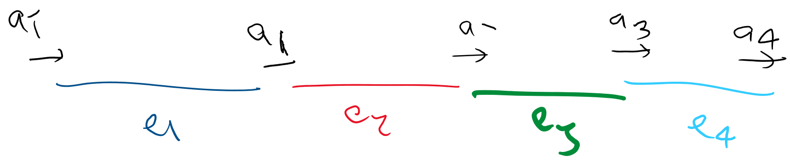
2023/11/01

Wednesday, November 1, 2023 11:11 AM

From last time:

Calculate e_1



$$f_D^{e_1} = K^{e_1} a^{e_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$M(e_1) = [\bar{1} \ 1]$

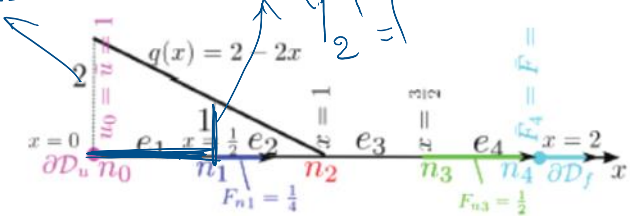
$$a^{e_1} = \begin{bmatrix} a_{\bar{1}} \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$r^{e_1} = \frac{L_{e_1}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\frac{1}{2}$

$$f_r^{e_1} = r^{e_1} q^{e_1}$$

$q_1^{e_1} = 2$ $\quad q_2^{e_1} = 1$

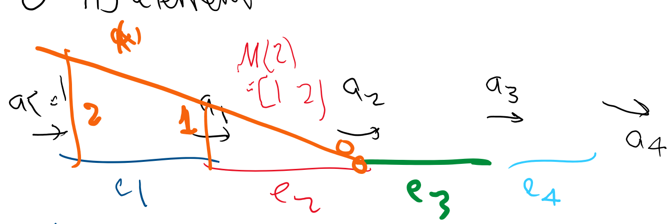


$$f_r^{e_1} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/12 \\ 4/12 \end{bmatrix}$$

last time

$$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1} = \begin{bmatrix} 5/12 \\ 4/12 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{matrix} \bar{1} \\ 1 \end{matrix} \begin{bmatrix} -19/12 \\ 28/12 \end{bmatrix}$$

0 1D element



$M(2) = [1 \ 2]$

$M(1) = [\bar{1}, 1]$ $\quad n_f = 4$

$$F^e = \begin{array}{c|c} 1 & 28/12 + 2/12 \\ 2 & 1/12 \\ 3 & \\ 4 & \end{array}$$

Element 2 $\quad f_D^{e_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

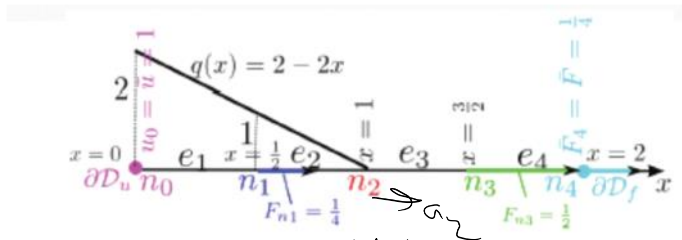
$$f^{e_2} = f_r^{e_2} = r^{e_2} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{L^{e_2}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/12 \\ 1/12 \end{bmatrix}$$

$$f^{e_2} = f_r^{e_2} = r^{e_2}\begin{bmatrix}q_1\\q_2\end{bmatrix} = \frac{L^{e_2}}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}q_1\\q_2\end{bmatrix} = \frac{1}{12}\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2/12\\1/12\end{bmatrix}$$

$$f^{e_2} = f_r^{e_2} + \underbrace{f_N^{e_2}}_{0} - \underbrace{f_D^{e_2}}_{0}$$

$$f^{e_3} = \begin{bmatrix}0\\0\end{bmatrix} \qquad f^{e_4} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$F_e = \begin{bmatrix}3/12\\1/12\\0\\0\end{bmatrix}$$



$$F_n = \begin{bmatrix}1/4\\0\\1/2\\1/4\end{bmatrix} \qquad F = F_e + F_n = \begin{bmatrix}11/4\\1/12\\1/2\\1/4\end{bmatrix}$$

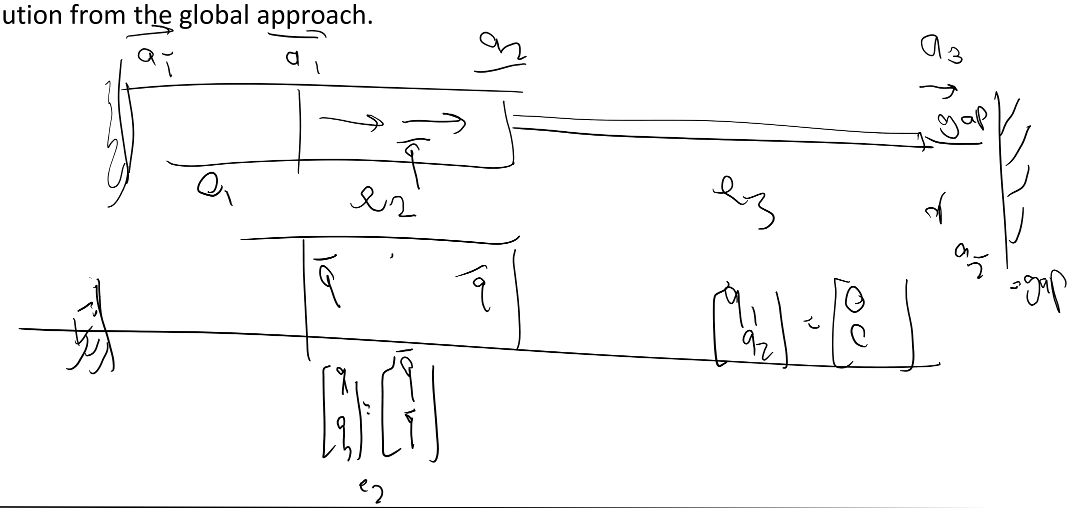
$$a = K^{-1}F = \begin{bmatrix}43/24\\53/24\\31/24\\65/24\end{bmatrix} \rightarrow a_2$$

?

This matches the solution from the global approach.

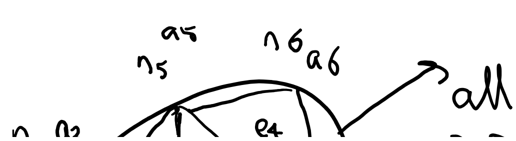
HW4

q(x)



Why element forces assemble the way we saw above
At the element level we have 3 forces:

$$f^e = \underbrace{f_r^e}_{\text{Source}} + \underbrace{f_N^e}_{\text{Neumann}} - \underbrace{f_D^e}_{\text{Dirichlet}}$$



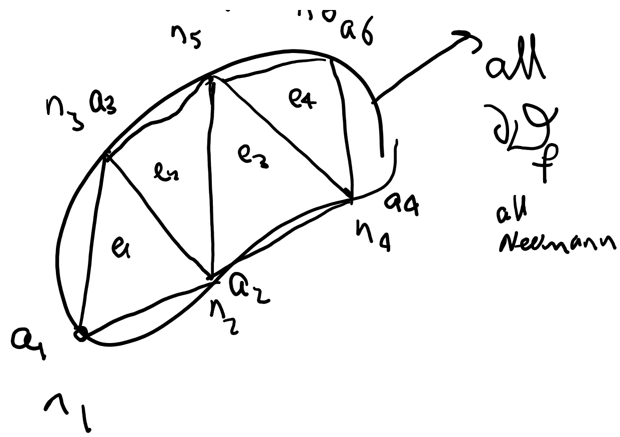
Source Neumann Dirichlet

$n_f = 6$

$n_p = 0$ (- - -)

$$F_f = \int_{\Omega} \begin{bmatrix} N_1 \\ \vdots \\ N_6 \end{bmatrix} Q \, dv$$

heat conduction problem: $r = Q$



2D heat conduction

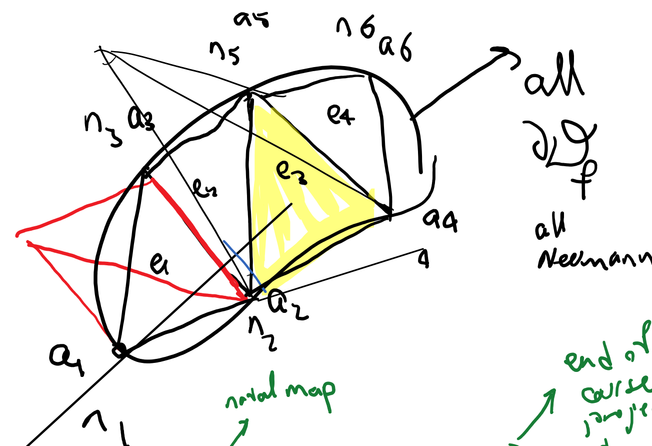
T (temperature) unknown

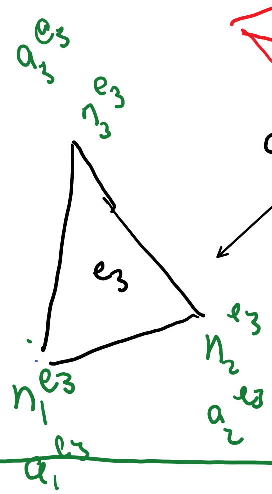
1 dof / node

$$= \underbrace{\int_{e_1} N Q \, dv}_{F_f^{e_1}} \; \cdots \; + \int_{e_4} N Q \, dv$$

$$F_f^{e_3} = \int_{e_3} \begin{bmatrix} \cancel{N_1} \to 0 \\ N_2 \\ \cancel{N_3} \\ N_4 \\ N_5 \\ \cancel{N_6} \to 0 \end{bmatrix} Q \, dv$$

$$= \int_{e_3} \begin{bmatrix} 0 \\ N_2 \\ 0 \\ N_4 \\ N_5 \\ 0 \end{bmatrix} Q \, dv$$





$$= \int_{e_3} \begin{bmatrix} 0 \\ N_1^{e_3} \\ 0 \\ N_2^{e_3} \\ N_3^{e_3} \\ 0 \end{bmatrix} Q \, dv$$

nodal map

$LEM_{e_3} = [2, 4, 5]$ → end of course project text input

$M_{e_3} = [2, 4, 5]$

dof map

$$f_f^{e_3} = \int \begin{bmatrix} N_1^{e_3} \\ N_2^{e_3} \\ N_3^{e_3} \end{bmatrix} Q \, dv = \begin{array}{c} 2 \\ 4 \\ 5 \end{array} \begin{bmatrix} \int_{e_3} N_1^{e_3} Q \, dv \\ \int N_2^{e_3} Q \, dv \\ \int N_3^{e_3} Q \, dv \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \to \int_{e_3} N^{e_3} Q \, dv \\ 0 \\ \int_{e_3} N_2^{e_3} Q \, dv \\ \int N_3^{e_3} Q \, dv \\ 0 \end{bmatrix}$$

the local f_f^e assembled to global F_f matches contributions of element e

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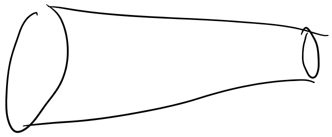
Calculating the stiffness matrix for a 1D bar

Calculating the stiffness matrix for a 1D bar
In the very first class I demonstrated

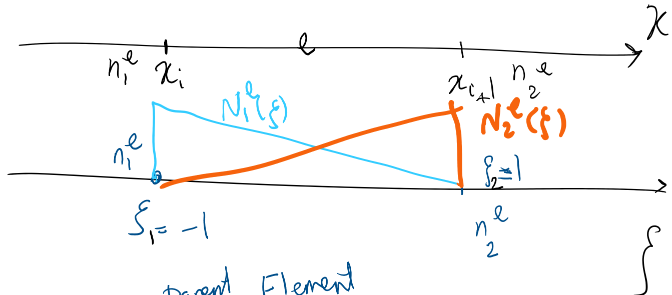
$F = \frac{AE}{L} \Delta u \rightarrow \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

General formula for K in these cases



A(x) or E(x)

$$\begin{cases} N_1^e(\xi = -1) = 1 \\ N_1^e(\xi = 1) = 0 \end{cases}$$



$$N_1^e(\xi) = \underbrace{a_0 + a_1 \xi}_{\text{sufficient}} + \cancel{a_2 \xi^2 + \dots}$$

$N_1^e(\xi = -1) = a_0 - a_1 = 1$

$N_1^e(\xi = 1) = a_0 + a_1 = 0$

$a_0 = \frac{1}{2} \quad a_1 = -\frac{1}{2}$

$$\boxed{N_1^e(\xi) = \frac{1-\xi}{2}}$$

same process

$$\boxed{N_2^e(\xi) = \frac{1+\xi}{2}}$$

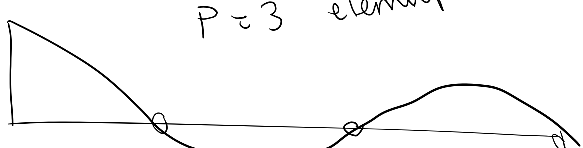
Easier way (Lagrange polynomials)

$N_1^e(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{\xi - (1)}{-1 - 1} = \frac{1-\xi}{2}$

$N_2^e(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \frac{\xi - (-1)}{1 - (-1)} = \frac{\xi + 1}{2}$

Later

P = 3 element



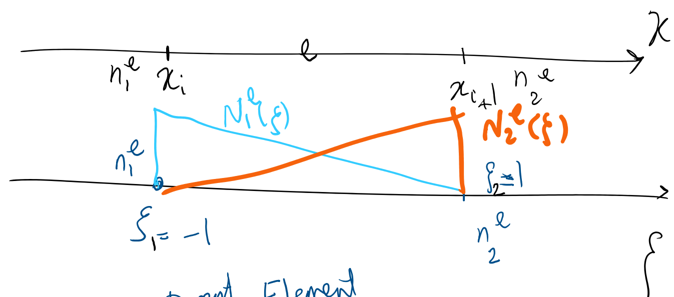
this approach right away gives the shape functions

$\int \omega' EA u' dx$

$\mathcal{D} \; h_m(\;)' \quad D = EA$

$= \frac{\partial}{\partial x}$

$$\boxed{k^e = \int_{x_i}^{x_{i+1}} B^{e^T} D B^e \, dx} \quad (1)$$



$N^e = [N_e^1 \;\; N_e^2]$

$$B^e = \frac{d}{dx}[N_e^1(\xi) \;\; N_e^2(\xi)] = \underbrace{\frac{d[N_1^e(\xi) \;\; N_2^e(\xi)]}{d\xi}}_{B^e_\xi} \cdot \frac{d\xi}{dx}$$

$x \quad \cancel{X}$

$x \quad \cancel{\xi}$

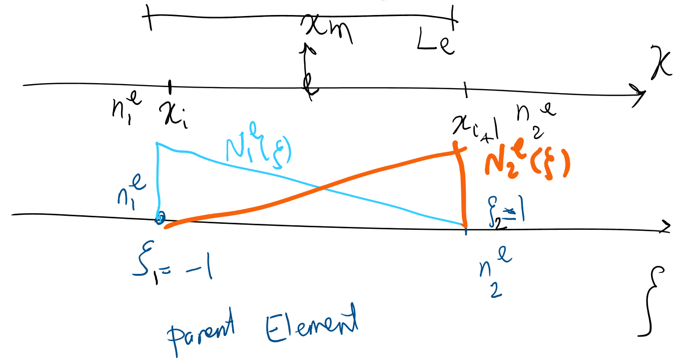
$$(1) \quad : \boxed{K^e = \int_{\xi=-1}^{\xi=1} B^{eT} \, D \, \overbrace{B^e}^{B^e_\xi} \Big(\underbrace{\frac{dx}{d\xi}}_{J = \frac{dx}{d\xi}} \Big) d\xi} \quad (3)$$

$$\boxed{B^e = \frac{1}{\frac{dx}{d\xi}} B^e_\xi} \quad (2)$$

we need $x(\xi)$

$\xi_1 = -1 \longrightarrow x_i$

$\xi_2 = 1 \longrightarrow x_{i+1}$



$$x(\xi) = a_0 + a_1 \xi$$

$$\left.\begin{aligned} a_0 + a_1(-1) &= x_i \\ a_0 + a_1(1) &= x_{i+1} \end{aligned}\right\} \rightarrow \quad a_0 = \frac{x_i + x_{i+1}}{2} = x_m, \qquad a_1 = \frac{x_{i+1} - x_i}{2} = \frac{L_e}{2}$$

$$\boxed{\begin{aligned} x(\xi) &= x_m + \frac{L_e}{2}\xi \\ J &= \frac{dx}{d\xi} = \frac{L_e}{2} \end{aligned}} \quad (4)$$

$$(3),(2) \Rightarrow \quad K^e = \int_{\xi=-1}^{\xi=1} \underbrace{\left(\frac{1}{J} \begin{bmatrix} B_{\xi_1} \\ B_{\xi_2} \end{bmatrix} \right)}_{B_e^T} \underbrace{EA(\xi)}_{D(\xi)} \left(\frac{1}{J} \begin{bmatrix} B_{\xi_1} & B_{\xi_2} \end{bmatrix} \right) \underbrace{(J d\xi)}_{dx}$$

$$K^e = \int_{\xi=-1}^{\xi=1} \frac{EA(\xi)}{J} \begin{bmatrix} B_{\xi_1} \\ B_{\xi_2} \end{bmatrix} \begin{bmatrix} B_{\xi_1} & B_{\xi_2} \end{bmatrix} d\xi = \int_{\xi=-1}^{\xi=1} \frac{EA(\xi)}{\underset{\searrow \frac{L_e}{2}}{J}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} d\xi \Rightarrow$$

note

$$N = \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$$

$$B_\xi = \frac{dN}{d\xi} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

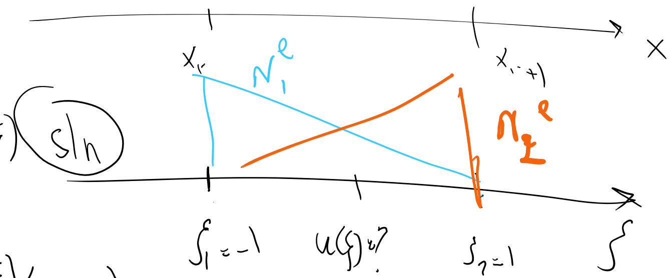
(5) Finite Element Stiffness matrix

$$K^e = \frac{\int_{-1}^{1} AE(\xi) \, d\xi}{2L^e} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

for constant AE

$$k^e = \left(\frac{AE}{L}\right)_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Side note: is any easier way to form $X(\xi)$?



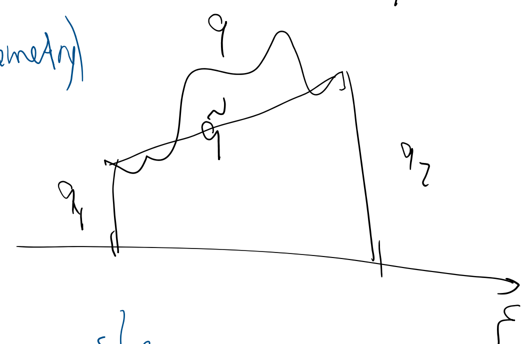
1) $u(\xi) = a_1^e N_1(\xi) + a_2^e N_2(\xi)$ s/n

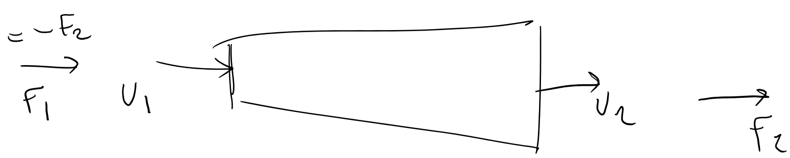
2) $q(\xi) = q_1 N_1(\xi) + q_2 N_2(\xi)$ / source term

3) $x(\xi) = \underbrace{x_1}_{\text{coordinate @ node}} N_1(\xi) + x_2 N_2(\xi)$ (geometry)

$= x_i \left(\frac{1-\xi}{2}\right) + x_{i+1}\left(\frac{1+\xi}{2}\right)$

$= \frac{x_i + x_{i+1}}{2} + \xi\left(\frac{x_{i+1} - x_i}{2}\right) = x_M + \xi \frac{L_e}{2}$



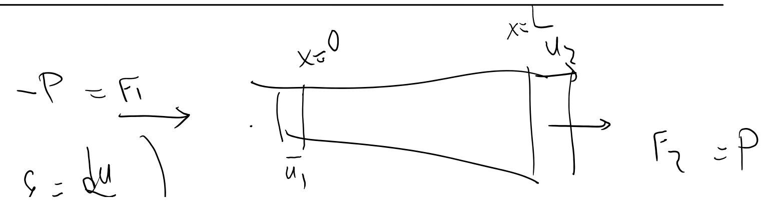


$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \underbrace{\left(\frac{\int_{-1}^{1} AE(\xi)\, d\xi}{2L_e}\right)}_{k^{FEM}} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow F_2 = k^{FEM} \underbrace{(u_2 - u_1)}_{\Delta u}$$

$F_1 = -F_2$

k^{FEM} stiffness is not exact for $AE \neq$ constant

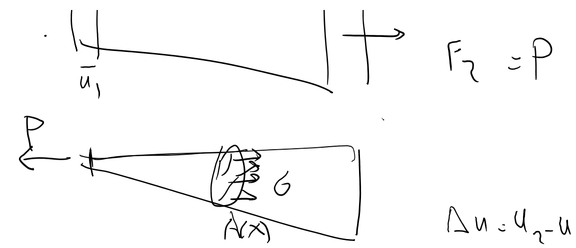
Exact solution



$\Delta u = u_2 - u_1 = \int_0^L \varepsilon\, dx \quad \left(\varepsilon = \frac{du}{dx} \right)$

$$\Delta u = u_2 - u_1 = \int_0^L \varepsilon \, dx \qquad \left(\varepsilon = \frac{du}{dx} \right)$$

$$\int du = \int \varepsilon \, dx$$



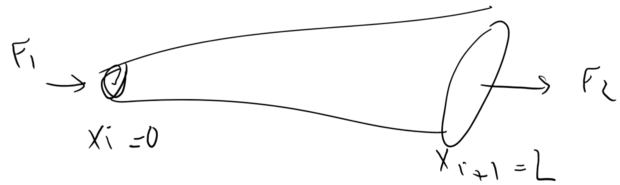
$$\Delta u = u_2 - u_1 = \int_0^L \varepsilon \, dx = \int_0^L \left(\frac{\sigma}{E} \right) dx = \int_0^L \frac{P}{A(x) E(x)} dx \qquad \sigma = \frac{P}{A}$$

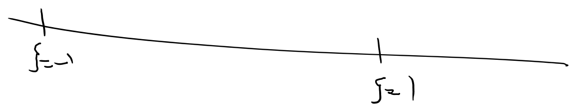
stress

$$\Rightarrow \Delta u = \left(\int_0^L \frac{dx}{AE(x)} \right) P \Longrightarrow k^{exact} = \frac{P}{\Delta u} = \frac{1}{\int_0^L \frac{dx}{AE(x)}}$$

F_2

Comparisons





$$K^{FEM} = \frac{\int_{\xi=-1}^{1} AE(\xi)\,d\xi}{2L} = \frac{\int_0^L AE(x)\,dx}{L^2}$$

$$K^{exact} = \frac{1}{\int_0^L \frac{dx}{AE(x)}}$$

$$K^e = k\begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

6

FEM

$$u = \overset{a_1}{\tilde{u}_1} N_1 + u_2 N_1$$

$$\varepsilon = \frac{du}{dx}$$

$$\sigma = E\varepsilon$$

$$\underline{F = \sigma A}$$

$$u = \int d\varepsilon$$

$$\varepsilon = \varepsilon = \frac{\sigma}{E}$$

$$\sigma = \frac{P}{A}$$

$$P = \text{const}$$

Exact