal{L}_m = (\ )'$$



$n_f = 4$

$n_p = 1 \qquad a_p (= \bar{a}) = [1]$

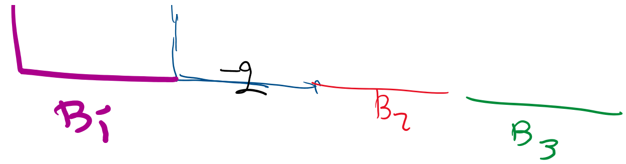
$$a = (a^f) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \Rightarrow \text{free}$$



$\int^2$

$$K = \int_0^2 B_f^t \underbrace{D}_{EA} B_f \, dx$$

$$B_f = \tfrac{d}{dx}[N_1 \; N_2 \; N_3 \; N_4]$$
$$= \frac{d}{dx}[N_1 \; N_2 \; N_3 \; N_4]$$
$$= [N_1' \; N_2' \; N_3' \; N_4']$$



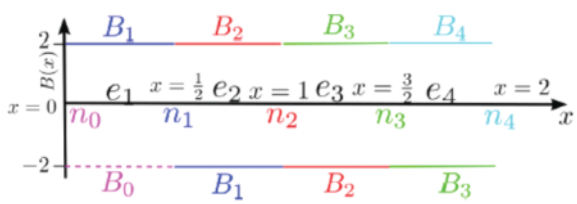
$$K = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \underbrace{1}_{EA} [B_1 \; B_2 \; B_3 \; B_4] \, dx$$

$$K_{11} = \int_0^2 B_1 B_1 \, dx = \int_{e_1} B_1 B_1 \, dx + \int_{e_2} B_1 B_1 \, dx =$$

$$\int_0^{1/2} (2)(2) \, dx + \int_{1/2}^{1} (-2)(-2) \, dx = \frac{4}{2} + \frac{4}{2} = 4$$

$$K_{12} = \int_0^2 B_1 B_2 \, dx = \int_{e_2} B_1 B_2 \, dx$$
$$= \int_{1/2}^{1} (-2)(2) \, dx = -2$$

Bar Example: Step 1: Stiffness matrix



$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA\,[B_1 \quad B_2 \quad B_3 \quad B_4]\, dx = \begin{bmatrix} \int_0^2 B_1B_1\,dx & \int_0^2 B_1B_2\,dx & \int_0^2 B_1B_3\,dx & \int_0^2 B_1B_4\,dx \\ & \int_0^2 B_2B_2\,dx & \int_0^2 B_2B_3\,dx & \int_0^2 B_2B_4\,dx \\ \text{sym.} & & \int_0^2 B_3B_3\,dx & \int_0^2 B_3B_4\,dx \\ & & & \int_0^2 B_4B_4\,dx \end{bmatrix}$$

$$= \begin{bmatrix} \int_{e_1} B_1B_1\,dx + \int_{e_2} B_1B_1\,dx & \int_{e_2} B_1B_2\,dx & 0 & 0 \\ & \int_{e_2} B_2B_2\,dx + \int_{e_3} B_2B_2\,dx & \int_{e_3} B_2B_3\,dx & 0 \\ \text{sym.} & & \int_{e_3} B_3B_3\,dx + \int_{e_4} B_3B_3\,dx & \int_{e_4} B_3B_4\,dx \\ & & & \int_{e_4} B_4B_4\,dx \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 & 0 \\ & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 \\ \text{sym.} & & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) \\ & & & \frac{1}{2}.(2).(2) \end{bmatrix} \Rightarrow$$

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix} \tag{316}$$

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