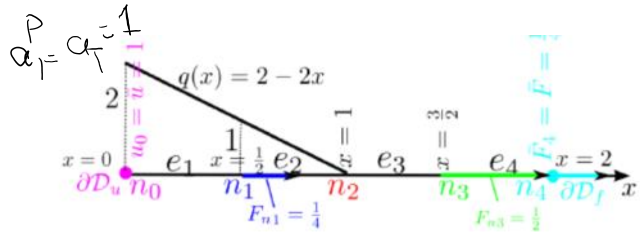
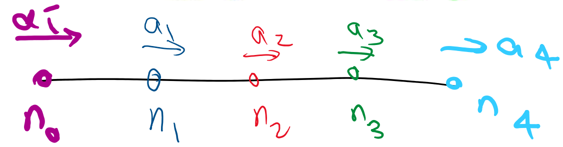
2023/10/25

Wednesday, October 25, 2023 11:16 AM

Problem description, slide 253



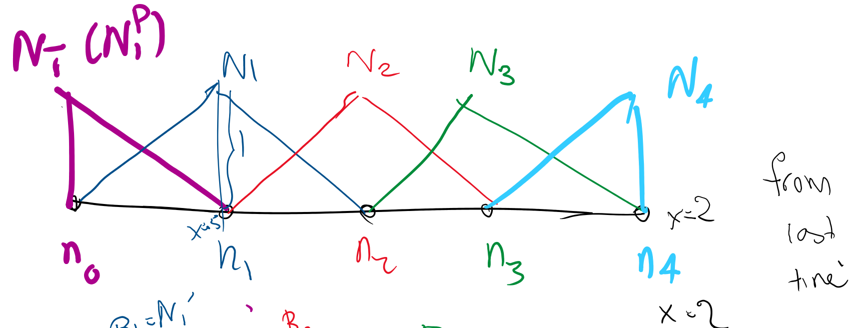



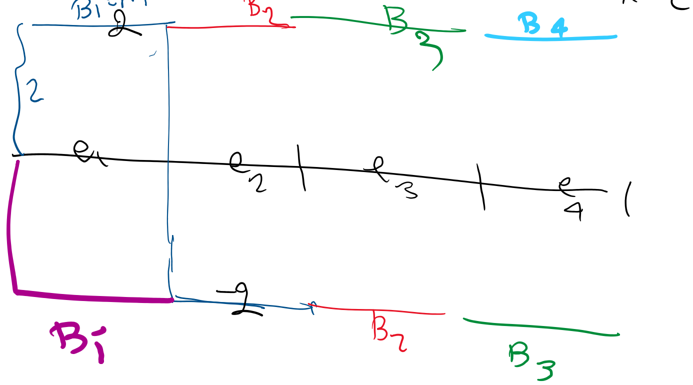

$F_D = ?$

$$F_D = K^{fp} a^p$$

$$K^{fp} = \int_\Omega (B^f)^t D B^p \, dv$$

$$= \int_0^2 \underbrace{\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}}_{(B^f)^t} \underbrace{EA}_{1} \left[B_{\bar 1}^p\right] dx$$



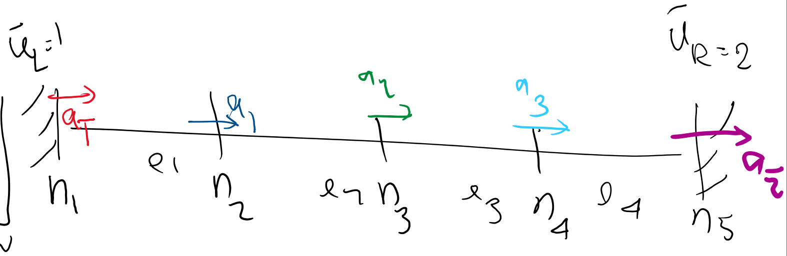

$$= \int_{e_1} \begin{bmatrix} B_1 B_{\bar 1} \\ 0 \\ 0 \\ 0 \end{bmatrix} dx = \int_0^{1/2} \begin{bmatrix} 2 \times (-2) \\ 0 \\ 0 \\ 0 \end{bmatrix} dx = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{K^{fp}}$$

$$a_1^p = [1]$$

$$F_D = K^{fp} a^p = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} (1) = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

What is scenario

$$\underbrace{(F_D)}_{\substack{3\times1 \\ n_f \times 1}} = \underbrace{\left(\int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} EA \left[B_1^p \; B_2^p\right] dx\right)}_{\substack{K^{fp}_{3\times2} \\ n_f \times n_p}} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\substack{a_{p\,2\times1} \\ n_p \times 1}}$$

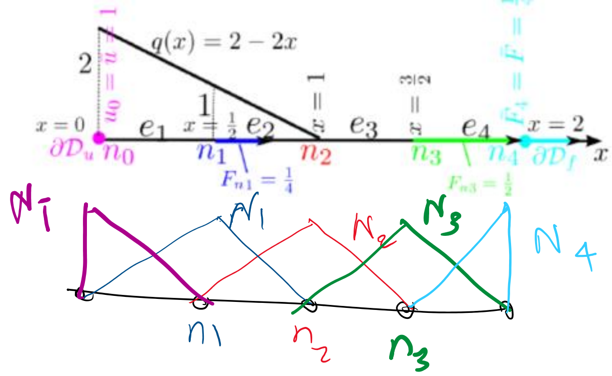



$n_f = 3$

$n_p = 2$

K and FD are calculated. We want to calculate source term and Neumann BC
We break the source term to two parts:

1. Distributed force -> Fr
2. Point forces -> Fn

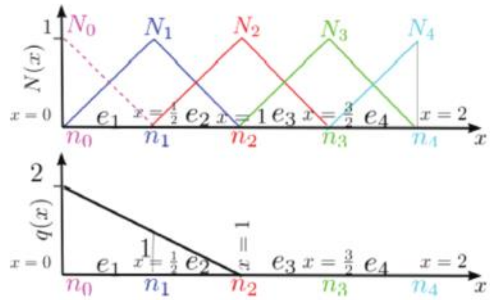



$$F_r = \int_0^2 \begin{bmatrix} N_1 \\ \vdots \\ N_4 \end{bmatrix} q\,dx =$$

$$= \begin{bmatrix} \int_0^2 N_1 q\,dx \\ \int_0^2 N_2 q\,dx \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \int_{e_1} N_1 q\,dx + \int_{e_2} N_1 q\,dx \\ \int_{e_2} N_2 q\,dx \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/12 \\ 0 \\ 0 \end{bmatrix}$$

for this q(x)

## Bar Example: Step 2.1: Source term force




From (312a),

$$\mathbf{F}_r = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix} q\,dx = \begin{bmatrix} \int_0^2 N_1(x)q(x)\,dx \\ \int_0^2 N_2(x)q(x)\,dx \\ \int_0^2 N_3(x)q(x)\,dx \\ \int_0^2 N_4(x)q(x)\,dx \end{bmatrix} = \begin{bmatrix} \int_{e_1} N_1(x)q(x)\,dx + \int_{e_2} N_1(x)q(x)\,dx \\ \int_{e_2} N_2(x)q(x)\,dx \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1/2}{6}\left((2).(0).(2) + (2).(1).(1) + (0).(1) + (1).(2)\right) + \frac{1/2}{6}\left((2).(1).(1) + (2).(0).(0) + (1).(0) + (0).(1)\right) \\ \frac{1/2}{6}\left((2).(0).(1) + (2).(1).(0) + (0).(0) + (1).(1)\right) \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$$\mathbf{F}_r = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} \tag{317}$$

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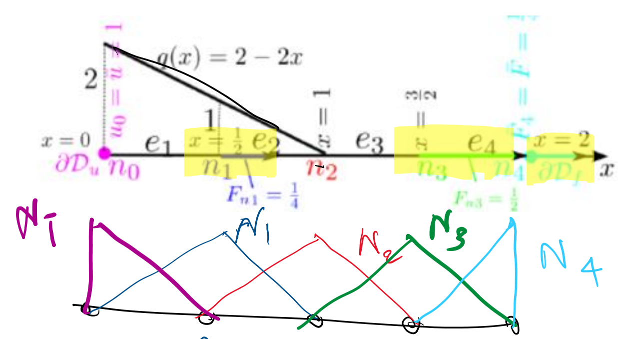
2nd Part Fr from point forces

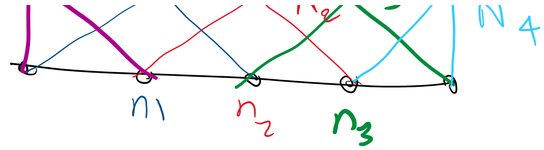
In 1D problems, Neumann BC will be a (collection of) points
So we treat it as a point force




point force

$$q^{pf}(x) = \underbrace{\frac{1}{4}\delta\left(x - \frac{1}{2}\right)} + \frac{1}{2}\delta\left(x - \frac{3}{2}\right)$$

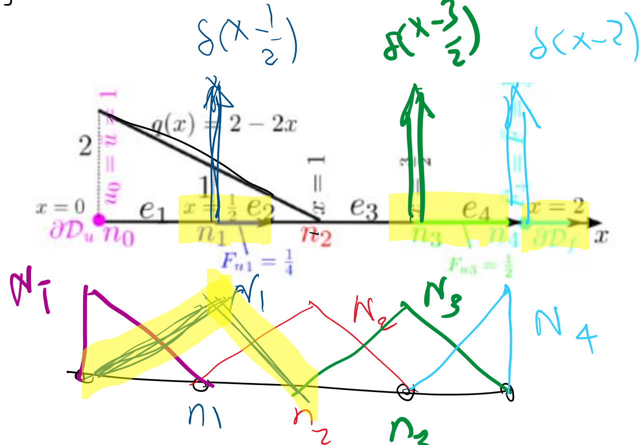
$$q^{pp}(x) = \underbrace{\frac{1}{4}\delta\left(x-\frac{1}{2}\right)}_{F_{n_1}\delta(x-x_{n_1})} + \frac{1}{2}\delta\left(x-\frac{3}{2}\right) + \frac{1}{4}\delta(x-2)$$




$$F_n = \int_0^2 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q^{pp}(x)\,dx = \int_0^2 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \left[\frac{1}{4}\delta\left(x-\frac{1}{2}\right) + \frac{1}{2}\delta\left(x-\frac{3}{2}\right) + \frac{1}{4}\delta(x-2)\right] dx$$

$F_n$ → nodal force vector

Example $(F_n)_1$




$$= \int_0^2 N_1 \left[\frac{1}{4}\delta\left(x-\frac{1}{2}\right) + \frac{1}{2}\delta\left(x-\frac{3}{2}\right) + \frac{1}{4}\delta(x-2)\right] dx$$

$$= \frac{1}{4} \underbrace{N_1(\underbrace{\tfrac{1}{2}}_{n_1})}_{1} + \frac{1}{2} \underbrace{N_1(\underbrace{\tfrac{3}{2}}_{n_3})}_{0} + \frac{1}{4} \underbrace{N_1(\underbrace{2}_{n_4})}_{0}$$

$$\int_0^2 f(x)\,\delta(x - \underbrace{\tfrac{1}{2}}_{x_0})\,dx = \underbrace{f(\tfrac{1}{2})}_{f(x_0)}$$

$= \frac{1}{4} = F_1 = \frac{1}{4}$ nodal force @ node 1

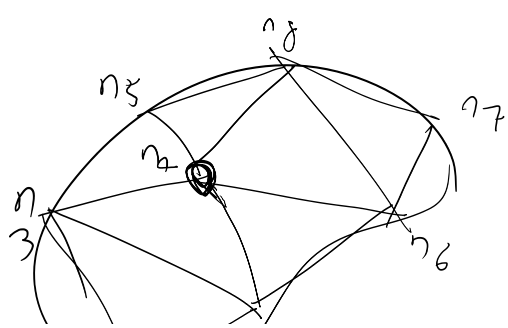
$(F_n)_2 = F_2 = 0 \qquad (F_n)_3 = F_3 = \frac{1}{2} \qquad (F_n)_4 = F_4 = \frac{1}{4}$

$$\boxed{F_n = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}}$$

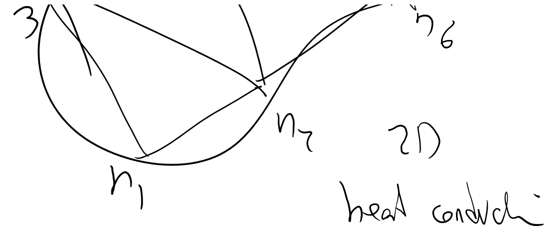
For any problem $F_n$ is the collection of nodal "forces"

$$(F_n)_{8\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_4 \\ 0 \\ 0 \end{bmatrix}$$

$F_4$ → point heat source



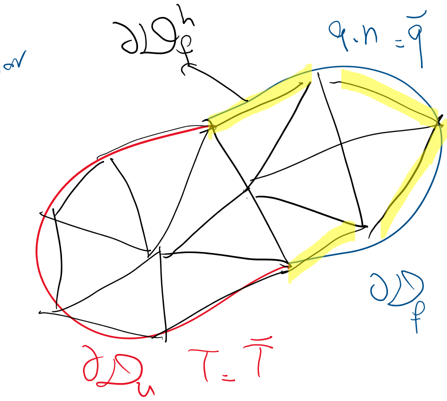
$$\left(\ \cdot_n\ \right)_{8\times1} \qquad \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{heat source}$$




$F_N$ = Neumann BC

$$F_N = \int_{\partial D_f^h} \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} \bar{F}\, ds$$

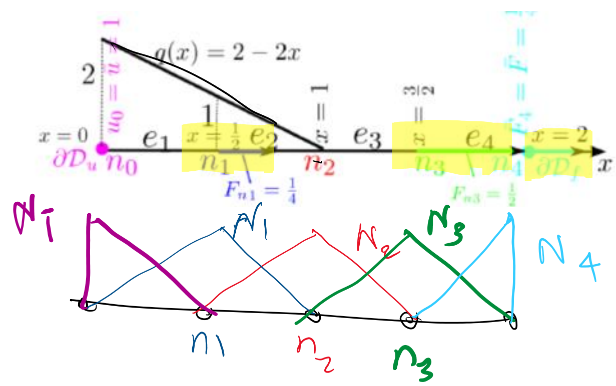
eq $\bar{q}$ for heat eqn




1D our problem

$$F_N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \left(\frac{1}{4}\right)\Bigg|_{x=2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

0, 0, 0, 1




In 1D we'll not form $F_n$ & include it in $F_n$

$$K = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \qquad F = F_n + \cancel{F_N}^{\,1D} + F_r - F_D$$

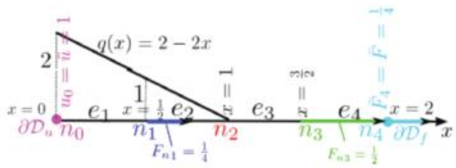
$$= \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{6} \\ 1/12 \\ 0 \\ \vdots \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{11}{4} \\ 1/12 \\ 1/2 \\ \vdots \end{bmatrix}$$

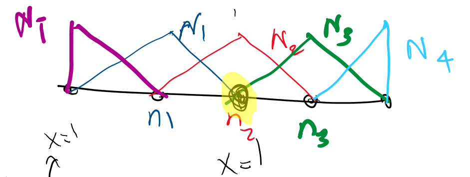
$$\begin{bmatrix} 43/24 \\ 53/24 \\ \vdots \end{bmatrix}$$

$$a = K^{-1}F = \begin{bmatrix} 43/24 \\ 53/24 \\ 31/24 \\ 65/24 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \vdots \\ 1/2 \\ 1/4 \end{bmatrix}$$

## Bar Example: FEM Solution




- From (311) we have

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D$$

- Obtaining the individual values from (317), (318), (319), and (320) we obtain,

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{4} \\ \frac{1}{12} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

- Recalling the value for the stiffness matrix (316) and $\mathbf{Ka} = \mathbf{F}$ we obtain,

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \frac{11}{4} \\ \frac{1}{12} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \quad \Rightarrow \quad \boxed{\mathbf{a} = \begin{bmatrix} \frac{43}{24} \\ \frac{53}{24} \\ \frac{31}{12} \\ \frac{65}{24} \end{bmatrix}} \qquad (321)$$

$$u^h(n_2) = \phi_p(n_2) + \sum_{i=1}^{4} a_i N_i(n_2)$$

$$= a_{\bar{1}} N_{\bar{1}}(n_2) + a_1 N_1(n_2) + a_2 \overbrace{N_2(n_2)}^{1}$$

$$+ a_3 \underbrace{N_3(n_2)}_{0} + a_4 \underbrace{N_4(n_2)}_{0}$$

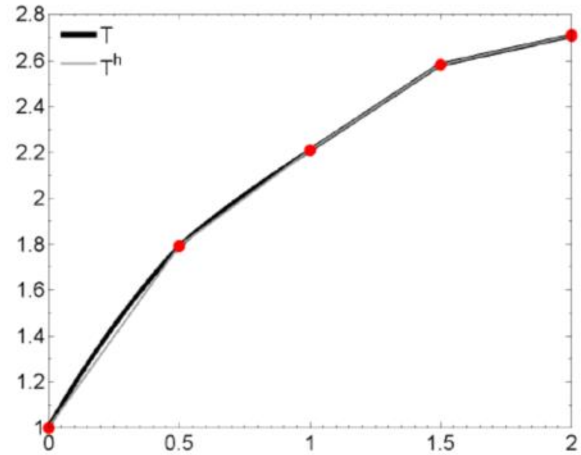
$$= a_2$$



$u^h(n_j) = a_j$ because of $\delta$ property of $N$'s

$N_i(n_j) = \delta_{ij}$

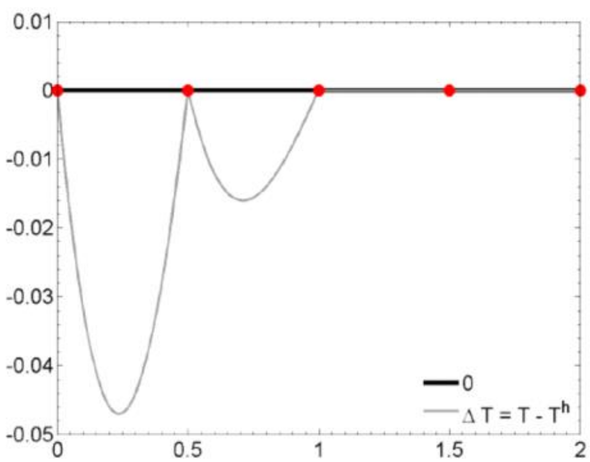
## Bar Example: solution values



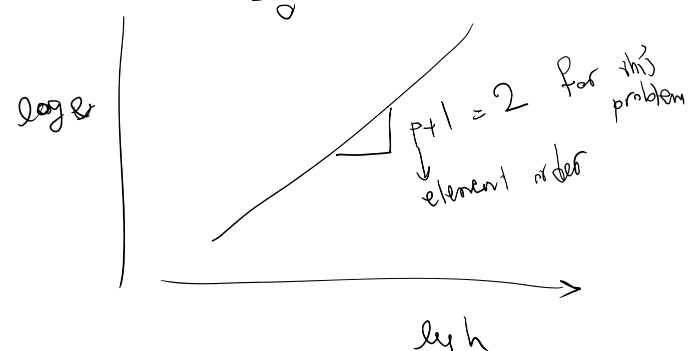

- $u^h$ and $u$ match at all nodes $n_0, n_1, n_2, n_3$, and $n_4$. This holds for 1D solid elements with uniform $AE$ and **does not hold** in general.



## Bar Example: error in solution values



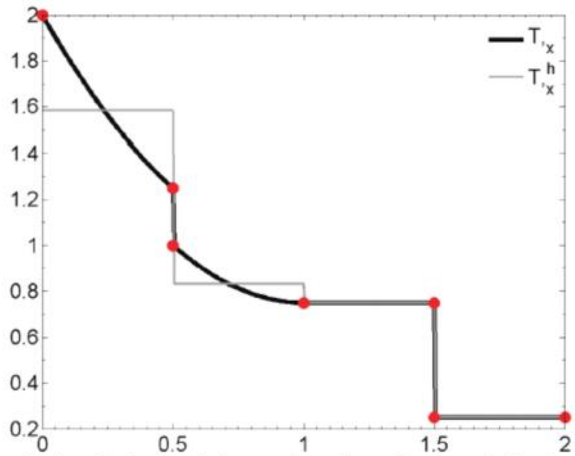

$$e(u) = \sqrt{\int_0^2 \left(u^h(x) - u^{exact}(x)\right)^2 dx}$$



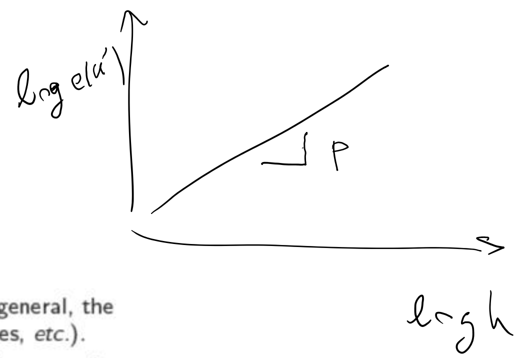

$e = C h^{p+1}$

- As mentioned before, the solution error at all nodes $n_0, n_1, n_2, n_3$, and $n_4$ is zero. This does not hold in general for FEM method.

## Bar Example: solution derivatives ($\propto$ axial force)




$$e(u') = \sqrt{\int_0^2 \left(u^{h\prime}(x) - u^{exact\prime}(x)\right)^2 dx}$$




- The errors in solution derivative is larger than those in the solution itself. In general, the accuracy of FE solution decreases for solution derivatives (*e.g.*, strains, stresses, *etc.*).
- Approximate solution $u^h$ exhibits jumps in $\frac{du^h}{dx}$ at all interior nodes. This is because the solution is piece-wise constant in $H^1([0\ 2])$.
- Even the exact solution exhibits jumps in $\frac{du}{dx}$ at $n_1$ and $n_3$ from the concentrated forces.
- The $H^1([0\ 2])$, rather than $C^1([0\ 2])$, is the right solution space for $u$ and $u^h$ as none of them belong to the latter space.

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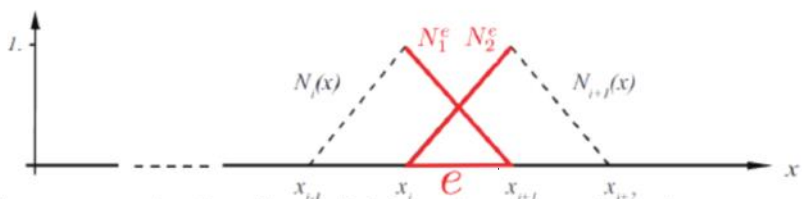
## Global shape functions to element shape functions




- Finite Elements are the domain subdivisions that are used for the construction of the shape functions
- Restriction of (global) shape functions to elements form the elements' shape functions (local).
- To distinguish element level and global level quantities, any element level value is decorated by $(.)^e$.
- Local node numbers in the element start from 1 to number of nodes in element $n_{\rm n}^e$ and are denoted by $n_1^e, \ldots, n_{n_{\rm n}^e}^e$.
- Similarly local dof start from 1 to the number of dof in element $n_{\rm dof}^e$.
- For example in the figure both $n_{\rm n}^e$ and $n_{\rm dof}^e$ are both 2 and the range for local node number and dof is from 1 to 2.
- Element shape functions satisfy the condition,

$$N_i^e(n_j^e) = \delta_{ij} \tag{325}$$

- More generally (*e.g.*, beam elements), shape function $i$ has a value 1 at dof $i$ while has a value zero at all other element dofs.

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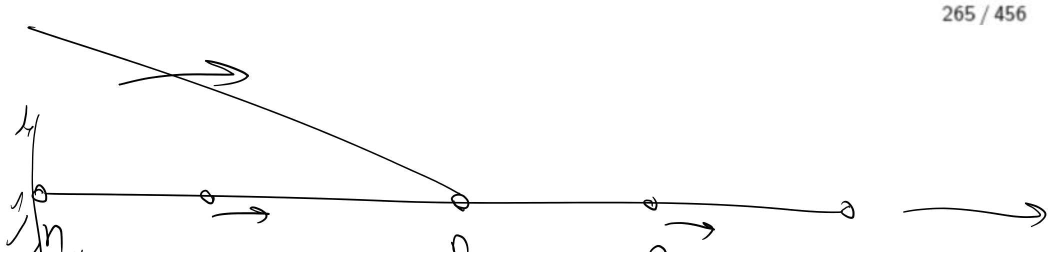
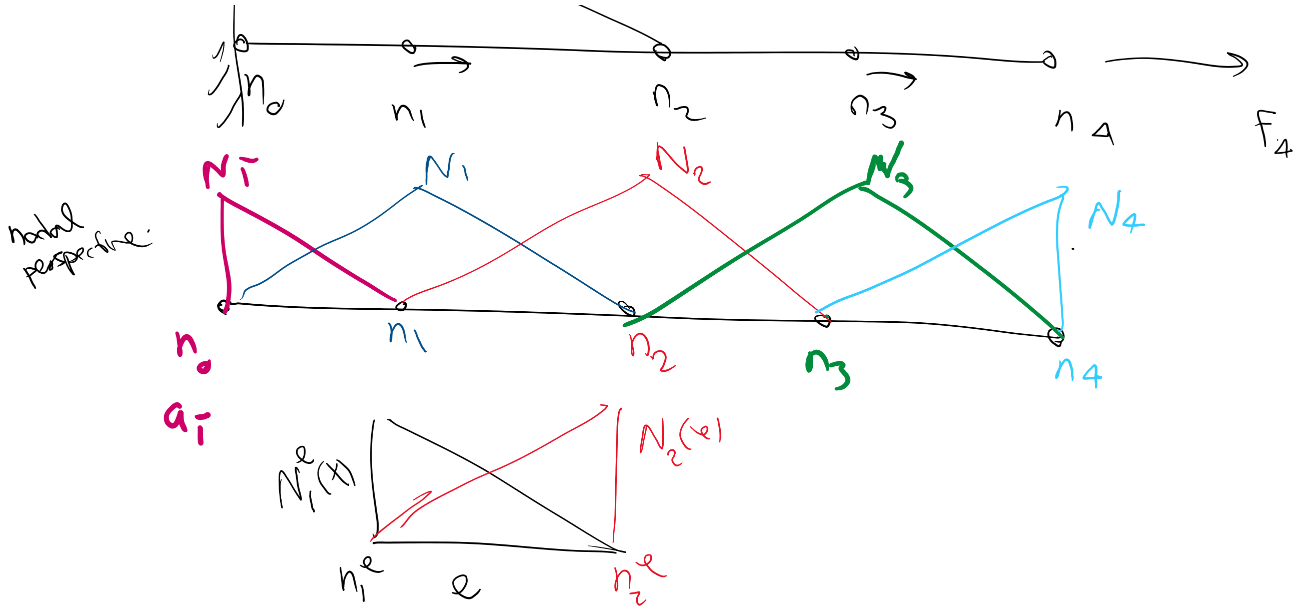
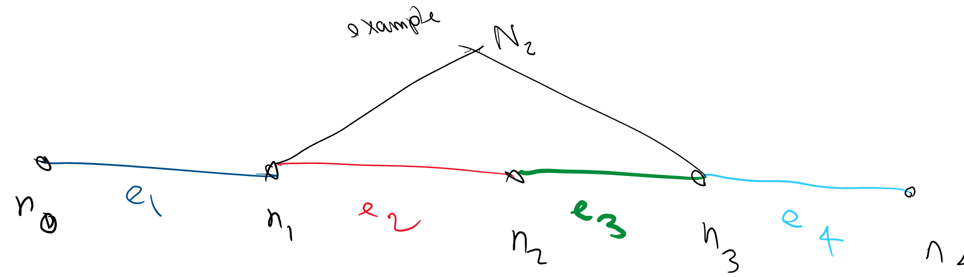





Calculating the stiffness matrix using the element-centered approach:



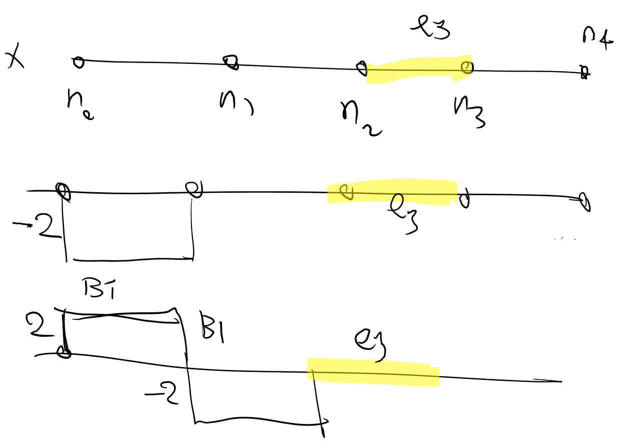

$$K = \int_0^2 \underbrace{\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \,[B_1\ B_2\ B_3\ B_4]}_{I\ \text{(integrand)}}\, dx = \underbrace{\int_{e_1} I\,dx}_{K^{e_1}} + \underbrace{\int_{e_2} I\,dx}_{K^{e_2}} + \underbrace{\int_{e_3} I\,dx}_{K^{e_3}} + \underbrace{\int_{e_4} I\,dx}_{K^{e_4}}$$

## Example $K^{e_3}$

$$K^{e_3} = \int_{e_3} \begin{bmatrix} \cancel{B_1} \\ B_2 \\ B_3 \\ \cancel{B_4} \end{bmatrix} \underbrace{(EA)}_{1} [\cancel{B_1}\ \ B_2\ \ B_3\ \ \cancel{B_4}]\, dx$$

"\" zero in $e_3$

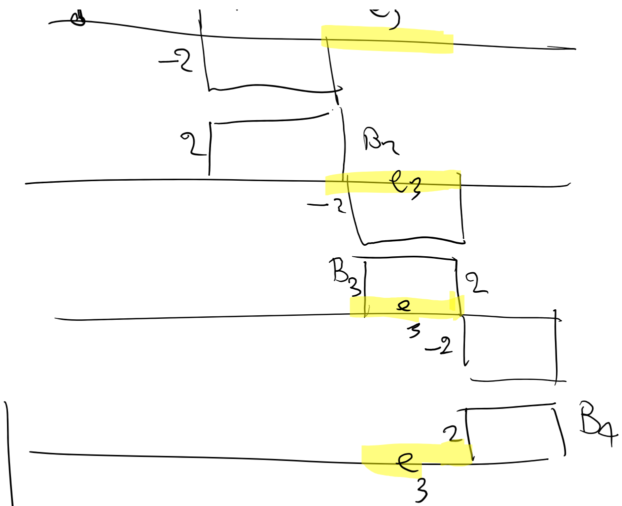
$$= \int \begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix} [0\ \ B_2\ \ B_3\ \ 0]\, dx =$$



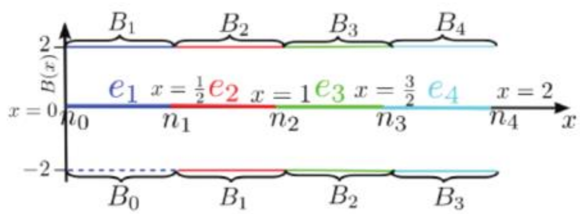
$$= \int_{e_3} \begin{bmatrix} 0 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & B_2 & B_3 & 0 \end{bmatrix} dx =$$

$$\int_{e_3} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 & 0 \end{bmatrix} dx$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$




## Local approach (element-centered)




$$\mathbf{K} = \int_0^2 \mathbf{B}^T EA\mathbf{B}\, dx = \overbrace{\int_{e_1} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e_1}} + \overbrace{\int_{e_2} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e_2}} + \overbrace{\int_{e_3} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e_3}} + \overbrace{\int_{e_4} \mathbf{B}^T EA\mathbf{B}\, dx}^{\mathbf{K}^{e_4}} = \tag{327}$$

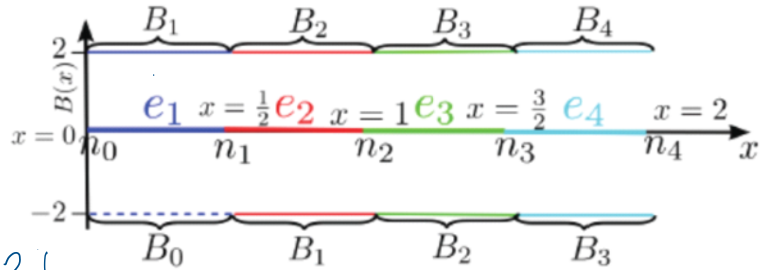
$$= \begin{bmatrix} \int_{e_1} B_1 B_1\, dx & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} \int_{e_2} B_1 B_1\, dx & \int_{e_2} B_1 B_2\, dx & 0 & 0 \\ & \int_{e_2} B_2 B_2\, dx & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \int_{e_3} B_2 B_2\, dx & \int_{e_3} B_2 B_3\, dx & 0 \\ \text{sym.} & & \int_{e_3} B_3 B_3\, dx & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & \int_{e_4} B_3 B_3\, dx & \int_{e_4} B_3 B_4\, dx \\ & & & \int_{e_4} B_4 B_4\, dx \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}.(2).(2) & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 & 0 \\ & \frac{1}{2}.(2).(2) & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 \\ \text{sym.} & & \frac{1}{2}.(2).(2) & 0 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) \\ & & & \frac{1}{2}.(2).(2) \end{bmatrix} \Rightarrow$$

# Local approach (element-centered)




$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\mathbf{K}^{e_1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix} \qquad \mathbf{K}^{e_2} = \begin{bmatrix} 2 & -2 & 0 & 0 \\ & 2 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

$$\mathbf{K}^{e_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ \text{sym.} & & 2 & 0 \\ & & & 0 \end{bmatrix} \qquad \mathbf{K}^{e_4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 2 & -2 \\ & & & 2 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{K}^{e_1} + \mathbf{K}^{e_2} + \mathbf{K}^{e_3} + \mathbf{K}^{e_4} \quad \Rightarrow$$

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix}$$