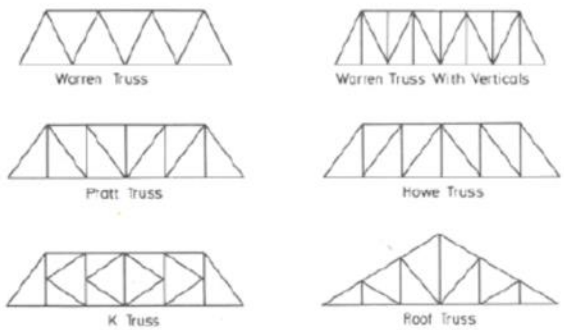
2023/11/06

Monday, November 6, 2023 11:20 AM

Trusses

Trusses

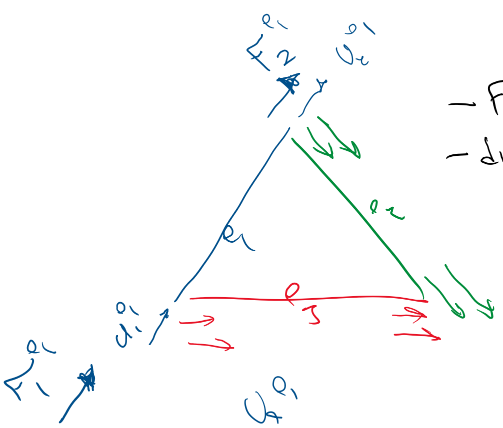


Types of simple Plane truss

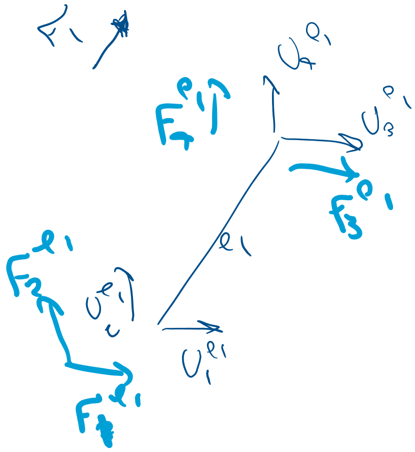
- Trusses are 2D or 3D ensemble of bars.
- The main load transfer mechanism of these bars is axial force as the hinge connection at nodes prevent generation and transfer of moments.
- Although in bar elements we could have body force, in trusses we do not apply any type of load between the nodes (except the weight of bars themselves which may be neglected in many applications).
- Generally, top and bottom bars carry the moments and middle diagonal and vertical bars carry shear forces if we think of truss as a big bar.

312 / 456

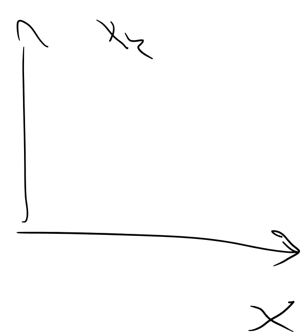
New concept for trusses: Coordinate transformation (rotation) between element and global coordinate systems



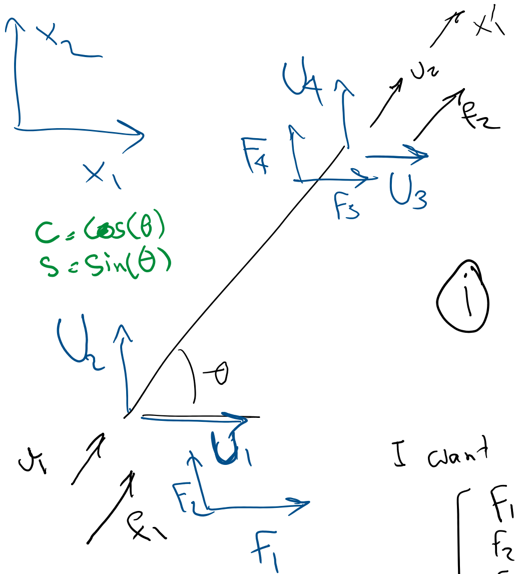
- Forces cannot easily be added
- displacement continuity cannot easily be enforced



now everything is in a common coordinate system & "hand shake is possible"



Goal: Obtain the element stiffness matrix in the global coordinate system
Basically truss elements are bars that are arbitrary aligned in 2D/ 3D



local coordinate values, lower case u & f

we already know k in local coordinate system (bar stiffness)

(i) $$\underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}_{2\times1}}_{f} = k_{2\times2} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u} {}_{2\times1}$$

$$k = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

I want

$$\underbrace{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}_{4\times1}}_{F} = K_{4\times4} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{U} {}_{4\times1}$$

i) $f = ku$

what if we had $u = \underbrace{(T_{uU})}_{\text{Transfer from } U \text{ to } u} U$ $\Rightarrow f = (k T_{uU}) U$ (ii)

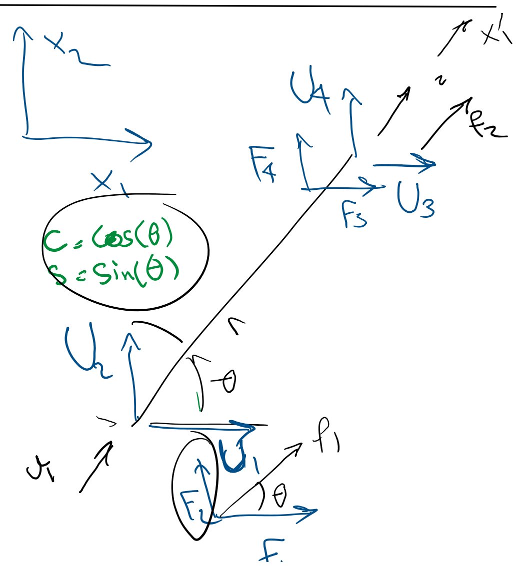
(iii)

$$F = \overset{\text{if we have this}}{T_{Ff}}\, f \quad \text{from ii we have} \quad F = T_{Ff} f = \underbrace{(T_{Ff}\, k\, T_{uU})}_{\text{this will be our } K \text{ in global coordinate system}} U$$

$T_{uU} = ?$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}_{2\times4} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{4\times1}$$

(iv) $\rightarrow (T_{uU})_{2\times4}$



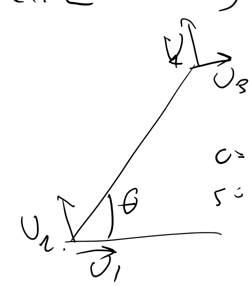
$(T_{Ff})_{4\times2}$

4×2

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \underbrace{\begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}_{4\times2}}_{(v)\quad T_{Ff} = (T_{uU})^t} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Plug T_{uU} (eq (iv)) & T_{Ff} eqn (v)) into (iii) to ~~get~~ get

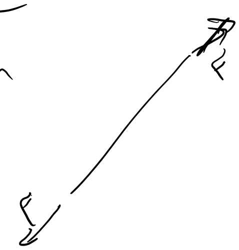
$$K = \frac{AE}{L}\left[\begin{array}{c|c} k_b & -k_b \\ \hline -k_b & k_b \end{array}\right] \qquad \underset{2\times2}{k_b} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \tag{1}$$



$c = \cos\theta$
$s = \sin\theta$

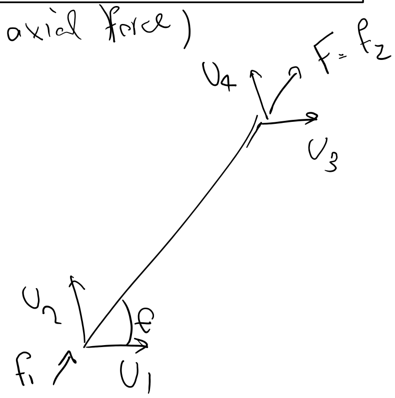
Compare this with

$F = \frac{AE}{L}\Delta u$



BTW we also like to have the value of F (axial force)

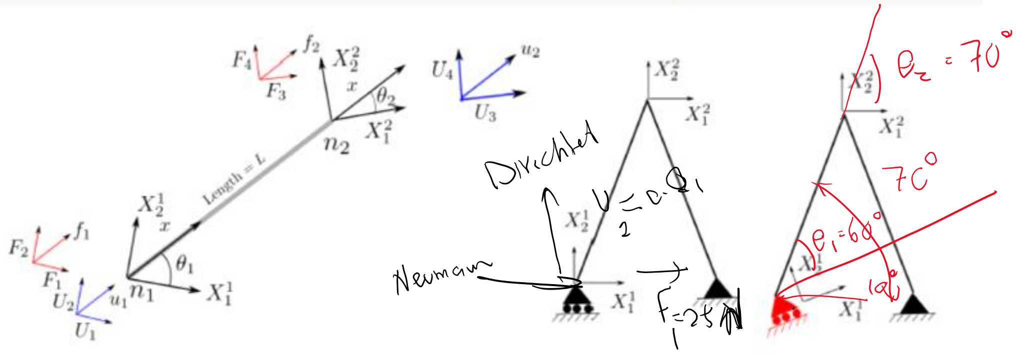
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = (k)_{2\times2} (T_{uU})_{2\times4} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$ eqn (ii) above

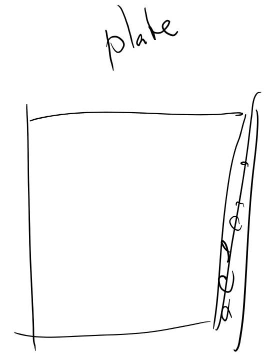


$$\boxed{f_2 = F = \frac{AE}{L}\Big(c\,(U_3 - U_1) + s\,(U_4 - U_2)\Big)} \quad (2)$$

There are cases that the angles at the two ends are not equal (we don't have the same "global coordinate system" at the two ends of the truss element

Truss element /two different coordinate systems

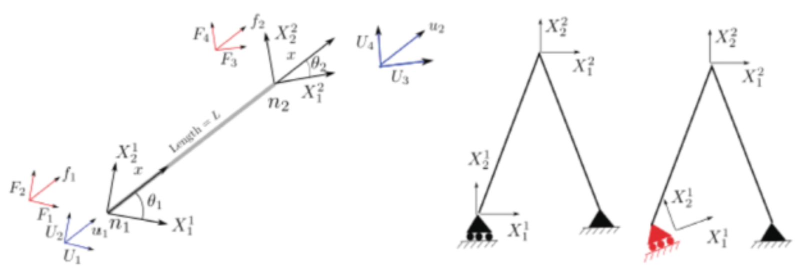




- In some instances we need to employ two different coordinate systems at the end points of a bar or in general coordinate system(s) that are not aligned with global coordinate system.
- For example the support highlighted in red in the right figure, do decouple displacement at the support and set the normal displacement to zero (Dirichlet BC) and tangential one free (Neumann BC) we need to employ the rotated coordinate system X_1^1, X_2^1.
- We have two different angles, θ_1 and θ_2. We define,

$$c_1 = \cos(\theta_1) \qquad s_1 = \sin(\theta_1)$$
$$c_2 = \cos(\theta_2) \qquad s_2 = \sin(\theta_2)$$

Truss element /two different coordinate systems



- As before $\mathbf{T} := \mathbf{T_{uU}} = \mathbf{T_{Ff}}$ and in this case is given by,

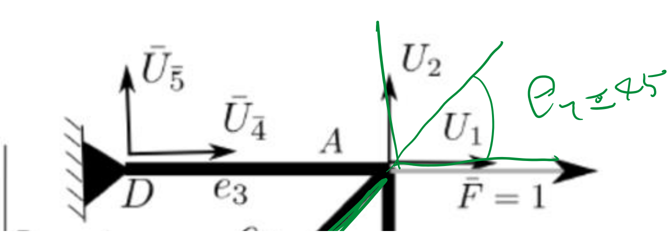
$$\mathbf{T} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & c_2 & s_2 \end{bmatrix} \tag{393}$$

- Accordingly, from $\mathbf{K} = \mathbf{T}^{\mathrm{T}}\mathbf{k}\mathbf{T}$ we obtain,

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} c_1^2 & c_1 s_1 & -c_1 c_2 & -c_1 s_2 \\ c_1 s_1 & s_1^2 & -c_2 s_1 & -s_1 s_2 \\ -c_1 c_2 & -c_2 s_1 & c_2^2 & c_2 s_2 \\ -c_1 s_2 & -s_1 s_2 & c_2 s_2 & s_2^2 \end{bmatrix} \tag{394}$$

- Finally the axial tensile force in the bar, which is the second line of $\mathbf{k}\mathbf{T_{uU}} = \mathbf{k}\mathbf{T}$ is (compare to one global coordinate in (387)):

$$\boxed{T = AE/L\,(-c_1 U_1 - s_1 U_2 + c_2 U_3 + s_2 U_4)} \tag{395}$$



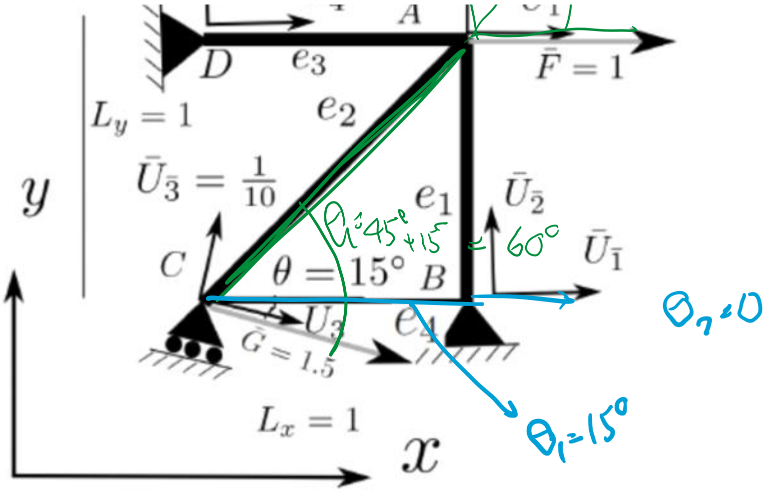
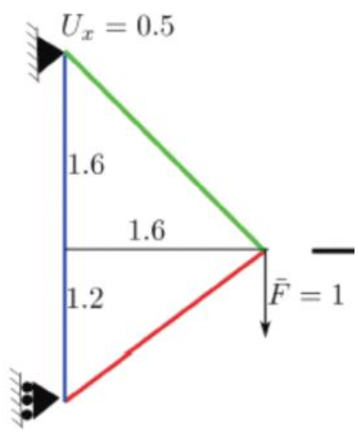
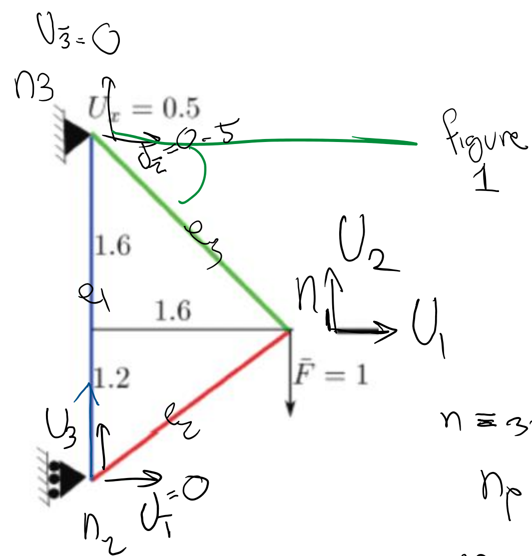


Figure 2: 3 dof truss with an angled support

Don't submit this problem in HW4

Truss example





3 nodes, 2 dof/node

$n = 3 * 2 = 6$ dof

$n_p = 3$

$n_f = n - n_p = 3$

$LEM(e_1) = [2, 3]$

$LEM(e_2) = [2, 1]$

$LEM(e_3) = [3, 1]$ $\quad L_{e_3} = \sqrt{(1.6)^2 + (1.6)^2} = (\sqrt{2})\,1.6$

$\theta_{e_3} = -45°$

$M(e_3) = [\bar{2}, \bar{3}, 1, 2]$ dof Map

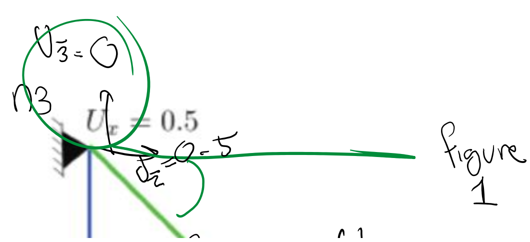
Element 3 calculations:

$\theta_{e_3} = -45°$

$L_{e_3} = \sqrt{2}(1.6)$

$(AE)_{e_3} = 1$

$$K_{e_3} = \left(\frac{AE}{L}\right)_{e_3} \left[\begin{array}{cc|c} c^2 & sc & \\ sc & s^2 & \\ \hline - & & + \end{array}\right]$$



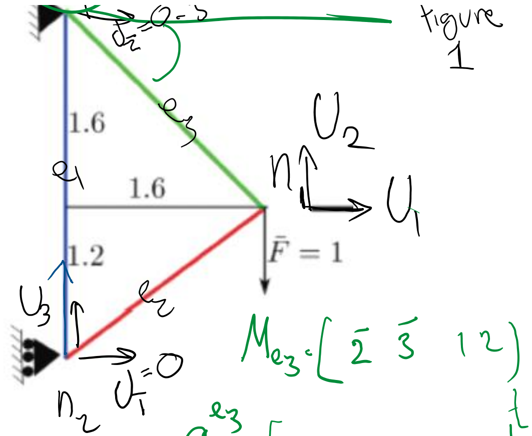
$(AE)_{e_3} = 1$

$c = \cos 45° = \frac{\sqrt{2}}{2}$

$s = \sin 45° = \frac{\sqrt{2}}{2}$

$= \frac{1}{1.6\sqrt{2}} \left[\begin{array}{cc|cc} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \hline & & & \end{array}\right]$

$K^{e_3} = \left[\begin{array}{cc|c} .221 & -.221 & - \\ -.221 & .221 & \\ \hline - & & + \end{array}\right]$



$M_{e_3} = [\bar{2} \;\; \bar{3} \;\; 1 \;\; 2]$

$a^{e_3} = [.5 \;\; 0 \;\; ? \;\; ?]^t$

$a^{e_3} = [.5 \;\; 0 \;\; 0 \;\; 0]^t$

free dofs

forces for e_3:

$f^{e_3} = \cancel{f_r^{e_3}} + \cancel{f_N^{e_3}} - f_D^{e_3}$

0

1D element

"Neumann BC" = collection of nodes

$\longrightarrow F_n$ (global nodal forces)

$f_D^{e_3} = K^{e_3} a^{e_3} = \left[\begin{array}{cc|c} .221 & -.221 & - \\ -.221 & .221 & \\ \hline - & & + \end{array}\right] \begin{bmatrix} .5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow f_D^{e_3} = \begin{bmatrix} .1105 \\ -.1105 \\ -.1105 \\ .1105 \end{bmatrix} \longrightarrow f^{e_3} = \cancel{f_r} + \cancel{f_N} - f_D = \begin{bmatrix} -.1105 \\ .1105 \\ .1105 \\ -.1105 \end{bmatrix}$

0 0

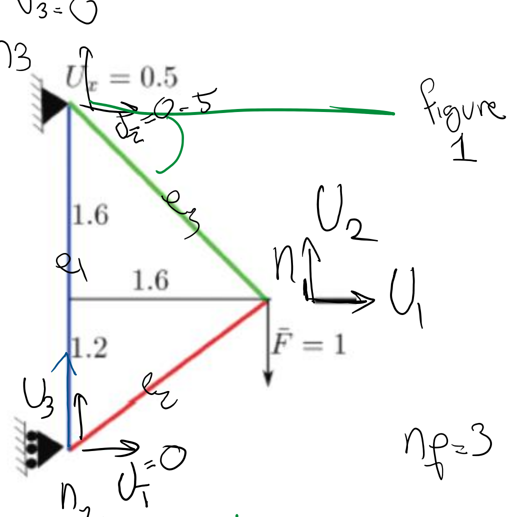
Assemble element 3

$K = \left[\begin{array}{c|c|c|c} .221 & -.221 & & \\ \hline -.221 & .221 & & \\ \hline & & & \end{array}\right]$

$$k^{e_3} = \begin{array}{c|cccc} & \bar{2} & \bar{3} & 1 & 2 \\ \hline \bar{2} & & & & \\ \bar{3} & & & & \\ 1 & & & .22 & -.22 \\ 2 & & & -.22 & .22 \end{array}$$

$U_{\bar{3}} = 0$

n3



$F = F_n + F_e = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{F_n} + f_e$

$F_e = \begin{bmatrix} +.1105 \\ -.1105 \\ \hline \\ \end{bmatrix}$

$n_f = 3$

$M(e_3) = [\bar{2} \;\; \bar{3} \;\; 1 \;\; 2]$

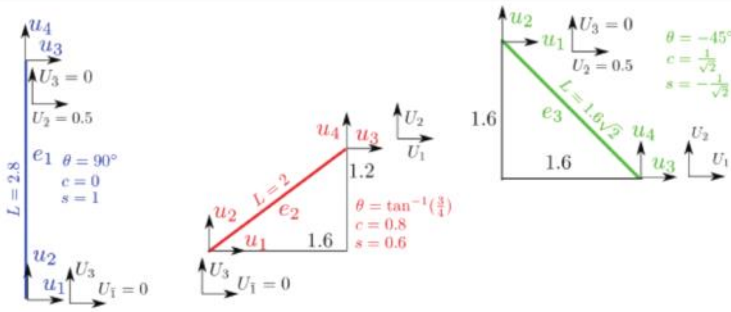
$f^{e_3} = \begin{array}{c|c} \bar{2} & -.1105 \\ \bar{3} & .1105 \end{array}$

$$f_D^{e_3} = \begin{array}{c} \bar{2} \\ \bar{3} \\ 1 \\ 2 \end{array} \begin{bmatrix} -.1105 \\ .1105 \\ .1105 \\ -.1105 \end{bmatrix}$$

$u_2 \quad u_1$

$M(e_3) = [\bar{2} \;\; \bar{3} \;\; 1 \;\; 2]$

Truss Example



- Table below summarizes parameters for each element:

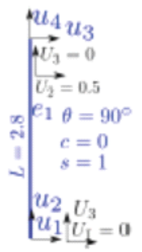
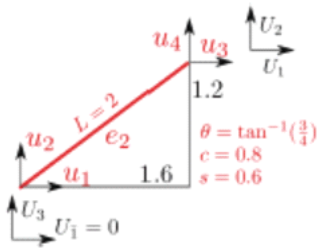
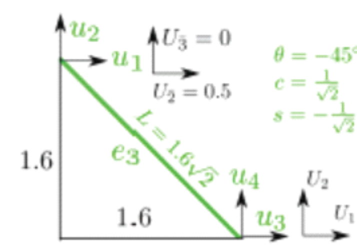
e	L^e	θ	c	s	$\mathbf{M}_t^e$
e_1	2.8	90°	0	1	$[\bar{1} \;\; 3 \;\; \bar{2} \;\; \bar{3}]$
e_2	2	$\tan^{-1}(\frac{3}{4})$	0.8	0.6	$[\bar{1} \;\; 3 \;\; 1 \;\; 2]$
e_3	$1.6\sqrt{2}$	−45°	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$[\bar{2} \;\; \bar{3} \;\; 1 \;\; 2]$

- Local stiffness matrices are given by (390):

$$\mathbf{k}^e = \frac{AE}{L}\left[\begin{array}{c|c} \mathbf{k}_b & -\mathbf{k}_b \\ \hline -\mathbf{k}_b & \mathbf{k}_b \end{array}\right], \; \mathbf{k}_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \; : \; \mathbf{k}^e = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

- As mentioned for trusses generally $\mathbf{f}_r^e = 0$ (no body force), similar to bars we lump natural BC into nodal forces, and finally $\mathbf{f}_D^e = \mathbf{k}^e\mathbf{a}^e$.

Truss example: Assembly of global system

e	e_1	e_2	e_3
			
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{(1)(1)}{2.8}\begin{bmatrix}0&0&0&0\\0&1&0&-1\\0&0&0&0\\0&-1&0&1\end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar{1} & 3 & \bar{2} & \bar{3}\\ \hline \bar{1} & 0 & 0 & 0 & 0\\ 3 & 0 & 0.3571 & 0 & -0.3571\\ \bar{2} & 0 & 0 & 0 & 0\\ \bar{3} & 0 & -0.3571 & 0 & 0.3571\end{array}$	$\mathbf{k}^{e_2} = \frac{(1)(1)}{2}\begin{bmatrix}0.64&0.48&-0.64&-0.48\\0.48&0.36&-0.48&-0.36\\-0.64&-0.48&0.64&0.48\\-0.48&-0.36&0.48&0.36\end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar{1} & 3 & 1 & 2\\ \hline \bar{1} & 0.32 & 0.24 & -0.32 & -0.24\\ 3 & 0.24 & 0.18 & -0.24 & -0.18\\ 1 & -0.32 & -0.24 & 0.32 & 0.24\\ 2 & -0.24 & -0.18 & 0.24 & 0.18\end{array}$	$\mathbf{k}^{e_3} = \frac{(1)(1)}{1.6\sqrt{2}}\begin{bmatrix}0.5&-0.5&-0.5&0.5\\-0.5&0.5&0.5&-0.5\\-0.5&0.5&0.5&-0.5\\0.5&-0.5&-0.5&0.5\end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar{2} & \bar{3} & 1 & 2\\ \hline \bar{2} & 0.221 & -0.221 & -0.221 & 0.221\\ \bar{3} & -0.221 & 0.221 & 0.221 & -0.221\\ 1 & -0.221 & 0.221 & 0.221 & -0.221\\ 2 & 0.221 & -0.221 & -0.221 & 0.221\end{array}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e = \begin{bmatrix}0&0&0&0\\0&0.3571&0&-0.3571\\0&0&0&0\\0&-0.3571&0&0.3571\end{bmatrix}\begin{bmatrix}0\\0\\0.5\\0\end{bmatrix} = \begin{array}{c\|c} \bar{1} & 0\\ 3 & 0\\ \bar{2} & 0\\ \bar{3} & 0\end{array}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e = \begin{bmatrix}0.32&0.24&-0.32&-0.24\\0.24&0.18&-0.24&-0.18\\-0.32&-0.24&0.32&0.24\\-0.24&-0.18&0.24&0.18\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{array}{c\|c} \bar{1} & 0\\ 3 & 0\\ 1 & 0\\ 2 & 0\end{array}$	$\mathbf{k}^{e_3}\mathbf{a}_3^e = \begin{bmatrix}0.221&-0.221&-0.221&0.221\\-0.221&0.221&0.221&-0.221\\-0.221&0.221&0.221&-0.221\\0.221&-0.221&-0.221&0.221\end{bmatrix}\begin{bmatrix}0.5\\0\\0\\0\end{bmatrix} = \begin{array}{c\|c} \bar{2} & 0.1106\\ \bar{3} & -0.1105\\ 1 & -0.1105\\ 2 & 0.1106\end{array}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{array}{c\|c} \bar{1} & 0\\ 3 & 0\\ \bar{2} & 0\\ \bar{3} & 0\end{array}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c\|c} \bar{1} & 0\\ 3 & 0\\ 1 & 0\\ 2 & 0\end{array}$	$\mathbf{f}_e^{e_3} = \mathbf{f}_r^{e_3} + \mathbf{f}_N^{e_3} - \mathbf{f}_D^{e_3} = \begin{array}{c\|c} \bar{2} & -0.1105\\ \bar{3} & 0.1105\\ 1 & 0.1105\\ 2 & -0.1105\end{array}$

we did this above

$$\mathbf{K} = \begin{bmatrix} 0.32+0.221 & 0.24-0.221 & -0.24\\ 0.24-0.221 & 0.18+0.221 & -0.18\\ -0.24 & -0.18 & 0.3571+0.18 \end{bmatrix} = \begin{bmatrix} 0.5410 & 0.019 & -0.24\\ 0.019 & 0.401 & -0.18\\ -0.24 & -0.18 & 0.5371 \end{bmatrix}$$

$$\mathbf{F} = \mathbf{F}_N + \mathbf{F}_e = \begin{bmatrix}0\\-1\\0\end{bmatrix} + \begin{bmatrix}0.1105\\-0.1105\\0\end{bmatrix} = \begin{bmatrix}0.1105\\-1.1105\\0\end{bmatrix} \Rightarrow \boxed{\mathbf{U} = \mathbf{K}^{-1}\mathbf{F} = \begin{bmatrix}U_1\\U_2\\U_3\end{bmatrix} = \begin{bmatrix}-0.2123\\-3.2980\\-1.200\end{bmatrix}}$$

Post-processing:
Axial forces

F^{e_3} $U_2^{e_3}=0$
$U_1^{e_3}=0.5$

$U_4^{e_3} = -3.2980$

$U_3^{e_3} = .2123$

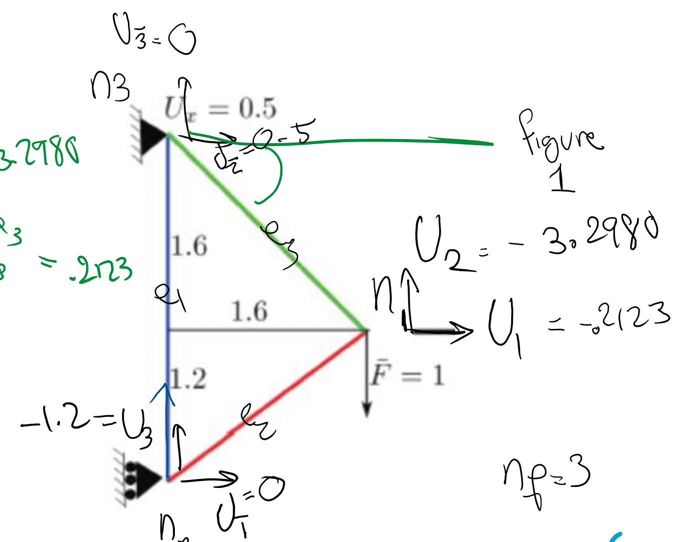
F^{e_3}

$$F^{e_3} = \left(\frac{AE}{L}\right)_{e_3}\left(c\left(U_3^{e_3}-U_1^{e_3}\right) + s\left(U_4^{e_3}-U_2^{e_3}\right)\right)$$

(from eqn 2)

$$F^{e_3} = \frac{1}{1.6\times\sqrt{2}}\left(\frac{\sqrt{2}}{2}\left(.2123-.5\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(-3.2980-0\right)\right)$$

$$\rightarrow \boxed{F^{e_3} = 0.8064}$$



$M(e_3) = [\bar{2}\ \bar{3}\ 1\ 2]$

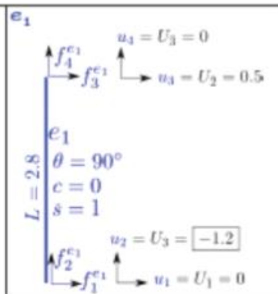
global solution $U = [-.2123\ \ -3.2980\ \ -1.2]$

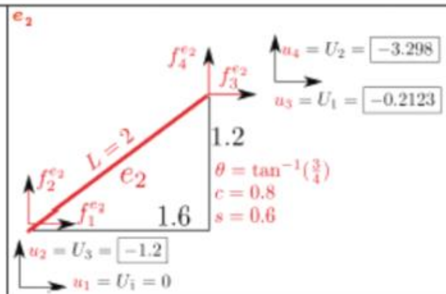
$\rightarrow \boxed{F^{e_3} = 0.8064}$

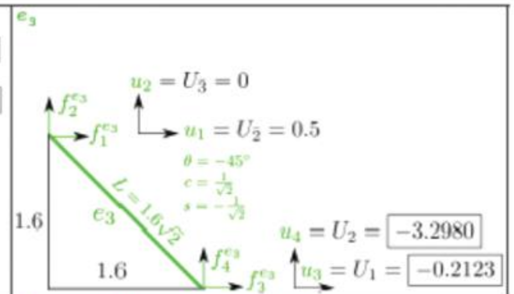
global solution $U = [-.2123, \; -3.2980, \; -1.2]$

$U^{e_3} = [0.5 \;\; 0, \;\; -.2123 \;\; -3.298c]$

Truss Example: Axial force and element local forces



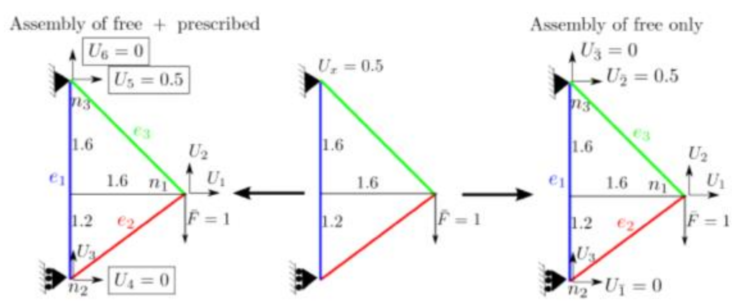




e	e_1	e_2	e_3
$\mathbf{u}^e$	$\begin{bmatrix} 0 \\ -1.2 \\ 0.5 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -1.2 \\ -0.2123 \\ -3.2980 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ 0 \\ -0.2123 \\ -3.2980 \end{bmatrix}$
T^e	$T^{e_1} = \frac{1 \times 1}{2.8}\{0 \times (0.5 - 0) + 1 \times (0 + 1.2)\} = 0.4286$	$T^{e_2} = \frac{1 \times 1}{2}\{0.8 \times (-0.2123 - 0) + 0.6 \times (-3.2928 + 1.2)\} = -0.7128$	$T^{e_3} = \frac{1 \times 1}{1.6\sqrt{2}}\{\frac{1}{\sqrt{2}} \times (-0.2123 - 0.5) - \frac{1}{\sqrt{2}} \times (-3.2928 - 0)\} = 0.8064$

Please read these slides:

Assembly of free + prescribed dofs vs. free only



- All we covered so far was the **assembly of free dof only**.
- We can assemble all dofs (free + prescribed) as shown in figure on the left.
- The numbering of dof when assembling free + prescribed dof is exactly like before with the difference that **we first number free dof followed by prescribed dof** as shown in the figure. For each group (f & p) we start from node n_1 to n_{n_n}.

329 / 456

To slide 335