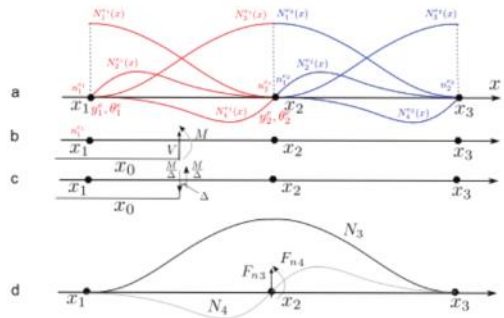
2023/11/13

Beam elements: Concentrated load



- Our main goal is to obtain the contribution of concentrated vertical force and moment to global force vector for the beam problem.
- For each node, there are two dofs: vertical displacement and rotation. We expect the corresponding nodal forces to be concentrated vertical load and concentrated moment as they are dual quantities.
- In the figure (b) a concentrated vertical force V and moment M are exerted at point x.
- The source term force is:

$$\mathbf{F}_r = \int_{\mathcal{D}} \mathbf{N}^{\mathrm{T}}.\mathbf{r}\,\mathrm{d}v,\ \mathcal{D} = [0\ L], \mathbf{r} = q \text{ (vertical distributed load)} \quad \Rightarrow$$

$$F_{r\,i} = \int_0^L N_i(x) q(x)\,\mathrm{d}x$$

365 / 456

for concentrated force

$$f_r = \int_e N^T q\, dx$$

$$= \int_e \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} V\,\delta(x - x_0)\,dx$$

$$= V \begin{bmatrix} N_1(\xi_0) \\ N_2(\xi_0) \\ N_3(\xi_0) \\ N_4(\xi_0) \end{bmatrix}$$

where

$x_0 = \xi_0 \frac{L}{2} + x_{ave}$

$V \times$ values of N

for concentrated moment

$$f_r = M \frac{d}{dx} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} (\xi_0)$$

$$= M \frac{1}{\left(\frac{dx}{d\xi}\right)} \frac{d}{d\xi} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} (\xi_0)$$

Important for the last problem in HW4

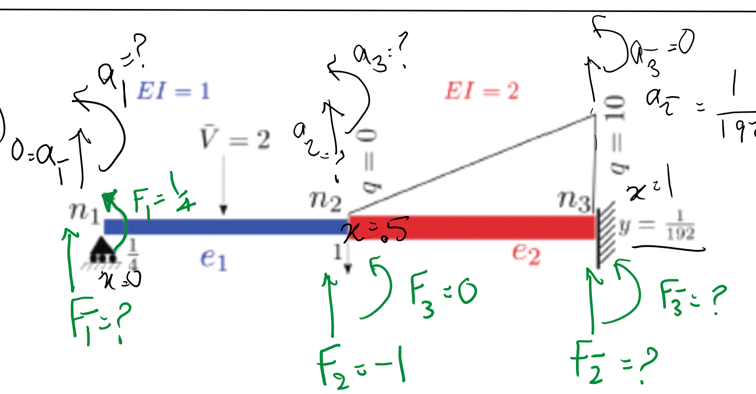
Sample problem:

fig 1

3 nodes × 2 dof / node → 6 dofs

— dofs (disp & rotation)

— "force" (vert force & moments)



$$F_n = \begin{bmatrix} 1/4 \\ -1 \\ 0 \end{bmatrix}$$

nodal map LEM(1) = [1, 2] LEM(2) = [2, 3]

dof map M(1) = [1, $\bar{1}$, 2, $\bar{2}$] M(2) = [2, 3, $\bar{2}$, $\bar{3}$]

calculation of e_2 $K^{e_2} = ?$ $EI = 2$, $L = 1$

Slide 371

calculation of e_2 $\quad k^{e_2} = ?$ $\quad EI = 2,\ L = \frac{1}{2}$

Slide 371

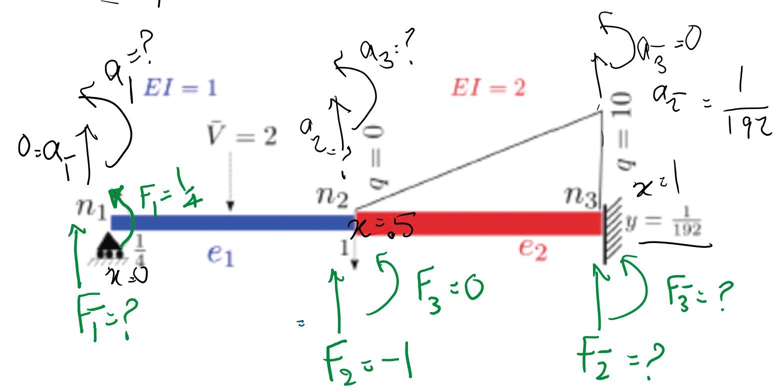
e	e_1	e_2
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{1}{(\frac{1}{2})^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$	$\mathbf{k}^{e_2} = \frac{2}{(\frac{1}{2})^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$
	$\begin{matrix} & \bar{1} & 1 & 2 & 3 \\ \bar{1} & 96 & 24 & -96 & 24 \\ 1 & 24 & 8 & -24 & 4 \\ 2 & -96 & -24 & 96 & -24 \\ 3 & 24 & 4 & -24 & 8 \end{matrix}$	$\begin{matrix} & 2 & 3 & \bar{2} & \bar{3} \\ 2 & 192 & 48 & -192 & 48 \\ 3 & 48 & 16 & -48 & 8 \\ \bar{2} & -192 & -48 & 192 & -48 \\ \bar{3} & 48 & 8 & -48 & 16 \end{matrix}$

$$K = \begin{bmatrix} 8 & -24 & 4 \\ & 96+192 & -24+48 \\ \text{sym} & & 8+16 \end{bmatrix}$$

$$\mathbf{k}^e = \frac{EI}{L^{e3}}\begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & 4L^{e2} & -6L^e & 2L^{e2} \\ & & 12 & -6L^e \\ \text{sym.} & & & 4L^{e2} \end{bmatrix} \quad \text{for constant } E \text{ and } I \qquad (427)$$

352 / 456

$$\mathbf{K} = \begin{bmatrix} 8 & -24 & 4 \\ & 96+192 & -24+48 \\ \text{sym.} & & 8+16 \end{bmatrix} = \begin{bmatrix} 8 & -24 & 4 \\ & 288 & 24 \\ \text{sym.} & & 24 \end{bmatrix}$$



Back to element 2 calculations:

$$f^{e_2} = f_r^{e_2} + f_N^{e_2} - f_D^{e_2}$$

$f_N^{e_2} \to 0$ 1D element

$$f_r^{e_2} = (r_e)_{4\times2}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = (r_e)_{4\times2}\begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

$LEM(1) = [1, 2]$ $\quad LEM(2) = [2, 3]$

$M(1) = [\bar{1}, 1, 2, 3]$ $\quad M(2) = [2, 3, \bar{2}, \bar{3}]$

$$\mathbf{f}_r^e \approx \mathbf{r}^e\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{where } \mathbf{r}^e = L^e\begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{20}L^e & \frac{1}{30}L^e \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{30}L^e & -\frac{1}{20}L^e \end{bmatrix} \quad \text{exact for linear } q \qquad (433)$$

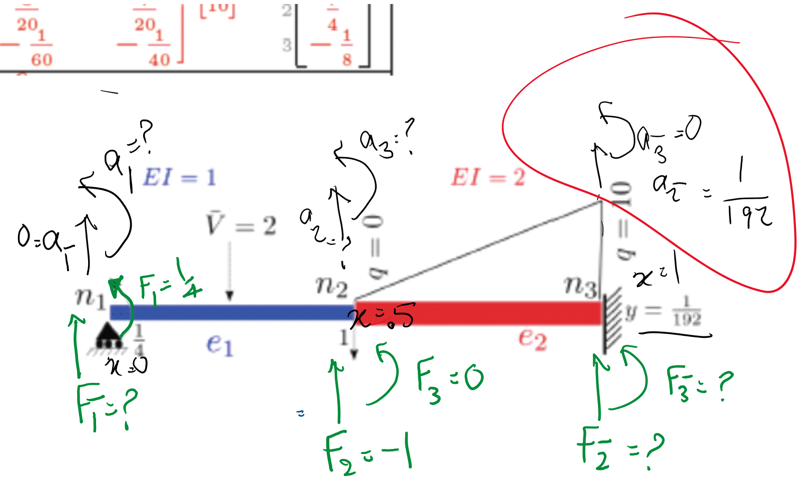
$$f_r^{e_1} = \bar{V}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}(\xi_0) = (-2)\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}(0)$$

ξ_0

$\mathbf{f}_r^e$	(440) (1st eqn) ($\xi = 0$)	equation (433); $\mathbf{r}^e\,[q_1 \quad q_2]^T$
	$\bar{V}\begin{bmatrix} N_1^{e_1}(\xi_0) \\ N_2^{e_1}(\xi_0) \\ N_3^{e_1}(\xi_0) \\ N_4^{e_1}(\xi_0) \end{bmatrix} = -2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{8} \\ \frac{1}{2} \\ -\frac{1}{8} \end{bmatrix} = \begin{matrix} \bar{1} \\ 1 \\ 2 \\ 3 \end{matrix}\begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{40} & \frac{1}{60} \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{60} & -\frac{1}{40} \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{matrix} 2 \\ 3 \\ \bar{2} \\ \bar{3} \end{matrix}\begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix}$

$$f_D^{e_2} = k^{e_2}_{4\times4}\, a^{e_2}_{4\times1}$$

$$= k^{e_2}_{4\times4}\begin{bmatrix} 0 \\ 0 \\ \frac{1}{192} \\ 0 \end{bmatrix} \begin{matrix} a_2 \\ a_3 \\ \to a_{\bar{2}} \\ \to a_{\bar{3}} \end{matrix}$$



LEM(1) = [1,2] LEM(2) = [2,3]

M(1) = [$\bar{1}$,1,2,3] M(2) = [2,3,$\bar{2}$,$\bar{3}$]

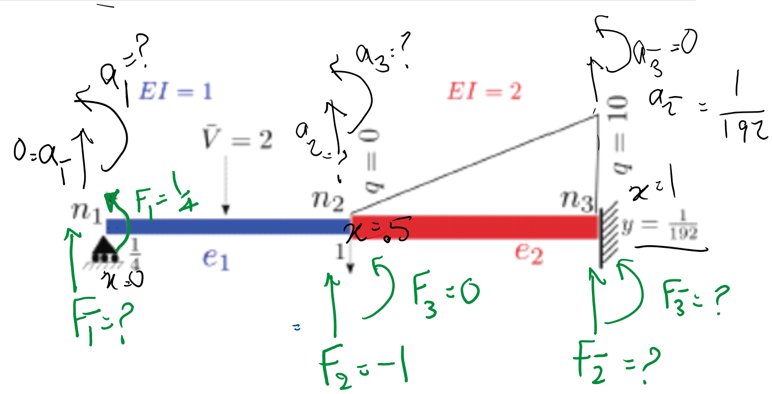
$$f_D^{e_1} = k^{e_1}_{4\times4}\, a^{e_1} = k^{e_1}_{4\times4}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \to a_{\bar{1}} \\ \to a_1 \\ \to a_2 \\ \to a_4 \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e = \begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{matrix} \bar{1} \\ 1 \\ 2 \\ 3 \end{matrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e = \begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \frac{1}{192} \\ 0 \end{bmatrix} = \begin{matrix} 2 \\ 3 \\ \bar{2} \\ \bar{3} \end{matrix}\begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \\ -\frac{1}{4} \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{matrix} \bar{1} \\ 1 \\ 2 \\ 3 \end{matrix}\begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{matrix} 2 \\ 3 \\ \bar{2} \\ \bar{3} \end{matrix}\begin{bmatrix} \frac{7}{4} \\ \frac{1}{3} \\ \frac{3}{4} \\ \frac{1}{8} \end{bmatrix}$

$$F^e = \begin{bmatrix} -\frac{1}{8} \\ -1 + \frac{7}{4} \\ \frac{1}{8} + \frac{1}{3} \end{bmatrix}$$

$$F^n = \begin{bmatrix} \frac{1}{4} \\ -1 \\ 0 \end{bmatrix}$$

$$F = F^e + F^n = \begin{bmatrix} -\frac{1}{8} \\ -\frac{3}{4} \\ \frac{11}{24} \end{bmatrix}$$



LEM(1) = [1,2] LEM(2) = [2,3]

M(1) = [$\bar{1}$,1,2,3] M(2) = [2,3,$\bar{2}$,$\bar{3}$]

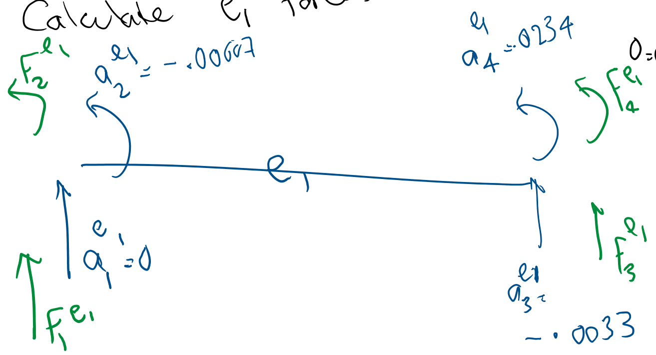
$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$$

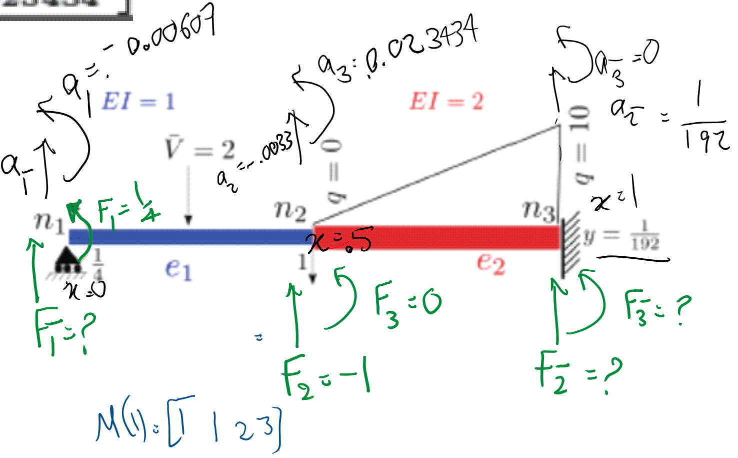
$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{\ldots} \end{bmatrix} = \begin{bmatrix} -.00607 \\ -.00332 \\ .023434 \end{bmatrix}$$

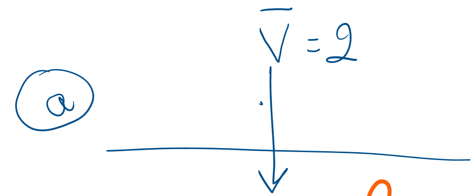
Now, we can calculate the element solutions:

Calculate e_1 forces





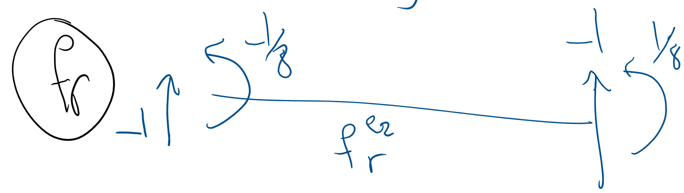
$$f_D^{e_1} = K^{e_1}_{4\times4}\, a^{e_1}_{4\times1} = K^{e_1}\begin{bmatrix} 0 \\ -.00607 \\ -.0033 \\ .0234 \end{bmatrix}$$



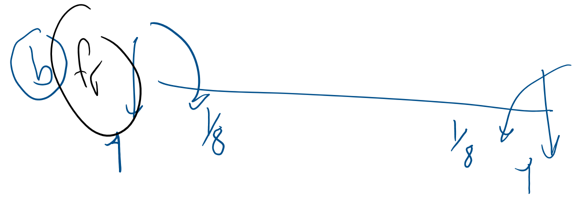
$$F^{e_1} = f_D^{e_1} + \underbrace{\begin{bmatrix} 1 \\ 1/8 \\ 1 \\ -1/8 \end{bmatrix}}_{-f_r^{e_1}}$$

$$f_r^{e_1} = \begin{bmatrix} -1 \\ -1/8 \\ -1 \\ 1/8 \end{bmatrix}$$

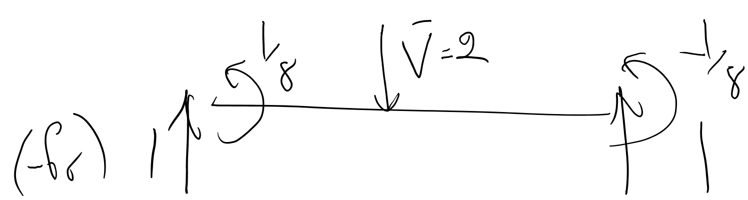
f_r equivalent to "distributed force"



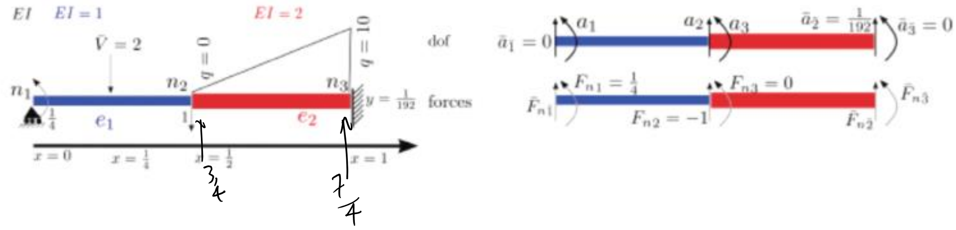
$$F^{e_1} = \underbrace{-\left(f_r^{e_1} + f_N^{e_1} - f_D^{e_1}\right)}$$



to balance the forces

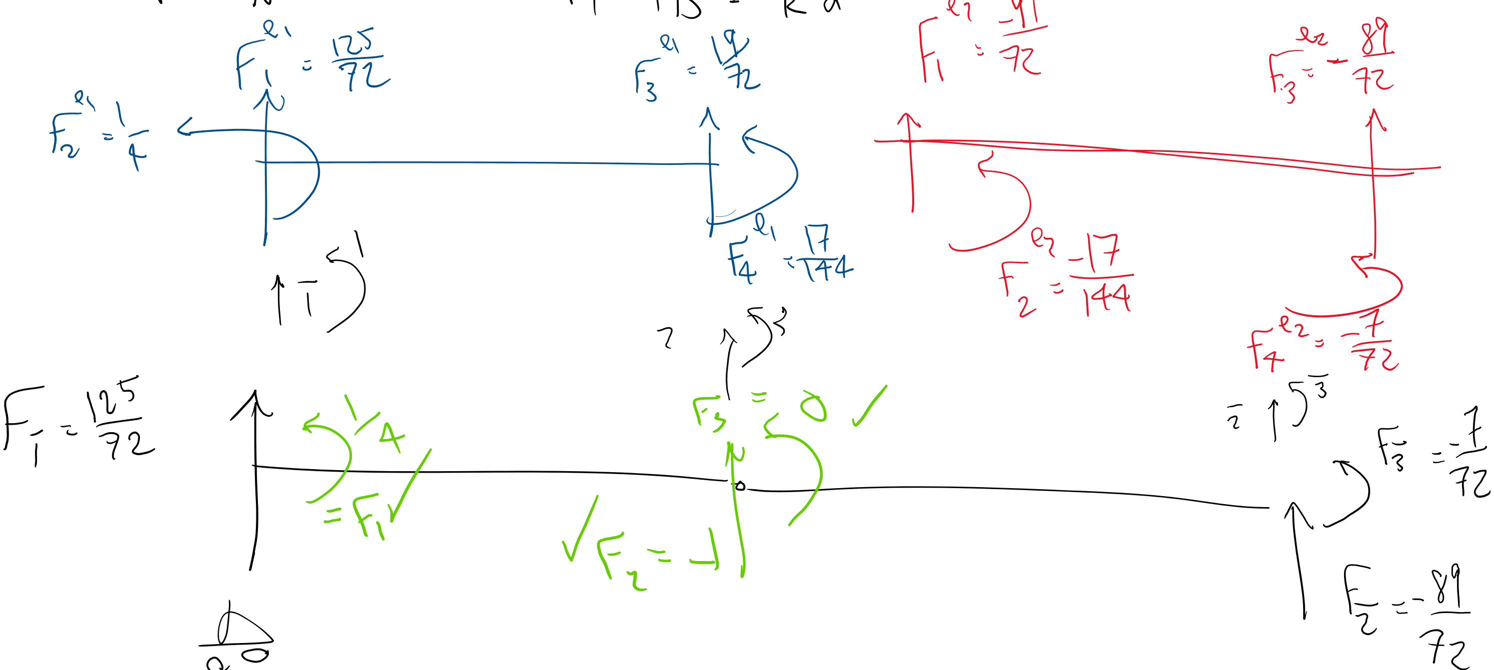


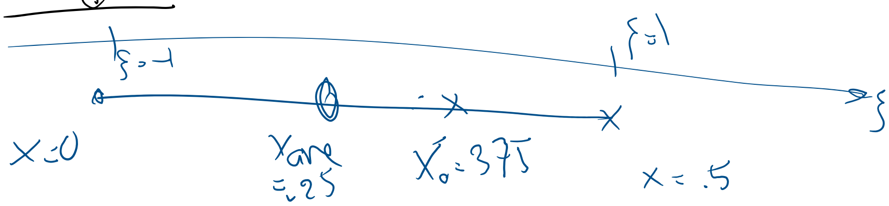
Beam Example: Resultant nodal forces



e	e_1	e_2
$\mathbf{u}^e$	$\begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix}$	$\begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix}$
$-f^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e - \mathbf{f}_r^{e_1} - \mathbf{f}_N^{e_1} =$ ($f_D^{e_1}$) $\begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix} - \begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix} = \begin{bmatrix} \frac{125}{72} \\ \frac{1}{4} \\ \frac{19}{72} \\ \frac{17}{144} \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e - \mathbf{f}_r^{e_2} - \mathbf{f}_N^{e_2} =$ $\begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix} \begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix} = \begin{bmatrix} -\frac{91}{72} \\ -\frac{17}{144} \\ -\frac{89}{72} \\ -\frac{7}{72} \end{bmatrix}$

To calculate element forces we need to calculate -fe = $-(f_r^e + f_N^e - f_D^e)$ in the absence of f_r^e & f_N^e this is simply $f_D^e = K^e a^e$





find $y(x_0)$, $\theta(x_0) = y'(x_0)$, $M(x_0) = EIy''(x_0)$, $V(x_0) = \frac{d}{dx}(EIy'')(x_0)$

$X_{ave} = .25$ $L = .5$ $X_0 = \xi_0 \frac{L}{2} + X_{ave} \rightarrow \xi_0 = .5$

for $\xi = \xi_0$

x`

Beam Example: Calculation of y, θ, M, V within element

- After the solution of global free dofs, they are transferred to elements.
- Once element dofs are know, we have the Displacement in the entire elements:

$$y(\xi) = N_1^e(\xi)a_1^e + N_2^e(\xi)a_2^e + N_3^e(\xi)a_3^e + N_4^e(\xi)a_4^e.$$

element shape functions are given in (421).

- Rotation: Obtained by differentiating previous equation w.r.t. x & noting that $\frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{L^e}{2}$:

$$\theta(\xi) = \frac{\mathrm{d}y}{\mathrm{d}x}(\xi) = \frac{\frac{\mathrm{d}y}{\mathrm{d}\xi}(\xi)}{\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi)} = \frac{2}{L^e}\left\{\frac{\mathrm{d}N_1^e}{\mathrm{d}\xi}(\xi)a_1^e + \frac{\mathrm{d}N_2^e}{\mathrm{d}\xi}(\xi)a_2^e + \frac{\mathrm{d}N_3^e}{\mathrm{d}\xi}(\xi)a_3^e + \frac{\mathrm{d}N_4^e}{\mathrm{d}\xi}(\xi)a_4^e\right\}$$

- Moment is directly obtained by differentiating the above equation:

$$M(\xi) = E(\xi)I(\xi)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(\xi) = E(\xi)I(\xi)\mathbf{B}^e(\xi)$$
$$= E(\xi)I(\xi)\left\{B_1^e(\xi)a_1^e + B_2^e(\xi)a_2^e + B_3^e(\xi)a_3^e + B_4^e(\xi)a_4^e\right\} \quad \textit{cf.}\ (424) \text{ for } \mathbf{B}^e$$

- Shear force is obtained by differentiating M w.r.t. x. It's a similar process to deriving θ from y with the difference that if EI are not constant we need to take it into account. For constant EI we have:

$$V(\xi) = \frac{\mathrm{d}M}{\mathrm{d}x}(\xi) = \frac{\frac{\mathrm{d}M}{\mathrm{d}\xi}(\xi)}{\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi)} = \frac{2EI}{L^e}\left\{\frac{\mathrm{d}B_1^e}{\mathrm{d}\xi}(\xi)a_1^e + \frac{\mathrm{d}B_2^e}{\mathrm{d}\xi}(\xi)a_2^e + \frac{\mathrm{d}B_3^e}{\mathrm{d}\xi}(\xi)a_3^e + \frac{\mathrm{d}B_4^e}{\mathrm{d}\xi}(\xi)a_4^e\right\}$$

- To obtain these fields for the entire beam we evaluate these equations for all elements.

- Free forces $\mathbf{F}_f$:

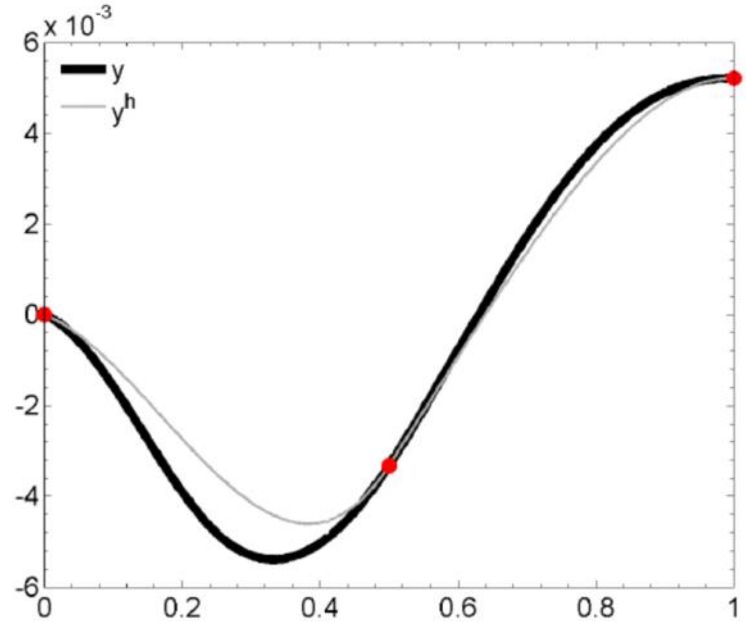
$$F_{n1} = f_2^{e1} = \frac{1}{4} \qquad = \text{moment exerted on the left support} \qquad (443a)$$

$$F_{n2} = f_3^{e1} + f_1^{e2} = \frac{19}{72} - \frac{91}{72} = -1 \qquad = \text{vertical load exerted at the center of the beam} \qquad (443b)$$

$$F_{n3} = f_4^{e1} + f_2^{e2} = \frac{17}{144} - \frac{17}{144} = 0 \qquad = \text{zero moment exerted at the center of the beam} \qquad (443c)$$

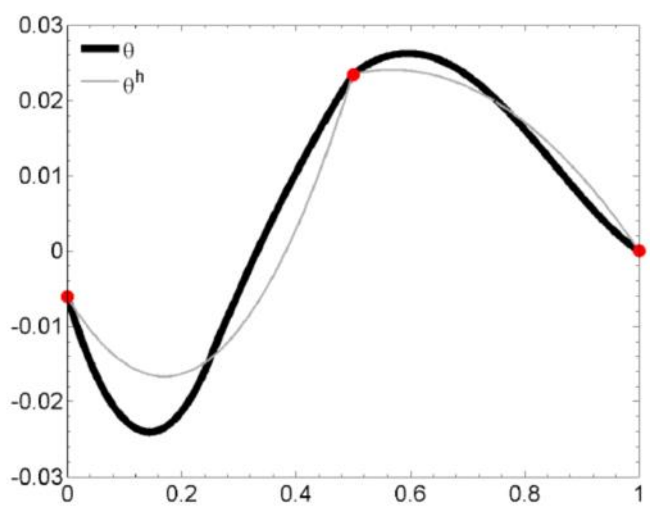
- As mentioned before, this step is not necessary and may just be used as a necessary (not sufficient) condition for the correctness of hand calculations.

Beam Example: FEM vs. Exact solution: y



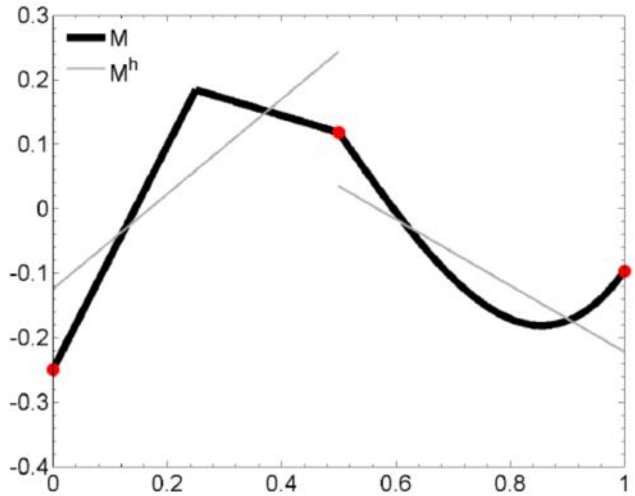
- Similar to bar element FEM and exact solutions match at nodes.
- This behavior is restricted to certain problems in 1D with constant material properties along the element and does not extend to more general cases.

Beam Example: FEM vs. Exact solution: θ



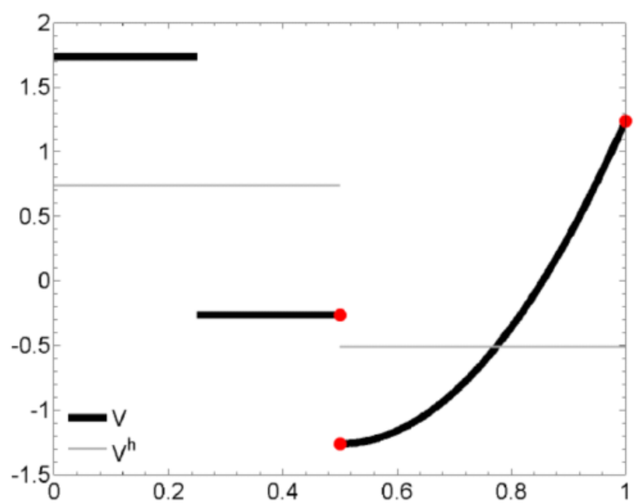
- Rotations at nodes (rotational dofs) match those from exact solution.
- Again we emphasize that while this behavior is shared for certain types of 1D problems, it does not extend to more general cases.

Beam Example: FEM vs. Exact solution: M



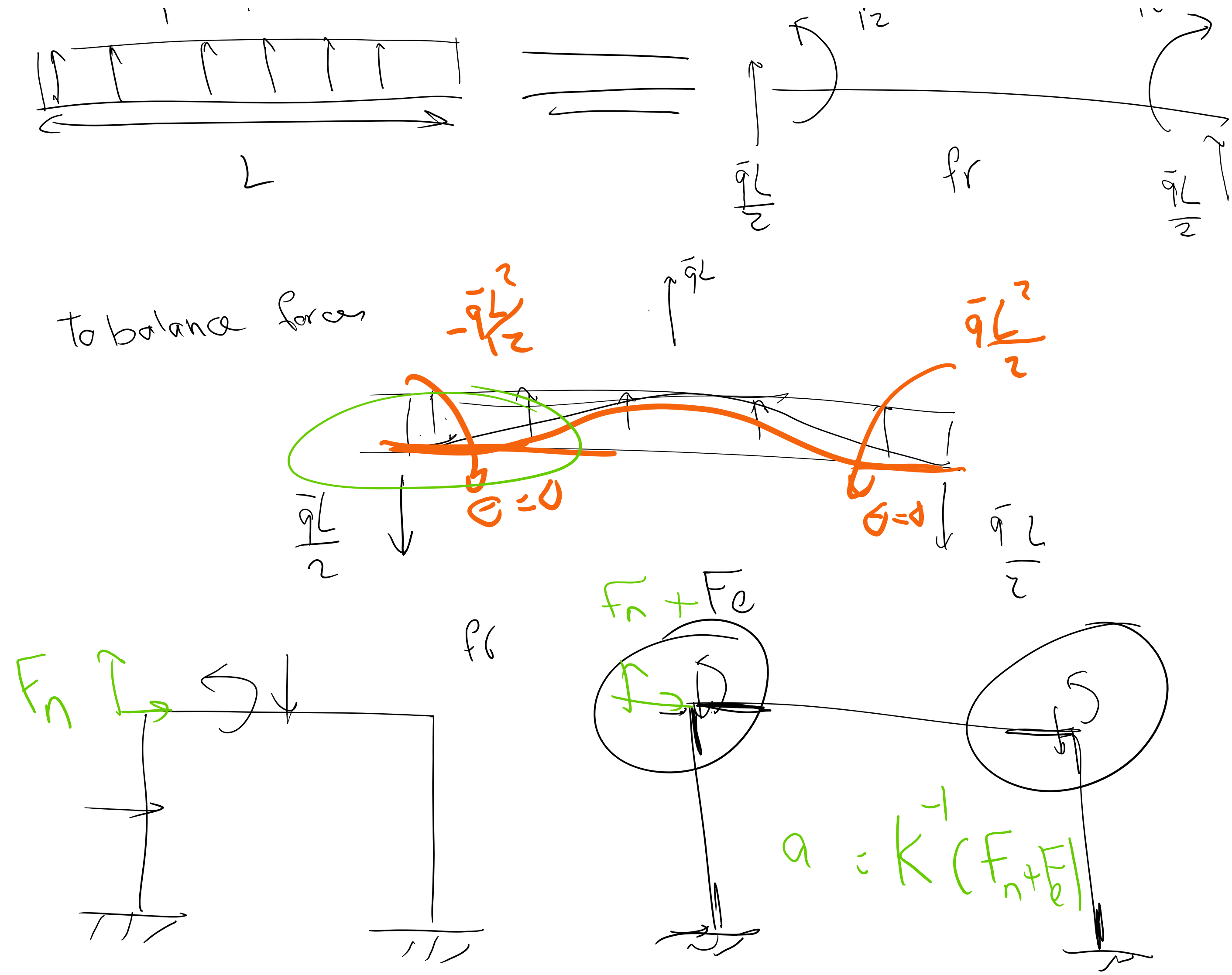
- It is clear that FEM solutions for M are much less accurate than those for y and θ when compared to exact solution.
- This is a general behavior where FEM accuracy decreases for solution derivatives.

Beam Example: FEM vs. Exact solution: V



Beam elements: Forces: A. Source term forces

$$\mathbf{f}_r^e = \mathbf{r}^e \begin{bmatrix}\bar{q}\\ \bar{q}\end{bmatrix} = \begin{bmatrix}\frac{\bar{q}L^e}{2}\\ \frac{\bar{q}L^{e\,2}}{12}\\ \frac{\bar{q}L^e}{2}\\ -\frac{\bar{q}L^{e\,2}}{12}\end{bmatrix} \quad \text{constant } q(x) = \bar{q} \qquad (434)$$



5. **80 Points** In figure 5 **Frame element** e_2 is hinged to **truss element** e_1. For the frame element, a concentrated moment $\bar{M} = 1$ is applied at $x = 0.75$ and a distributed load $q = x$ is applied over the length of the frame.

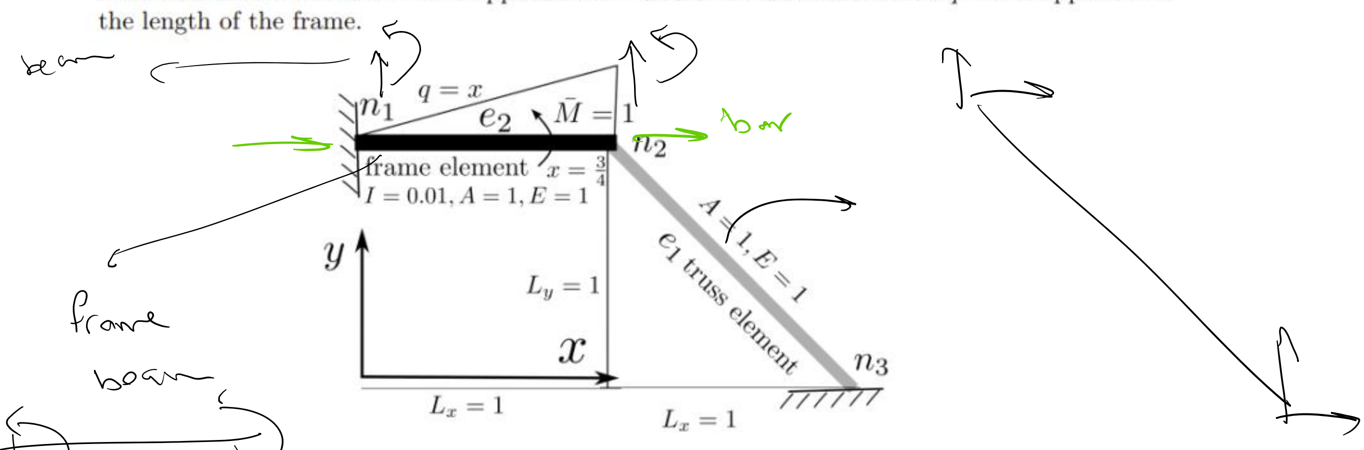
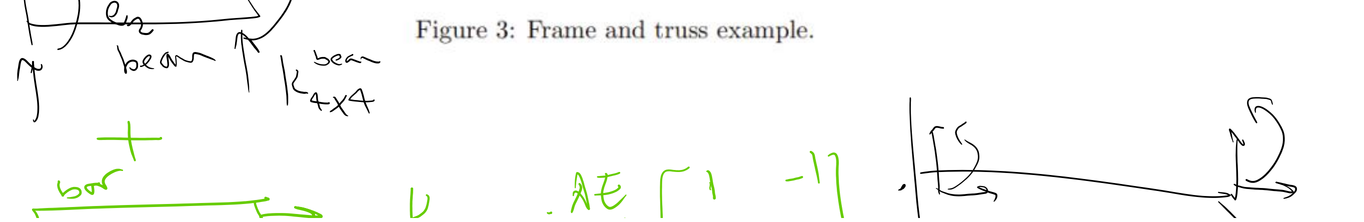
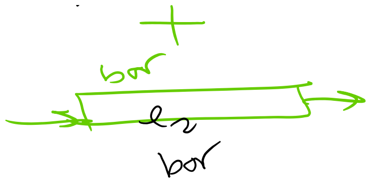


Figure 3: Frame and truss example.





$$K_{bar} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$