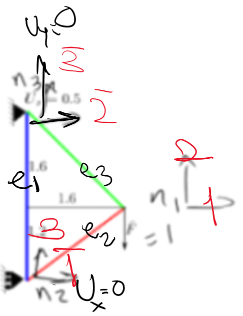
2023/11/27

Monday, November 27, 2023 11:19 AM

From last time

Step 14: Compute prescribed dof forces



$n_f = 3 \quad n_p = 3 \quad n = 6$

node	p	pos	v	f
1	0	1	-.2123	0
	0	2	-3.29	0
2	1	$\bar{1}$	0.0	0 ?
	0	3	-1.2	-1
3	1	$\bar{2}$	0.5	0 p
	1	$\bar{3}$	0.0	0 ?

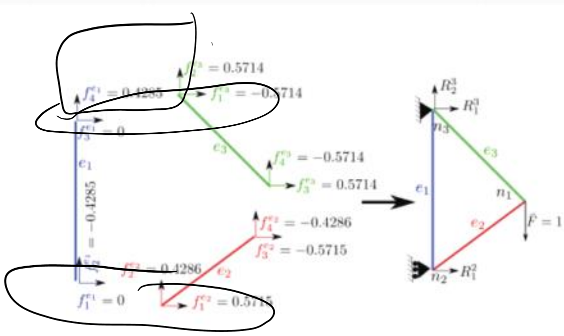
$$U_p = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} -.2123 \\ -3.29 \\ -1.2 \end{bmatrix} \qquad K_{n_f \times n_f} \qquad F_n = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

size n_p

$$F_p = \begin{array}{c} \bar{1} \\ \bar{2} \\ \bar{3} \end{array}\begin{bmatrix} \\ \\ \ \end{bmatrix}$$

e	LEM	dof Map	dofs	fe
1	[2 3]	[$\bar{1}$ 3 $\bar{2}$ $\bar{3}$]	[0.0 -1.2 .5 0]	
2	[2 1]	[$\bar{1}$ 3 1 2]	[0.0 -1.2 -.2123 -3.29]	
3	[3 1]	[$\bar{2}$ $\bar{3}$ 1 2]	[.5 0 -.2123 -3.29]	

Truss Example: Reaction Forces



- First, we compute reaction forces by adding up forces from individual elements that contribute to reaction forces:

$$R_1^2 = f_1^{e_1} + f_1^{e_2} = 0 + 0.5715 = 0.5715 \tag{397a}$$
$$R_1^3 = f_3^{e_1} + f_1^{e_3} = 0 + -0.5714 = -0.5714 \tag{397b}$$
$$R_2^3 = f_4^{e_1} + f_2^{e_3} = 0.4285 + 0.5714 = 0.9999 \tag{397c}$$

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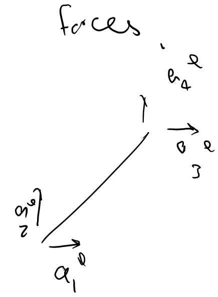
We first need to calculate each element's forces.

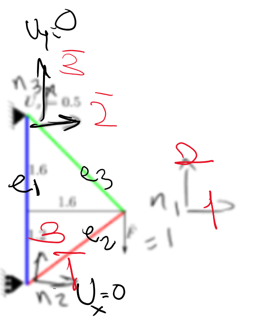


$$\underset{\text{for nodal force}}{f^e} = \underbrace{k^e a^e}_{f_D^e} - f_f^e - f_N^e$$

$$= -f_e = -(f_r^e + f_N^e - f_D^e)$$

$$= -\underbrace{f_e}_{\text{assembly}} = -(f_r + f_N - f_P^e)$$

for term project we simply have $f^e = k^e a^e$



$n_f = 3$ $n_p = 3$ $n = 6$

node	p	pos	v	f
1	0	1	$-.2123$	0
	0	2	-3.29	0
2	1	$\bar{1}$	0.0	.5715
	0	3	-1.2	-1
3	1	$\bar{2}$	0.5	$-.5715$
	1	$\bar{3}$	0.0	1.

$$U_p = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} -.2133 \\ -3.29 \\ -1.2 \end{bmatrix} \qquad K_{n_f \times n_f} \qquad F_n = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

size n_f

$$F_p = \begin{array}{c} \bar{1} \\ \bar{2} \\ \bar{3} \end{array}\begin{bmatrix} 0 + .5715 \\ 0 - .5715 \\ .4285 + .5714 \end{bmatrix} = \begin{bmatrix} .5715 \\ -.5715 \\ 1 \end{bmatrix}$$

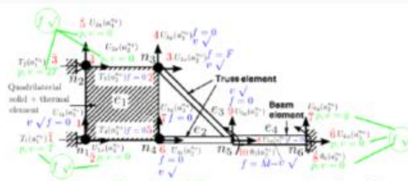
e	LEM	dof Map	dofs	fe
1	[2 3]	[$\bar{1}$ 3 $\bar{2}$ $\bar{3}$]	[0.0 -1.2 .5 0]	[0, $-.4285$ 0 .4285]
2	[2 1]	[$\bar{1}$ 3 1 2]	[0.0 -1.2 $-.2123$ -3.29]	[.5715 .4286 $-.5715$ $-.4286$]
3	[3 1]	[$\bar{2}$ $\bar{3}$ 1 2]	[.5 0 $-.2123$ -3.29]	[$-.5714$ 0.5714 .5714 $-.5714$]

$f = k^e a^e$

for e = 1:ne loop over elements
  fee = feo element total force = element all forces except essential force
  for i = 1:nedof loop over rows of ke; nedof = element # dof
    l = dofMap(i) local to global dof map M_t^e
    if (l < 0) l corresponds to a prescribed dof, we skip **free dofs**
      for j = 1:nedof loop over columns of ke. ALL columns (dofs) of p and f used
        fee(i) = fee(i) - ke(i, j) * edofs(j) edofs: element dofs = a^e
      end
      Fp(-l) = Fp(-l) - fee(i)
        1. element's total force fee component i'th is computed→added to Fp(-l)
        2. -l used because l < 0: *prescribed* dof
        3. fee is **subtracted**
    end
  end
end

```
for n = 1:nNodes
   for dofi = 1: node(n).nndof num dof for node (n)
      if node(n).ndof(dofi).p == true prescribed dof
         posn = node(n).ndof(dofi).pos position of dof in global prescribed force F_p
         node(n).ndof(dofi).f = Fp(-posn)
            1. set prescribed dof force to corresponding force in global Fp (F_p)
            2. posn < 0; prescribed dof
      end
   end
end
```


Step 15: Compute/output nodes & elements: a) nodes



- After free dof values and prescribed dof forces are calculated, we can output the results.
- Node output is general, while element output is specific to its type. Sample pseudocode:

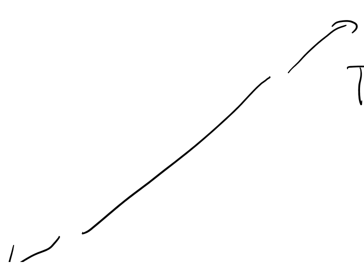
```
for n = 1:nNodes
  print: node (n) from (nNodes) nodes
  coord: (node(n).coordinate)
  for dofi = 1: node(n).nndof num dof for node (n)
    print dof:
      dof (dofi) from (node(n).nndof) dofs
        Field: (node(n).ndof(dofi).Field)
        Index: (node(n).ndof(dofi).index)
        Value: (node(n).ndof(dofi).value)
        Force: (node(n).ndof(dofi).force)
        prescribed: (node(n).ndof(dofi).p) may only be output for debugging purposes
        position: (node(n).ndof(dofi).position) may only be output for debugging purposes
  end
end
```

- Refer to slides 398 and 397 for other data members in node and dof objects.

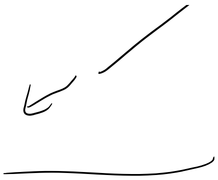
Nodes
- Value
- Force
- boolean (p)
- pos

verbose option

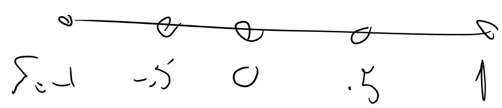
Elements f_i^e: k_a^{ae}
postprocess



$\frac{AE}{L}\left(c\left(U_3-U_1\right)+s\left(U_4-U_2\right)\right)$



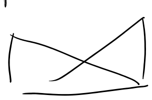
frame

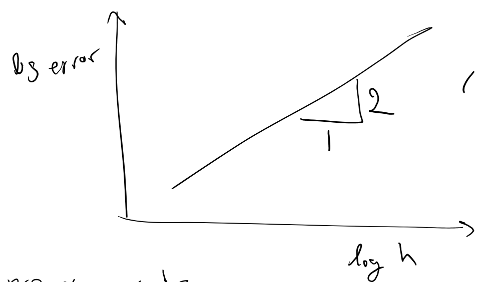


output $\begin{bmatrix} y \\ \theta \\ M \\ V \end{bmatrix}$

Higher order elements and quadrature

p=1 elements FEM

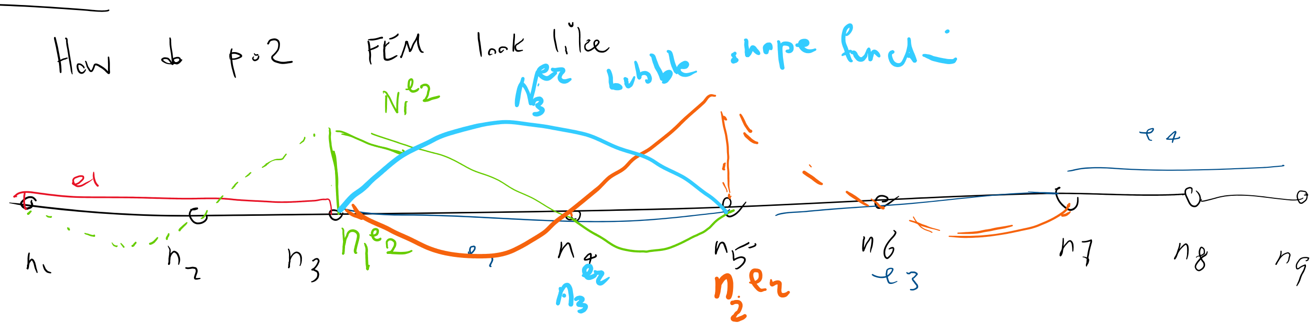




(P+1).

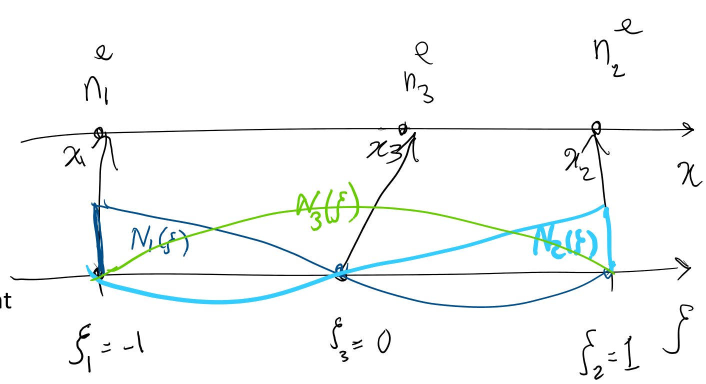
p=2 elements has faster convergence rate

How do p=2 FEM look like



Calculate shape functions for p=2 elements:

We express the shape functions in the parent element



$N_1(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2$ find α_i's

fig 1

$N_1 \ldots (\xi - \xi\ )(\xi - \xi\ )\ (\xi - 1)(\xi - 0)$

fig 1

$$N_1(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{\xi(\xi-1)}{2}$$

$N_1(\xi_1) = 1$ $\quad N(\xi_2) = 0$ $\quad N(\xi_3) = 0$

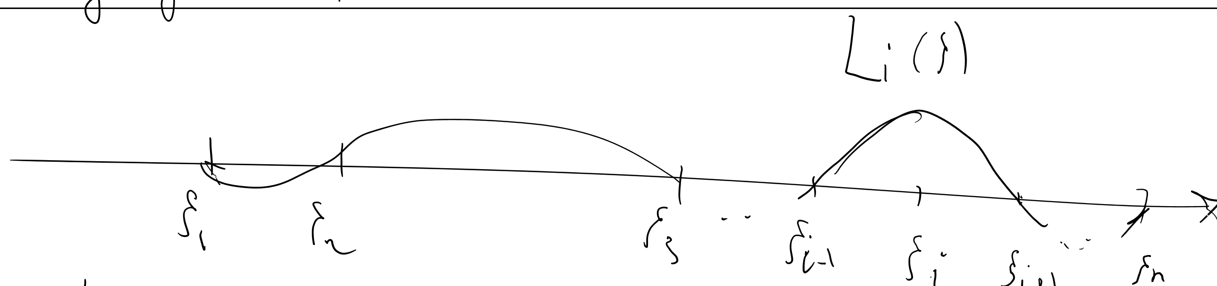
$$N_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)} = \frac{\xi(\xi+1)}{2}$$

$$N_3(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)} = 1-\xi^2$$

(1) $\boxed{N(\xi) = [N_1(\xi), N_2(\xi), N_3(\xi)] = \left[\frac{\xi(\xi-1)}{2}, \frac{\xi(\xi+1)}{2}, 1-\xi^2\right]}$

Lagrange Polynomials

$L_i(\xi)$



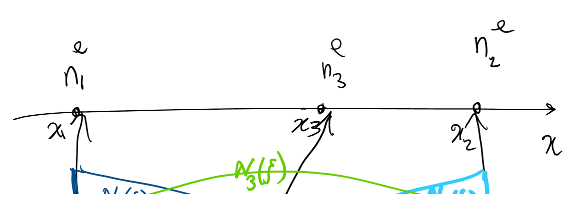
$$L_i(\xi) = \frac{\prod_{j\neq i}(\xi-\xi_j)}{\prod_{j\neq i}(\xi_i-\xi_j)}$$ (2)

$K = ?$ $\quad$ bar : weak $\quad \int w' \underbrace{EA}_{D = EA} u' \, dx$

$\underline{Lm = ( )'}$

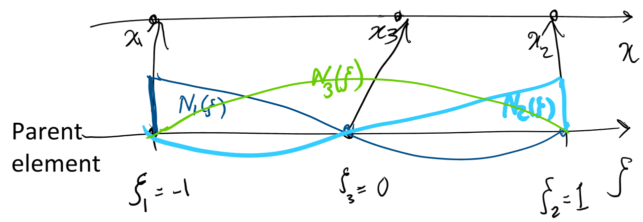
$$K^e = \int_{x_1}^{x_3} \underbrace{\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}}_{B^t} \underbrace{EA}_{D} [B_1 \; B_2 \; B_3] \, dx$$ (3)

$B = \frac{d}{dx} N$



$\int_{x_1} [\;\;] B^T \; D$

$$B = \frac{d}{dx} N$$



$\xi \longleftrightarrow x$

pull the integral back to ξ

$$B = \frac{d}{dx} N = \underbrace{\frac{1}{\left(\frac{dx}{d\xi}\right)}}_{J} \cdot \overbrace{\frac{d}{d\xi} N}^{B_\xi} \quad , \quad B = \frac{1}{J} B_\xi \qquad (4)$$

$$dx = \underbrace{\left(\frac{dx}{d\xi}\right)}_{J} d\xi \qquad (5)$$

plug (4) & (5) into (3)

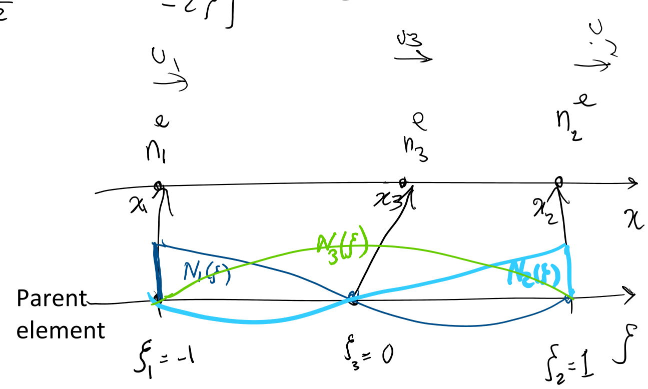
$$k^e = \int_{-1}^{1} \frac{1}{J} \begin{bmatrix} B_{\xi_1} \\ B_{\xi_2} \\ B_{\xi_3} \end{bmatrix} EA \, \frac{1}{J} \begin{bmatrix} B_{\xi_1} & B_{\xi_2} & B_{\xi_3} \end{bmatrix} (J \, d\xi) \qquad (6)$$

$B_\xi = ?$ $\quad B_\xi = \frac{d}{d\xi} N = \frac{d}{d\xi} \left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2 \right]$

$$\rightarrow \quad B_\xi = \left[\xi - \frac{1}{2} \qquad \xi + \frac{1}{2} \qquad -2\xi \right] \qquad (7)$$

$x(\xi) = ?$

Idea



u_1, u_2, u_3 are nodal displacements

$$u(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) + u_3 N_3(\xi)$$

eg $\xi = \xi_1 = -1$ $\quad u(\xi_1) = u_1 \underbrace{N_1(\xi_1)}_{1} + u_2 \underbrace{N_2(\xi_1)}_{0} + u_3 \underbrace{N_3(\xi_1)}_{0} = u_1$

$$\boxed{x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi)} \qquad (8)$$

$$J = \frac{dx}{d\xi} = x_1 \underbrace{\frac{dN_1}{d\xi}}_{B_{\xi_1}} + x_2 \underbrace{\frac{dN_2}{d\xi}}_{B_{\xi_2}} + x_3 \underbrace{\frac{dN_3}{d\xi}}_{B_{\xi_3}} = x_1\left(\xi - \frac{1}{2}\right) + x_2\left(\xi + \frac{1}{2}\right) + x_3(-2\xi)$$

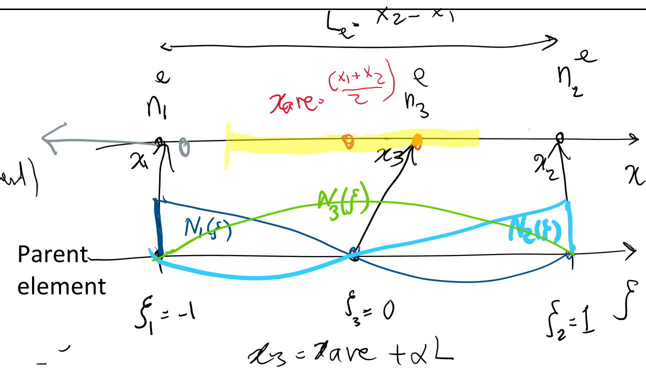
$$J(\xi) = 2\left(\frac{x^1+x^2}{2} - x^3\right)\xi + \frac{1}{2}(x_2 - X_1)$$

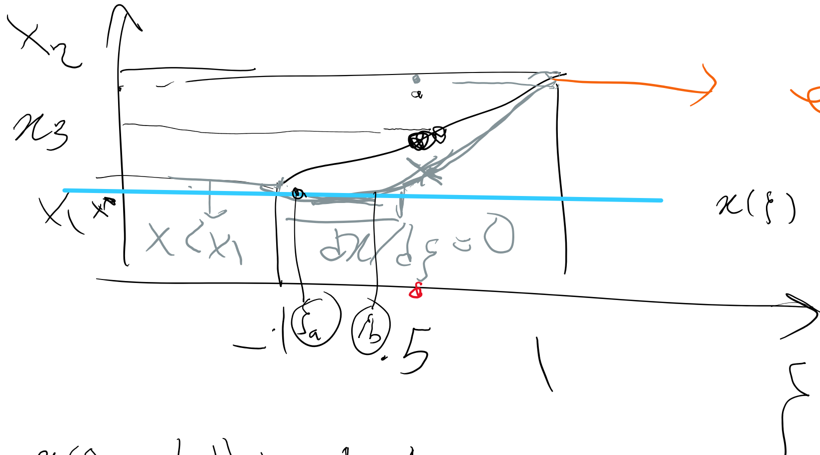
(9)

$$J(\xi) = 2(x_{ave} - x^3)\xi + \frac{L_e}{2}$$

. if $x_3 = x_{ave}$ (unskewed element)

$J(\xi) = \frac{L_e}{2}$ linear $\xi \to x$





eg the figure

$x(\xi)$ is not 1-1

$x(\xi)$ should be 1-1,

In HW you'll show that for $|\alpha| = .25$

$J(\xi) = 0$ @ one point @ $|\alpha| > .25$ is not acceptable

In fracture mechanics for alpha = 0.25 we can recover LEFM singular stress and strain fields with FEM

$$K^e = \int_{-1}^{1} \frac{EA(\xi)}{\underbrace{2\xi(x_{ave}-x_3)+\frac{L_e}{2}}_{J(\xi)}} \begin{bmatrix}\xi-\frac{1}{2}\\ \xi+\frac{1}{2}\\ -2\xi\end{bmatrix}\left[\xi-\frac{1}{2},\ \xi+\frac{1}{2},\ -2\xi\right] d\xi$$

if $EA(\xi)$ = constant Homogeneous material & prismatic

if $x_3 = x_{ave}$ unskewed element $J = \frac{L}{2}$

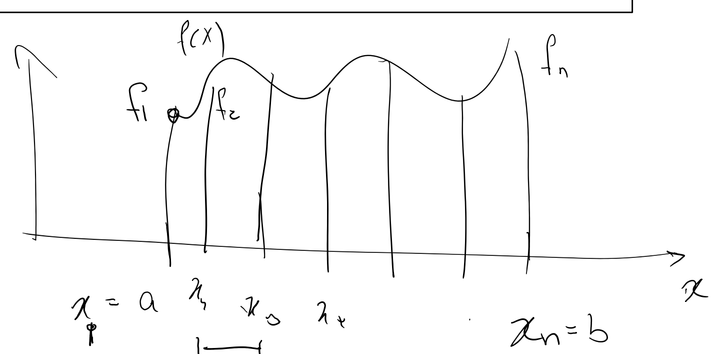
$$K^e = \frac{AE}{L}\begin{bmatrix} 7/3 & -8/3 & 1/3 \\ & 7/3 & -8/3 \\ sym & & 16/3 \end{bmatrix} \quad (10)$$

Numerical integration (quadrature)

Newton-Cotes method

$$\boxed{\int_a^b f(x)\,dx \approx L \sum_{i=1}^{n} w_i f(x_i)} \quad (11)$$

NC



$h = \frac{x_n - x_1}{n-1}$ $L = b - a$

n pts

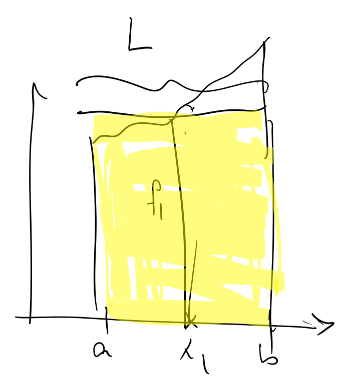
n-1 segments

for different n's we have different w's

$n=1$

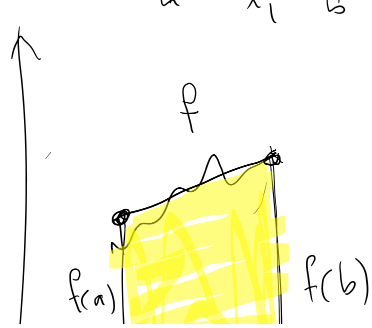
$$\int_a^b f(x)\,dx = L\, f(x_1)\times 1$$

rectangular rule $w_1 = 1$



$n=2$

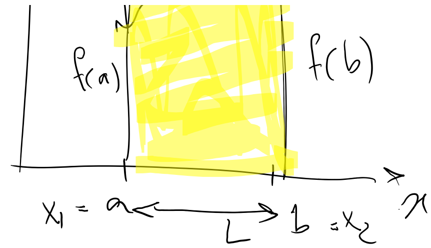
$$\int_a^b f(x)\,dx = L\left(\frac{f(a)+f(b)}{2}\right)$$



$$\int_a f(x)\,dx = L\left(\frac{\quad}{2}\right)$$

$$= L\left(w_1 f(x_1) + w_2 f(x_2)\right)$$

$$w_1 = w_2 = \frac{1}{2}$$



trapezoidal rule