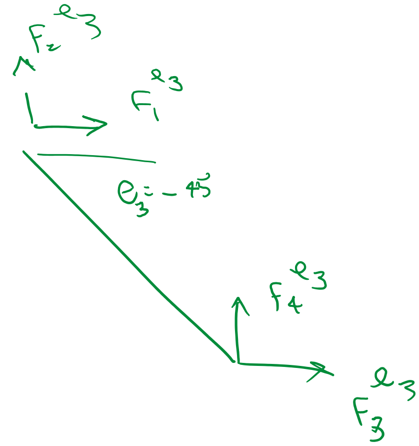
2023/11/08

Wednesday, November 8, 2023 11:18 AM

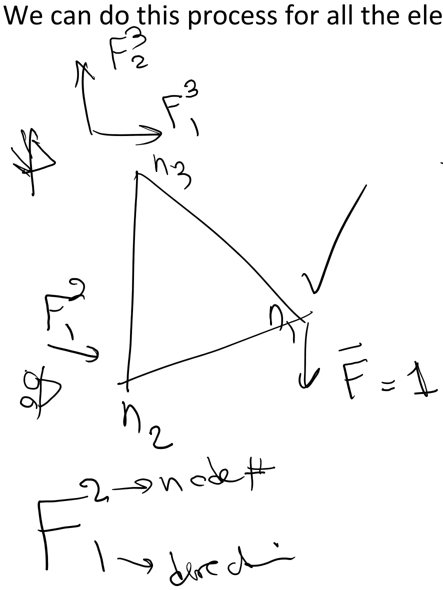
Final step of the last example: Calculate the reaction forces
I'll only do the element calculations for e3 here

$$F^{e3} = k^{e3} a^{e3}$$

$$= \begin{bmatrix} .221 & -.221 & \cdots \\ & \cdots & \\ & \cdots & \\ & \cdots & \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \\ -.2123 \\ -3.2980 \end{bmatrix} = \begin{bmatrix} -.5714 \\ .5714 \\ .5714 \\ -.5714 \end{bmatrix}$$



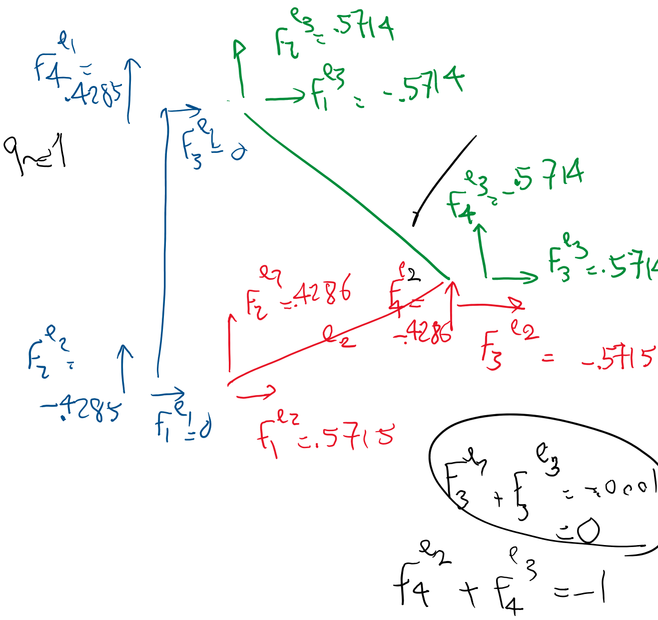
We can do this process for all the elements to get:



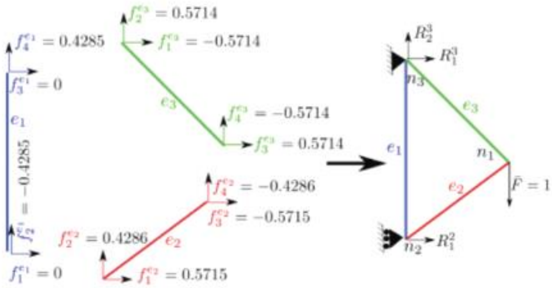
$$F_1^3 = F_3^{e2} + F_1^{e3} = -.5714$$

$$F_2^3 = F_4^{e1} + F_2^{e3} = .9999 \approx 1$$

$$F_1^2 = F_1^{e1} + F_1^{e2} = 0.5715$$



Truss Example: Reaction Forces



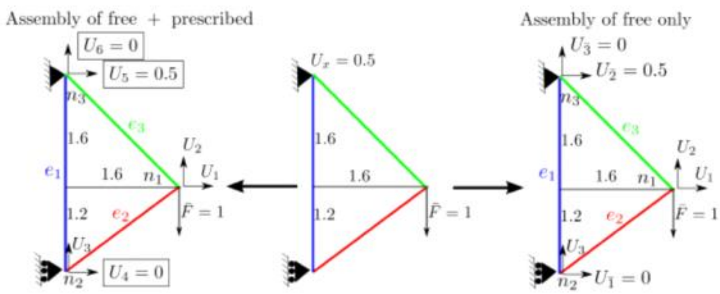
- First, we compute reaction forces by adding up forces from individual elements that contribute to reaction forces:

$$R_1^2 = f_1^{e_1} + f_1^{e_2} = 0 + 0.5715 = 0.5715 \quad (397a)$$

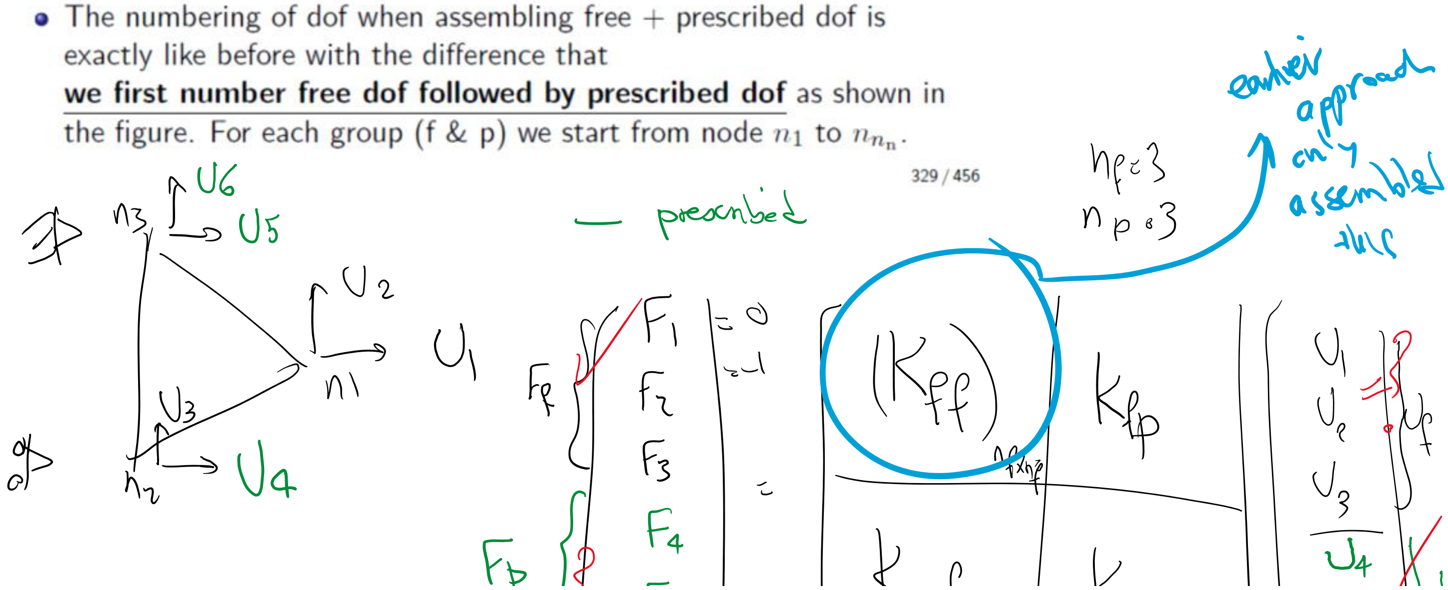
$$R_1^3 = f_3^{e_1} + f_1^{e_3} = 0 + -0.5714 = -0.5714 \quad (397b)$$

$$R_2^3 = f_4^{e_1} + f_2^{e_3} = 0.4285 + 0.5714 = 0.9999 \quad (397c)$$

Assembly of free + prescribed dofs vs. free only



- All we covered so far was the **assembly of free dof only**.
- We can assemble all dofs (free + prescribed) as shown in figure on the left.
- The numbering of dof when assembling free + prescribed dof is exactly like before with the difference that **we first number free dof followed by prescribed dof** as shown in the figure. For each group (f & p) we start from node n_1 to n_{n_n}.



$$F_p \begin{Bmatrix} ? \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} K_{pf} & K_{pp} \end{bmatrix}_{6\times 6} \begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} U_p \quad \text{knows}$$

$$F_f = K_{ff} U_f + K_{fp} U_p \longrightarrow U_f = K_{ff}^{-1}(F_f - K_{fp} U_p)$$

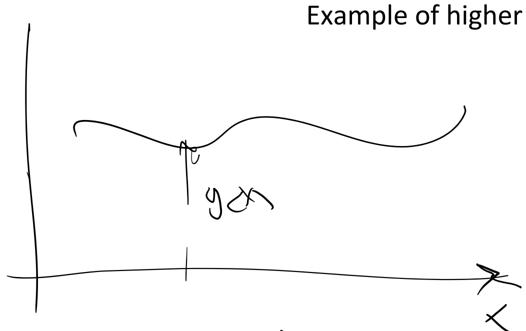
$$F_p = K_{pf} U_f + K_{pp} U_p$$

this will be calculated next

$$\tilde{K} = K_{pp} - K_{pf} K_{ff}^{-1} K_{fp}$$

Beam elements:

Example of higher order DE



DE

$$\frac{d^2}{dx^2} EI \left(\frac{d^2 y}{dx^2} \right) + q = 0 \longrightarrow \int_0 \left(\frac{d^2 \omega}{dx^2} \underbrace{EI}_{D} \underbrace{\frac{d^2 y}{dx^2}}_{\mathcal{L}_m = (\;)''} \right) dx = \ldots$$

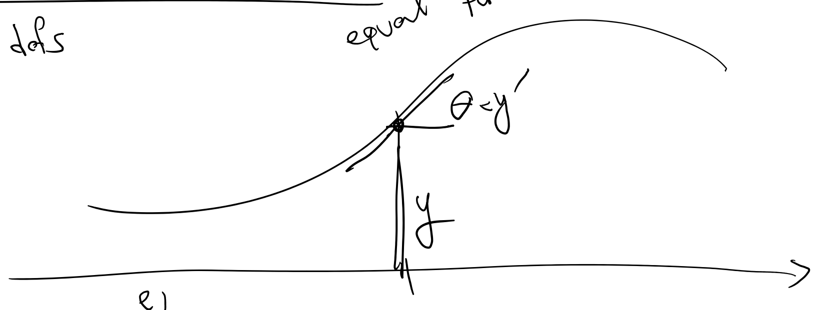
weak

$$M = 4 \qquad m = \frac{M}{2} = 2$$

$$B = \mathcal{L}_m N = N''$$

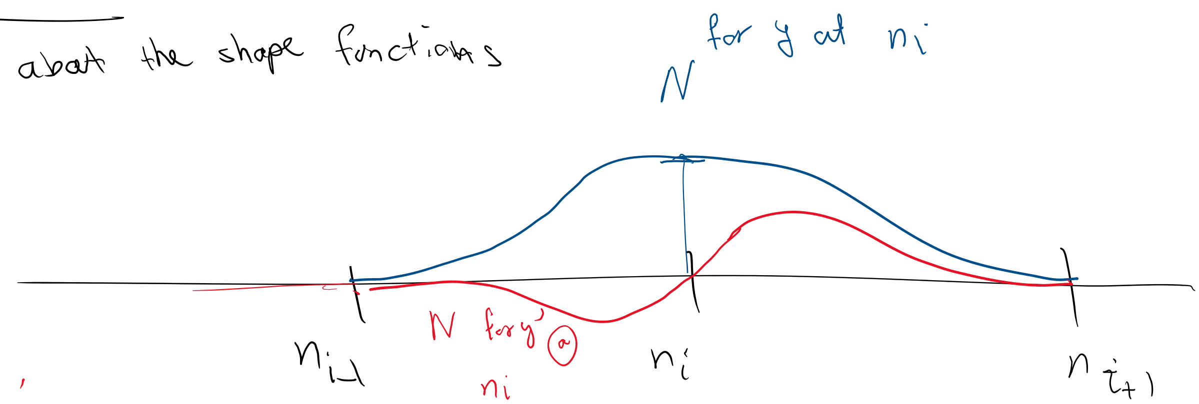
We need $C^{m-1} = C^1$ continuity

C^1 continuity & dofs

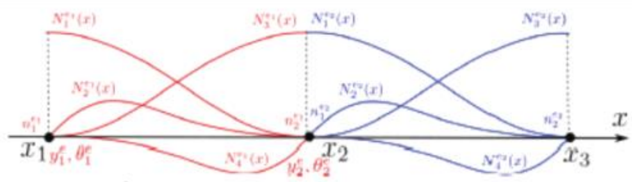


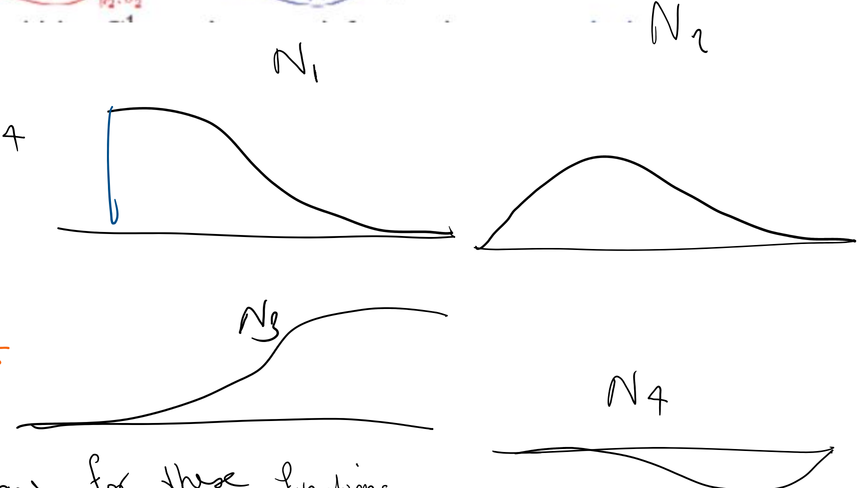
2 dofs (y, y') at each node

What about the shape functions



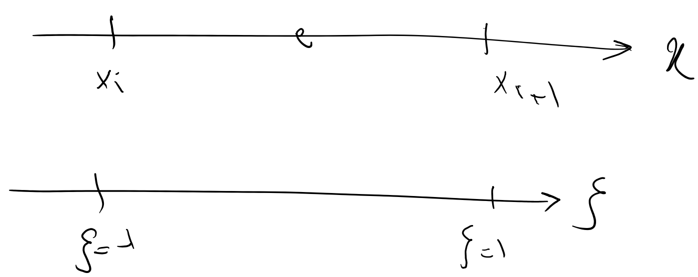
FEM formulation of beam elements: Shape functions





$a^e = [a_1, a_2, a_3, a_4]$

We want to have expressions for these functions



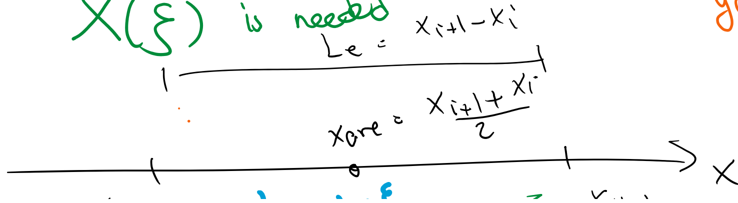
$N_i(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3$

$i = 1, 2, 3, 4$

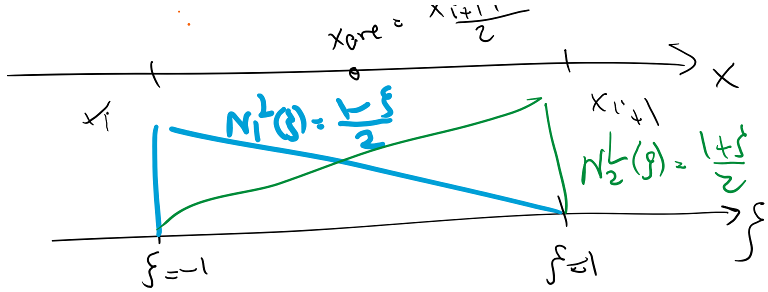
each shape function needs to satisfy 4 eqns

$y(\xi) = N_1(\xi) a_1 + N_2(\xi) a_2 + N_3(\xi) a_3 + N_4(\xi) a_4$

$X(\xi)$ is needed



$x(\xi) = x_i N_1^L(\xi) + x_{i+1} N_2^L(\xi)$



$$x(\xi) = X_i N_1^L(\xi) + X_{i+1} N_2^L(\xi)$$

(X_i: X_{n_1})

$$= X_{are} + \frac{L_e}{2}\xi$$

We see that the geometry (x) and solution (y(x)) are both interpolated by shape functions, but for x NL (order 2) for solution N of order 4 is used.

Now we want to derive formulas for N1 to N4

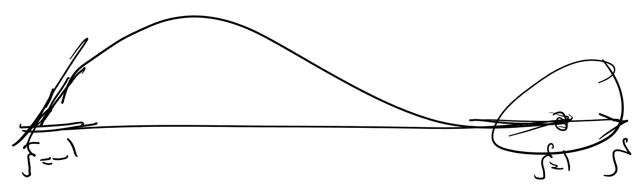
N_2

$$N_2(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3$$

$$\frac{dN_2(\xi)}{d\xi} = \alpha_1 + 2\alpha_2\xi + 3\alpha_3\xi^2$$

$$\frac{dN_2}{dx} = \frac{dN_2}{d\xi}\frac{d\xi}{dx} = \frac{dN_2}{d\xi}\frac{1}{\frac{dx}{d\xi}}$$

$$X(\xi) = X_{are} + \frac{L_e}{2}\xi \longrightarrow J = \frac{dx}{d\xi} = \frac{L_e}{2}$$



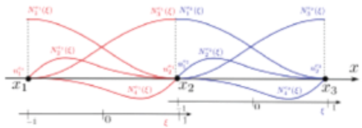
$$N_2(\xi = -1) = 0$$
$$\frac{dN_2}{dx}(\xi = -1) = 1$$
$$N_2(\xi = 1) = 0$$
$$\frac{dN_2}{dx}(\xi = 1) = 0$$

$$\frac{dN_2}{dx} = \frac{2}{L_e}\left(\alpha_1 + 2\alpha_2\xi + 3\alpha_3\xi^2\right)$$

$$\begin{cases} \alpha_0 + (-1)\alpha_1 + (-1)^2\alpha_2 + (-1)^3\alpha_3 = 0 \\ \frac{2}{L_e}\left(0\alpha_0 + 1\alpha_1 + 2(-1)\alpha_2 + 3(-1)^2\alpha_3\right) = 1 \\ \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \frac{2}{L_e}\left(0\alpha_0 + 1\alpha_1 + 2\alpha_2 + 3\alpha_3\right) = 0 \end{cases}$$

$\longrightarrow$

FEM formulation of beam elements: Shape functions



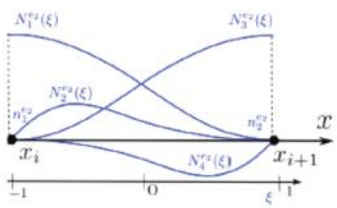
- Shape function N_i take the value of 1 at dof i and zero elsewhere:

$$\begin{array}{llllllll}
N_1(\xi=-1) & =1 & \frac{\mathrm{d}N_1}{\mathrm{d}x}(\xi=-1) & =0 & N_1(\xi=1) & =0 & \frac{\mathrm{d}N_1}{\mathrm{d}x}(\xi=1) & =0 \\
N_2(\xi=-1) & =0 & \frac{\mathrm{d}N_2}{\mathrm{d}x}(\xi=-1) & =1 & N_2(\xi=1) & =0 & \frac{\mathrm{d}N_2}{\mathrm{d}x}(\xi=1) & =0 \\
N_3(\xi=-1) & =0 & \frac{\mathrm{d}N_3}{\mathrm{d}x}(\xi=-1) & =0 & N_3(\xi=1) & =1 & \frac{\mathrm{d}N_3}{\mathrm{d}x}(\xi=1) & =0 \\
N_4(\xi=-1) & =0 & \frac{\mathrm{d}N_4}{\mathrm{d}x}(\xi=-1) & =0 & N_4(\xi=1) & =0 & \frac{\mathrm{d}N_4}{\mathrm{d}x}(\xi=1) & =1
\end{array} \tag{417}$$

- Similar to bar element, since N_i are expressed in terms of ξ we need to transfer $\frac{\mathrm{d}N_i}{\mathrm{d}\xi}$ to $\frac{\mathrm{d}N_4}{\mathrm{d}x}$ (*cf.* (415)):

$$\mathrm{d}x = \frac{L^e}{2}\mathrm{d}\xi \quad \Rightarrow \quad \frac{\mathrm{d}N_i}{\mathrm{d}x} = \frac{\frac{\mathrm{d}N_i}{\mathrm{d}\xi}}{\frac{\mathrm{d}x}{\mathrm{d}\xi}} \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}N_i}{\mathrm{d}x} = \frac{2}{L^e}\frac{\mathrm{d}N_i}{\mathrm{d}\xi}, \quad \frac{\mathrm{d}N_i}{\mathrm{d}\xi} = \frac{L^e}{2}\frac{\mathrm{d}N_i}{\mathrm{d}x}} \tag{418}$$

FEM formulation of beam elements: Shape functions



- From (417) and (418) we get:

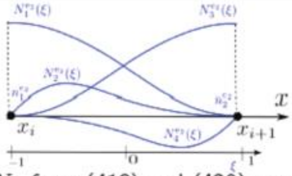
$$\begin{array}{llllllll}
N_1(\xi=-1) & =1 & \frac{\mathrm{d}N_1}{\mathrm{d}\xi}(\xi=-1) & =0 & N_1(\xi=1) & =0 & \frac{\mathrm{d}N_1}{\mathrm{d}\xi}(\xi=1) & =0 \\
N_2(\xi=-1) & =0 & \frac{\mathrm{d}N_2}{\mathrm{d}\xi}(\xi=-1) & =\frac{L^e}{2} & N_2(\xi=1) & =0 & \frac{\mathrm{d}N_2}{\mathrm{d}\xi}(\xi=1) & =0 \\
N_3(\xi=-1) & =0 & \frac{\mathrm{d}N_3}{\mathrm{d}\xi}(\xi=-1) & =0 & N_3(\xi=1) & =1 & \frac{\mathrm{d}N_3}{\mathrm{d}\xi}(\xi=1) & =0 \\
N_4(\xi=-1) & =0 & \frac{\mathrm{d}N_4}{\mathrm{d}\xi}(\xi=-1) & =0 & N_4(\xi=1) & =0 & \frac{\mathrm{d}N_4}{\mathrm{d}\xi}(\xi=1) & =\frac{L^e}{2}
\end{array} \tag{419}$$

- Since each N_i has four conditions, we interpolate them with cubic polynomials:

$$N_i = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 \quad \Rightarrow \quad \frac{\mathrm{d}N_i}{\mathrm{d}\xi} = \alpha_1 + 2\alpha_2\xi + 3\alpha_3\xi^2 \tag{420}$$

α_j are determined from the conditions in (419).

FEM formulation of beam elements: Shape functions



- For example, to determine N_1 from (419) and (420) we observe:

$$\left.\begin{aligned} N_1(\xi=-1) &= 1 \\ \frac{dN_1}{d\xi}(\xi=-1) &= 0 \\ N_1(\xi=1) &= 0 \\ \frac{dN_1}{d\xi}(\xi=1) &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \alpha_0-\alpha_1+\alpha_2-\alpha_3 &= 1 \\ \alpha_1-2\alpha_2+3\alpha_3 &= 0 \\ \alpha_0+\alpha_1+\alpha_2+\alpha_3 &= 0 \\ \alpha_1+2\alpha_2+3\alpha_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ -\frac{3}{4}\\ 0\\ \frac{1}{4}\end{bmatrix}$$

$$\boxed{N_1(\xi) = \frac{1}{4}(2-3\xi+\xi^3)}$$

- Similarly for N_2:

$$\left.\begin{aligned} N_2(\xi=-1) &= 0 \\ \frac{dN_2}{d\xi}(\xi=-1) &= \frac{L^e}{2} \\ N_2(\xi=1) &= 0 \\ \frac{dN_2}{d\xi}(\xi=1) &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \alpha_0-\alpha_1+\alpha_2-\alpha_3 &= 0 \\ \alpha_1-2\alpha_2+3\alpha_3 &= \frac{L^e}{2} \\ \alpha_0+\alpha_1+\alpha_2+\alpha_3 &= 0 \\ \alpha_1+2\alpha_2+3\alpha_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \frac{L^e}{8}\begin{bmatrix}1\\ -1\\ -1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \frac{L^e}{8}\begin{bmatrix}1\\ -1\\ -1\\ 1\end{bmatrix}$$

$$\boxed{N_2(\xi) = \frac{L^e}{8}(1-\xi-\xi^2+\xi^3)}$$

we did this above

$$N_2(\xi) = \frac{L^e}{8}(1-\xi-\xi^2+\xi^3) = \alpha_0+\alpha_1\xi+\alpha_2\xi^2+\alpha_3\xi^3$$

FEM formulation of beam elements: Shape functions

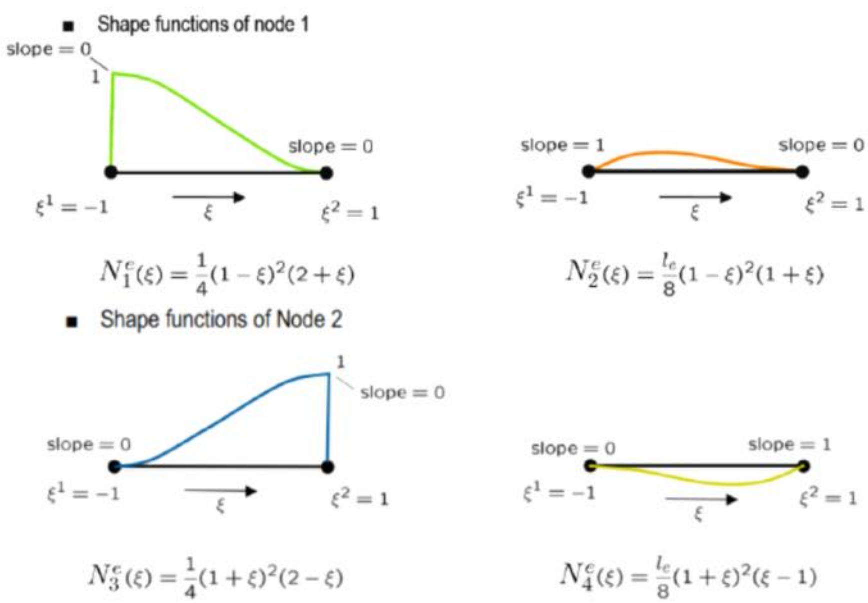


figure from F. Cirak

Now that we have the shape functions, we can calculate the stiffness matrix

weak $\int w'' \underbrace{EI}_{D} y'' \, dx \longrightarrow K_{4\times4} = \int_{x_i}^{x_{i+1}} \begin{bmatrix} B_1 \\ \vdots \\ B_4 \end{bmatrix} EI \, [B_1 \; \cdots \; B_4] \underbrace{dx}_{(J d\xi)}$

$\int_M = (\;)''$

$B = N''\left(\frac{d^2 N}{dx^2}\right)$

$J = \frac{dx}{d\xi} = \frac{L_e}{2}$

$K_e = \int B^t \underbrace{D}_{EI} B \overset{L_e/2}{\widetilde{J}} d\xi$

$B = [B_1 \; B_2 \; B_3 \; B_4] = \frac{d^2}{dx^2}[N_1(\xi) \; N_2(\xi) \; N_3(\xi) \; N_4(\xi)]$

For example $B_2 = \frac{d^2}{dx^2} N_2(\xi) = \frac{d}{dx}\left(\frac{d}{dx} N_2(\xi)\right) = \frac{d}{dx}\left(\frac{dN_2(\xi)}{d\xi} \frac{1}{\frac{dx}{d\xi}}\right)$ ($\frac{dx}{d\xi} = J$, constant $= \frac{L_e}{2}$)

$= \frac{1}{J} \frac{d}{dx}\left(\frac{dN_2(\xi)}{d\xi}\right) = \frac{1}{J}\left(\frac{d^2 N_2(\xi)}{d\xi^2} \frac{1}{\frac{dx}{d\xi}}\right) = \frac{1}{J^2}\frac{d^2 N}{d\xi^2}$ (const)

$N_2(\xi) = \frac{L_e}{8}(1 - \xi - \xi^2 + \xi^3)$ $\quad \frac{dN_2}{d\xi}(\xi) = \frac{L_e}{8}(-1 - 2\xi + 3\xi^2)$

$\longrightarrow B_{\xi_2} = \frac{d^2 N_2(\xi)}{d\xi^2} = \frac{L_e}{8}(-2 + 6\xi)$

$B_2(\xi) = \frac{1}{J^2}(B_{\xi_2}) = \frac{1}{\left(\frac{L_e}{2}\right)^2} \frac{L_e}{8}(6\xi - 2) = \frac{3\xi - 1}{L_e}$

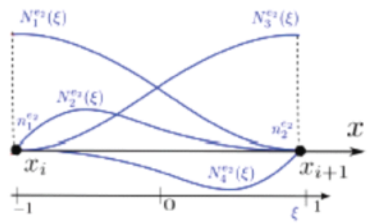
do this for other N's

$$B = \frac{d^2 N}{dx^2} = \frac{1}{J^2} \underbrace{\frac{d^2 N}{d\xi^2}}_{B_\xi} = \frac{4}{L_e^2} B_\xi = \left[\frac{6\xi}{L_e^2} \quad \frac{-1+3\xi}{L_e} \quad \frac{-6\xi}{L^2} \quad \frac{1+3\xi}{L_e}\right]$$

$K^e \quad \int_{-1}^{1} B^t \ldots B(\xi) \overset{L_e/2}{\widetilde{J}} d\xi$

$$K^e_{4\times4} = \int_{-1}^{1} B^t(\xi)\, EI(\xi)\, B(\xi)\, \tilde{J}\, d\xi \qquad \tilde{J} = \frac{L_e}{2}$$

FEM formulation of beam elements: Stiffness matrix



- From (424) and (425) we have,

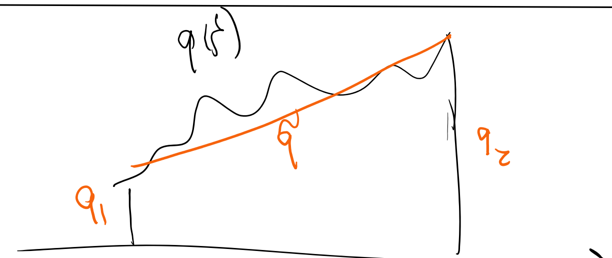
$$\mathbf{k}^e = \frac{1}{2L^e}\int_{-1}^{1} \begin{bmatrix} \frac{6\xi}{L^e} \\ -1+3\xi \\ -\frac{6\xi}{L^e} \\ 1+3\xi \end{bmatrix} E(\xi)I(\xi) \begin{bmatrix} \frac{6\xi}{L^e} & -1+3\xi & -\frac{6\xi}{L^e} & 1+3\xi \end{bmatrix} \mathrm{d}\xi \qquad (426)$$

- If E and I are constant, we can take those out of the equation and have:

$$\mathbf{k}^e = \frac{EI}{L^{e3}} \begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & 4L^{e2} & -6L^e & 2L^{e2} \\ & & 12 & -6L^e \\ \text{sym.} & & & 4L^{e2} \end{bmatrix} \quad \text{for constant } E \text{ and } I \qquad (427)$$

Source term force

$$\int_e w'' EI y'' dx = \int w q \, dx$$



$$f^e_r = \int_e \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q \, dx \approx \int_{\xi=-1}^{\xi=1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \overset{N}{q}(\xi) (\tilde{J} d\xi) = \left(\int \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} N_1^L(\xi) & N_2^L(\xi) \end{bmatrix} \tilde{J} d\xi \right) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

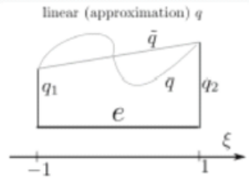
$$\overset{N}{q}(\xi) = N_1^L(\xi) q_1 + N_2^L(\xi) q_2$$

4×2

$(f^e_r) = (\ldots) \; [q_1 \ldots$

$$\left(f_r^e\right)_{4\times1} = \left(r_e\right)_{4\times2}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

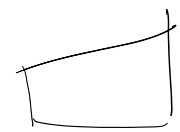
Beam elements: Forces: A. Source term forces



then the source term force is:

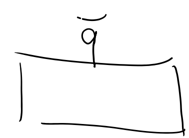
$$\mathbf{f}_r^e \approx \int_{-1}^{1} \mathbf{N}^e(\xi)^{\mathrm{T}}.\tilde{q}(\xi)\frac{L^e}{2}\mathrm{d}\xi = \mathbf{r}^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{where}$$

$$\mathbf{r}^e = \frac{L^e}{2}\int_{-1}^{1} \mathbf{N}^e(\xi)^{\mathrm{T}}.\mathbf{N}_L^e(\xi)\mathrm{d}\xi = \frac{L^e}{2}\int_{-1}^{1} \begin{bmatrix} \frac{1}{4}(2-3\xi+\xi^3) \\ \frac{L^e}{8}(1-\xi-\xi^2+\xi^3) \\ \frac{1}{4}(2+3\xi-\xi^3) \\ \frac{L^e}{8}(-1-\xi+\xi^2+\xi^3) \end{bmatrix} . \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \mathrm{d}\xi \quad \Rightarrow$$



$$\boxed{\mathbf{f}_r^e \approx \mathbf{r}^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{where } \mathbf{r}^e = L^e \begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{20}L^e & \frac{1}{30}L^e \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{30}L^e & -\frac{1}{20}L^e \end{bmatrix} \quad \text{exact for linear } q} \qquad (433)$$

q_1 and q_2 are shown in the right figure. For constant $q(x) = \bar{q}$, $q_1 = q_2 = \bar{q}$, from (433) we get:



$$\mathbf{f}_r^e = \mathbf{r}^e \begin{bmatrix} \bar{q} \\ \bar{q} \end{bmatrix} = \begin{bmatrix} \frac{\bar{q}L^e}{2} \\ \frac{\bar{q}L^{e2}}{12} \\ \frac{\bar{q}L^e}{2} \\ -\frac{\bar{q}L^{e2}}{12} \end{bmatrix} \quad \text{constant } q(x) = \bar{q} \qquad (434)$$