2023/10/30

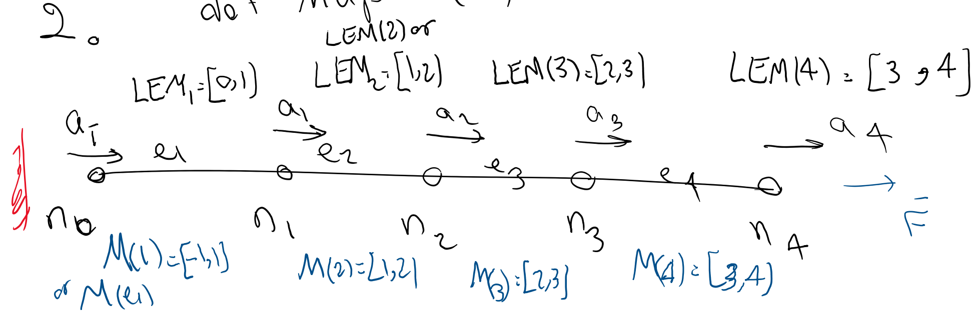
Friday, October 27, 2023 9:52 AM

Continuing the local FEM approach

To fully utilize the element-centered (local) approach, we need to define 2 maps:

1. Local Element Map (LEM) or nodal map

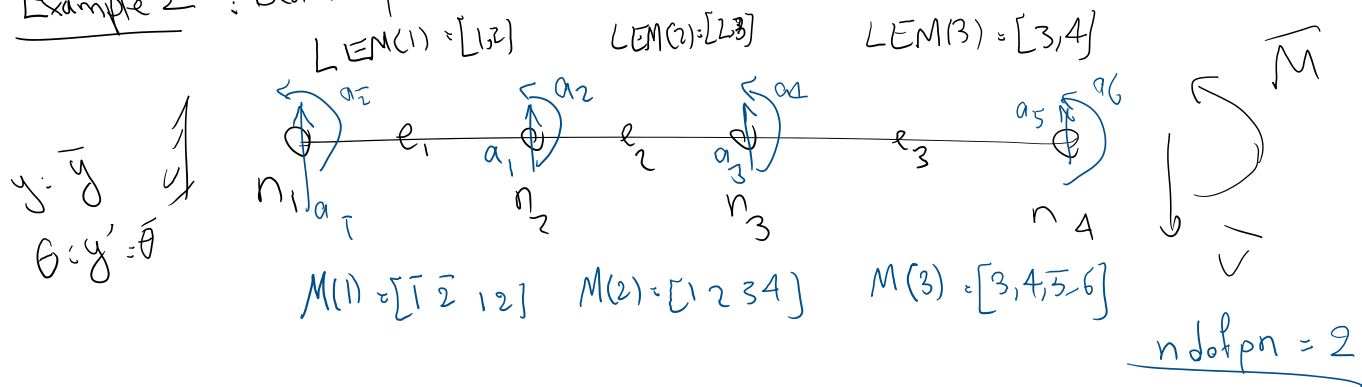
2. dof Map (M)



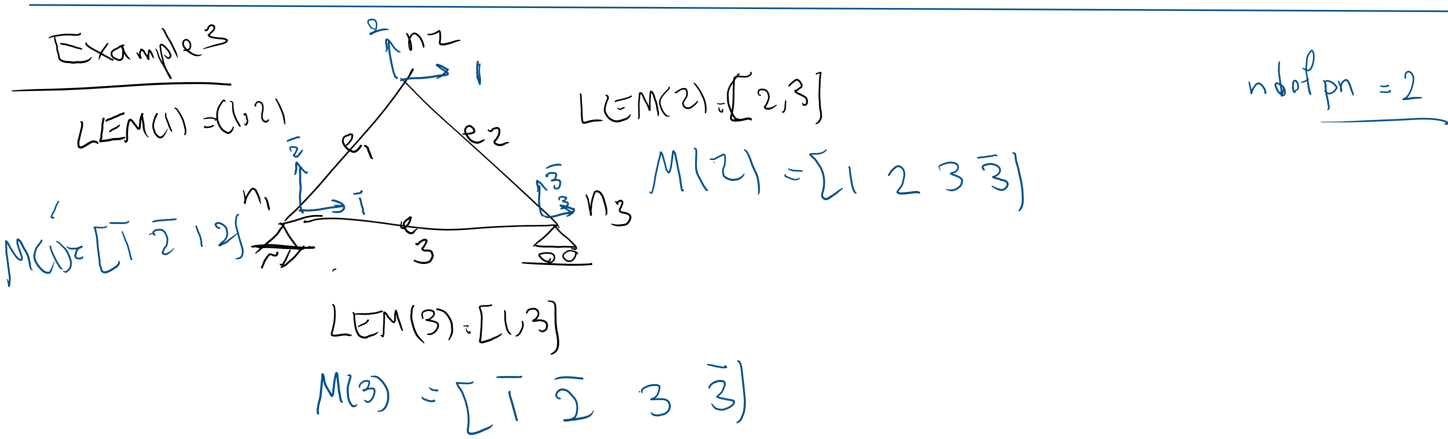
1) LEM: what nodes each element contain — Generally input from the user

2) M: which dof each element have (Software creates this map internally)

Example 2 : Beam problem

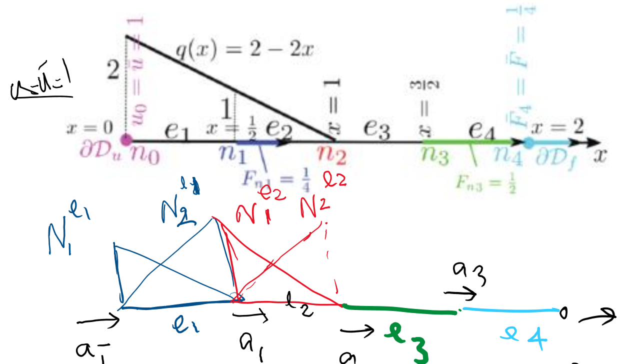


Example 3



For hand calculation, the main map that we need to form is M but in coding stage we need to use LEM to form M

Problem statement from slide 251:



$M(1) = [1,1]$ $M(2) = [1,2]$ $M(3) = [2,3]$ $M(4) = [3,4]$

We'll shortly see that

$$k^e = \frac{(AE)_e}{L_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

for all element $L_e = \frac{2}{4} = .5$

$(AE)^e = 1$

$$k^{e_1} = k^{e_2} = k^{e_3} = k^{e_4} = \frac{1}{.5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$$

$M(e_1)$

$$k^{e_1} = \begin{array}{c|cc} & 1 & 1 \\ \hline 1 & 2 & -2 \\ 1 & -2 & 2 \end{array}$$

$M(2)$

$$k^{e_2} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 2 & -2 \\ 2 & -2 & 2 \end{array}$$

$$k^{e_3} = \begin{array}{c|cc} & 2 & 3 \\ \hline 2 & 2 & -2 \\ 3 & -2 & 2 \end{array}$$

$M(e_4)$

$$k^{e_4} = \begin{array}{c|cc} & 3 & 4 \\ \hline 3 & 2 & -2 \\ 4 & -2 & 2 \end{array}$$

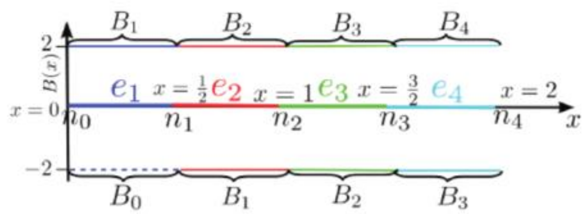
$$K = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 2+2 & -2 & & \\ 2 & -2 & 2+2 & -2 & \\ 3 & & -2 & +2+2 & -2 \\ 4 & & & -2 & +2 \end{array}$$

which is K^{ff}

this is called the assembly process

So, this matches with values we computed last time by integrating BT D B term over elements 1 to 4:

Local approach (element-centered)



$$\mathbf{K}=\int_0^2 \mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x=\overbrace{\int_{e_1}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x}^{\mathbf{K}^{e_1}}+\overbrace{\int_{e_2}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x}^{\mathbf{K}^{e_2}}+\overbrace{\int_{e_3}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x}^{\mathbf{K}^{e_3}}+\overbrace{\int_{e_4}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x}^{\mathbf{K}^{e_4}}= \tag{327}$$

$$=\begin{bmatrix}\int_{e_1}B_1B_1\,\mathrm{d}x & 0 & 0 & 0\\ & 0 & 0 & 0\\ \text{sym.} & & 0 & 0\\ & & & 0\end{bmatrix}+\begin{bmatrix}\int_{e_2}B_1B_1\,\mathrm{d}x & \int_{e_2}B_1B_2\,\mathrm{d}x & 0 & 0\\ & \int_{e_2}B_2B_2\,\mathrm{d}x & 0 & 0\\ \text{sym.} & & 0 & 0\\ & & & 0\end{bmatrix}$$

$$+\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \int_{e_3}B_2B_2\,\mathrm{d}x & \int_{e_3}B_2B_3\,\mathrm{d}x & 0\\ \text{sym.} & & \int_{e_3}B_3B_3\,\mathrm{d}x & 0\\ & & & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ & 0 & 0 & 0\\ \text{sym.} & & \int_{e_4}B_3B_3\,\mathrm{d}x & \int_{e_4}B_3B_4\,\mathrm{d}x\\ & & & \int_{e_4}B_4B_4\,\mathrm{d}x\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2}.(2).(2) & 0 & 0 & 0\\ & 0 & 0 & 0\\ \text{sym.} & & 0 & 0\\ & & & 0\end{bmatrix}+\begin{bmatrix}\frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 & 0\\ & \frac{1}{2}.(2).(2) & 0 & 0\\ \text{sym.} & & 0 & 0\\ & & & 0\end{bmatrix}$$

$$+\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0\\ \text{sym.} & & \frac{1}{2}.(2).(2) & 0\\ & & & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ & 0 & 0 & 0\\ \text{sym.} & & \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2)\\ & & & \frac{1}{2}.(2).(2)\end{bmatrix}\Rightarrow$$

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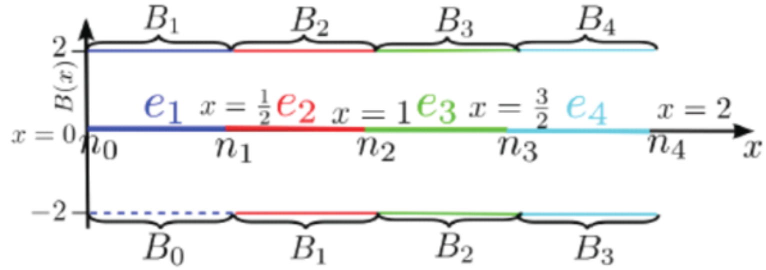
Local approach (element-centered)



$$\mathbf{K}^{e_1}=\begin{bmatrix}2 & 0 & 0 & 0\\ & 0 & 0 & 0\\ \text{sym.} & & 0 & 0\\ & & & 0\end{bmatrix}\qquad \mathbf{K}^{e_2}=\begin{bmatrix}2 & -2 & 0 & 0\\ & 2 & 0 & 0\\ \text{sym.} & & 0 & 0\\ & & & 0\end{bmatrix}$$

$$\mathbf{K}^{e_3}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 2 & -2 & 0\\ \text{sym.} & & 2 & 0\\ & & & 0\end{bmatrix}\qquad \mathbf{K}^{e_4}=\begin{bmatrix}0 & 0 & 0 & 0\\ & 0 & 0 & 0\\ \text{sym.} & & 2 & -2\\ & & & 2\end{bmatrix}$$

$$\mathbf{K}=\mathbf{K}^{e_1}+\mathbf{K}^{e_2}+\mathbf{K}^{e_3}+\mathbf{K}^{e_4}\quad\Rightarrow$$

$$\mathbf{K}=\begin{bmatrix}4 & -2 & 0 & 0\\ & 4 & -2 & 0\\ \text{sym.} & & 4 & -2\\ & & & 2\end{bmatrix}$$

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Comparing global versus local approaches:

Global (node-centered) approach	Local (element-centered) approach

Comparing global versus local approaches:

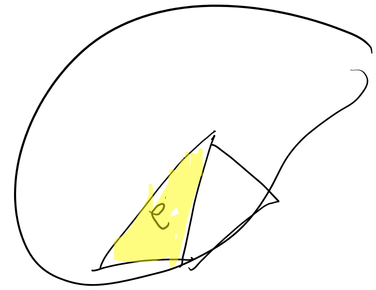
Global (node-centered) approach

$$K = K^{ff} = \int_{\Omega} B^{f^T} D B^f \, dv$$

we can drop f superscript

Local (element-centered) approach

$$k^e = \int_e B_e^T D B_e \, dv$$



$$F = F_e + F_n$$

nodal forces

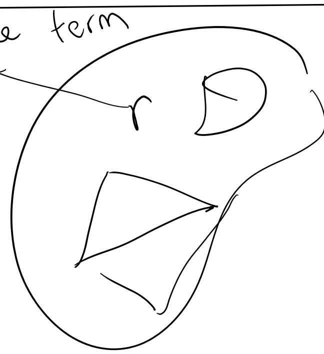
$$F_e = F_r + F_N - F_D$$

$$f_e^e = f_r^e + f_N^e - f_D^e$$

then assembe f_e^e to F^e

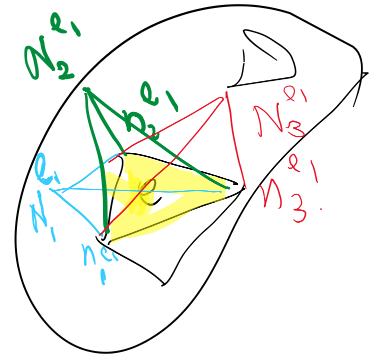
F_r

source term

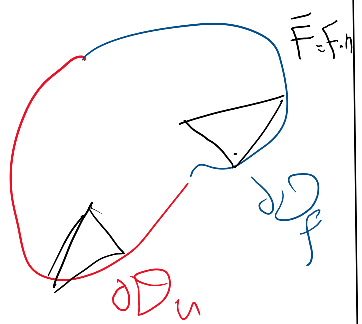


$$F_r = \int_D \begin{bmatrix} N_1^f \\ \vdots \\ N_{n_f}^f \end{bmatrix} r\, dV = \int_D (N^f)^t r\, dV$$

$$f_r^e = \int_e (N^e)^t r\, dV$$

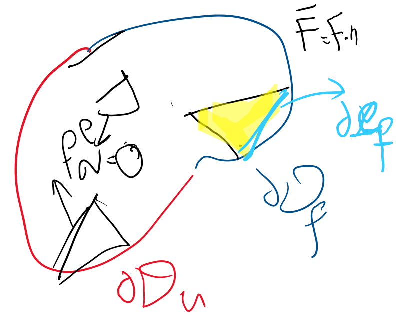


$$F_N = \int_{\partial D_f} \begin{bmatrix} N_1^f \\ \vdots \\ N_{n_f}^f \end{bmatrix} \bar F\, ds = \int_{\partial D_f} (N^f)^t \bar F\, ds$$

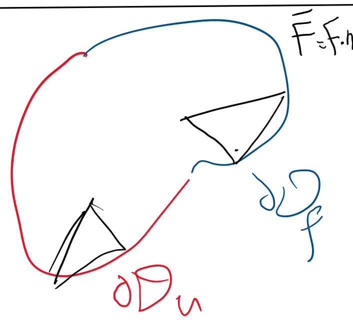


$$f_N^e = \int_{\partial e_f} (N^e)^t \bar F\, ds$$

$$\partial e_f = \partial e \cap \partial D_f$$

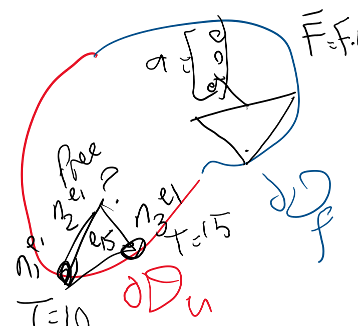


F_D



$$F_D = K^{fp} a^p$$

$$K^{fp} = \int_D B^{f\,T} D B^p\, dV$$



$$f_D^e = k^e a^e$$

$$a^{e_{15}} = \begin{bmatrix} 10 \\ 0 \\ 15 \end{bmatrix}$$

what we generally simply call K

global needs

K^{fp} ?

global approach needs this

$F_p = K^{fp} a^p$

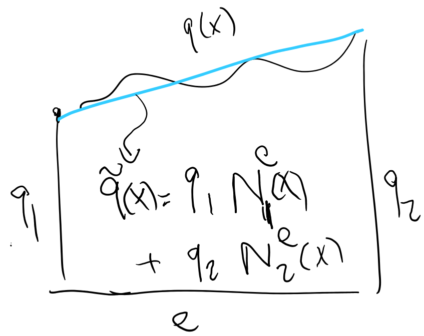
$$K = \begin{bmatrix} K^{ff} & K^{fp} \\ K^{pf} & K^{pp} \end{bmatrix}$$

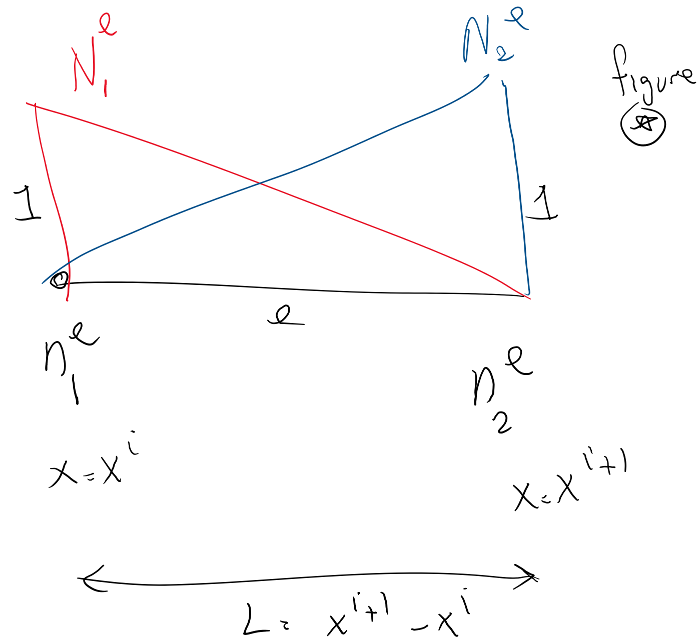
global approach eventually needs these to form reaction forces

Local appoach only works with K^{ff}

Formula for element f_r^e:

$$f_r^e = \int_e \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} q(x)\,dx$$





$$\tilde{q}(n_1) = q_1 N_1^e(n_1) + q_2 N_2^e(n_1) = q_1 \times 1 + q_2 \times 0 = q_1$$

$x_i \qquad x_{i+1}$

$$\tilde{q}(n_2) = q_2$$

$$f_r^e = \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} q(x)\,dx \approx \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} \tilde{q}(x)\,dx$$

$$f_r = \int_{x_i} \begin{bmatrix} N_2^e \end{bmatrix} \cdots \approx \int_{x_i} \begin{bmatrix} N_2^e \end{bmatrix} q(x)\,dx$$

$$= \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} \underbrace{\left(\begin{bmatrix} N_1^e & N_2^e \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \right)}_{\tilde{q}} dx =$$

$$r^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad r^e_{2\times 2} = \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} \begin{bmatrix} N_1^e & N_2^e \end{bmatrix} dx = \int_{x_i}^{x_{i+1}} \begin{bmatrix} \frac{x_{i+1} - x}{L_e} \\ \frac{x - x_i}{L_e} \end{bmatrix} \begin{bmatrix} \frac{x_{i+1} - x}{L_e} & \frac{x - x_i}{L_e} \end{bmatrix} dx$$

$\Longrightarrow$

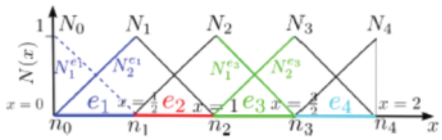
$$f_r^e \approx r^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$r^e = \int_e N^{e\,t} N^e dx = \frac{L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

This approximation of q by qTilde is perfectly fine because any potential error introduced in the final solution by this approximation is of the same or smaller order of finite element discretization (going to a finite number of unknowns)

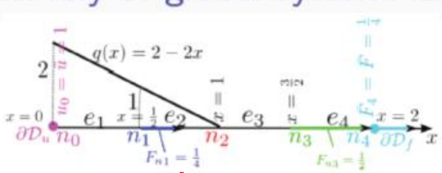
Solve the sample problem with the local approach:
We already solved for K

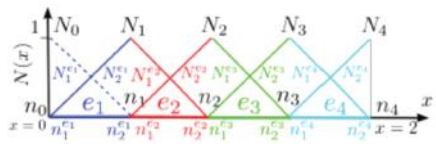
Example: Formation of $\mathbf{K}$ and $\mathbf{F}$



1. Local stiffness matrix: Since $A=1, E=1, L=\frac{1}{2}$ for all elements in the example shown, local stiffness matrix is:
$$\mathbf{k}^{e_i} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, i = 1,2,3,4.$$
2. Assembly to global system: Around the local stiffness matrix for e_3 we have the corresponding dof of the local dof in the global system:
$$\begin{array}{c c} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$
This means that for example $k_{11}^{e_3}$ will be added to K_{22}, $k_{12}^{e_3}$ to K_{23} and so forth:
$$\mathbf{K}^{e_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ \text{sym.} & & 2 & 0 \\ & & & 0 \end{bmatrix}$$

Assembly of global system: Bar example





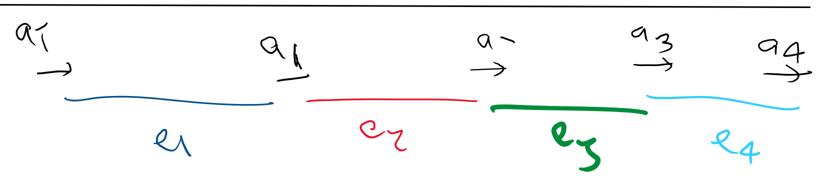
e	e_1	e_2	e_3	e_4
$\mathbf{k}^e$	$\begin{array}{cc} & 1\ 1 \\ 1 & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ 1 & \end{array}$	$\begin{array}{cc} & 1\ 2 \\ 1 & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ 2 & \end{array}$	$\begin{array}{cc} & 2\ 3 \\ 2 & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ 3 & \end{array}$	$\begin{array}{cc} & 3\ 4 \\ 3 & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ 4 & \end{array}$
$\mathbf{f}_r^e$	$\mathbf{r}^{e_1}\mathbf{q}^{e_1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{array}{c} 1 \\ 1 \end{array}\begin{bmatrix} \frac{5}{12} \\ \frac{4}{12} \end{bmatrix}$	$\mathbf{r}^{e_2}\mathbf{q}^{e_2} = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} \frac{2}{12} \\ \frac{1}{12} \end{bmatrix}$	$\mathbf{f}_r^{e_3} = \mathbf{r}^{e_3}\mathbf{q}^{e_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_r^{e_4} = \mathbf{r}^{e_4}\mathbf{q}^{e_4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{array}{c} 1 \\ 1 \end{array}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_3}\mathbf{a}_3^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_4}\mathbf{a}_4^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{array}{c} 1 \\ 1 \end{array}\begin{bmatrix} -\frac{19}{12} \\ \frac{28}{12} \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} \frac{2}{12} \\ \frac{1}{12} \end{bmatrix}$	$\mathbf{f}_e^{e_3} = \begin{array}{c} 2 \\ 3 \end{array}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_4} = \begin{array}{c} 3 \\ 4 \end{array}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\mathbf{K} = \begin{bmatrix} 2+2 & -2 & 0 & 0 \\ -2 & 2+2 & -2 & 0 \\ 0 & -2 & 2+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}, \mathbf{F} = \mathbf{F}_n + \mathbf{F}_e = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{28}{12} + \frac{2}{12} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{4} \\ \frac{1}{12} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \Rightarrow$$

not done yet

$$\mathbf{a} = \mathbf{K}^{-1}\mathbf{F} = \begin{bmatrix} \frac{43}{24} \\ \frac{53}{24} \\ \frac{31}{12} \\ \frac{65}{24} \end{bmatrix}$$ which matches our solution from global approach (321)

Calculate e_1



$f_D^{e_1} = k^{e_1} a^{e_1}$ $\qquad a^{e_1}$

$M(e_1) = [\bar{1} \ 1]$

$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$a^{e_1} = \begin{bmatrix} a_1^- \\ \ \end{bmatrix} = \begin{bmatrix} 1 \\ \ \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

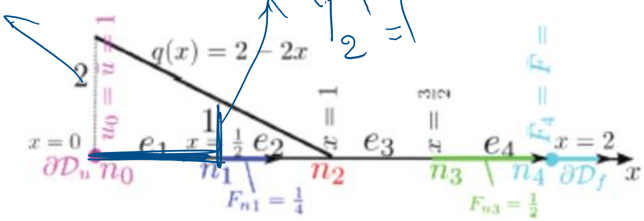
$$a^{e_1} = \begin{bmatrix} a_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$r^{e_1} = \frac{L_{e_1}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\frac{1}{2}$

$$f_r^{e_1} = r^{e_1} q^{e_1}$$

$q_1^{e_1} = 2$ $\quad$ $q_2^{e_1} = 1$



$$f_r^{e_1} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/12 \\ 4/12 \end{bmatrix}$$