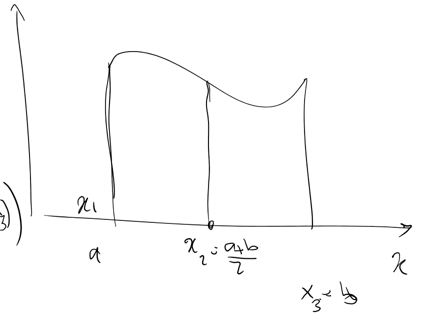
2023/11/29

Wednesday, November 29, 2023 11:22 AM

Newton-Cotes method continued

Simpson's rule

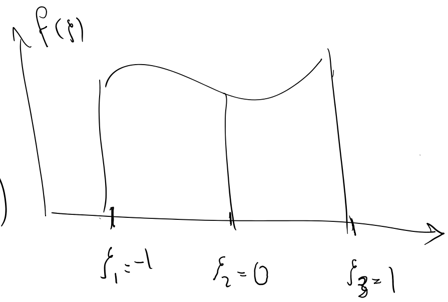
$$\int_a^b f(x)\,dx = (b-a)\left(\underset{\uparrow\ \frac{1}{6}}{\omega_1} f(x_1) + \underset{\uparrow\ \frac{4}{6}}{\omega_2} f(x_2) + \underset{\uparrow\ \frac{1}{6}}{\omega_3} f(x_3)\right)$$



why?

I'll demonstrate this for ξ coordinate

$$\int_{-1}^{1} f(\xi)\,d\xi = (1-(-1))(\omega_1 f(\underset{\xi_1}{-1}) + \omega_2 f(\underset{\xi_2}{0}) + \omega_3 f(\underset{\xi_3}{1})) \quad (1)$$



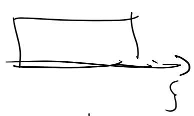
We have 3 unknowns w1 to w3: we need 3 equations to solve for w1 to w3

How about requiring that we integrate certain polynomials exactly?

$$f(\xi) = \underbrace{\alpha_0}_{eq1} + \underbrace{\alpha_1\xi}_{eq2} + \underbrace{\alpha_2\xi^2}_{eq3} + \alpha_3\xi^3 + \text{— —}$$

need to ensure 1, ξ, ξ^2 are integrated exactly (using eqn (1))

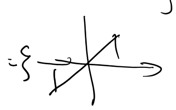
$f(\xi)=1$: $\int_{-1}^{1} \underset{f(\xi)}{1}\,d\xi = 2(\omega_1 f(-1) + \omega_2 f(0) + \omega_3 f(1)) = 2(\omega_1 + \omega_2 + \omega_3) = 2$



(i) $\boxed{\omega_1 + \omega_2 + \omega_3 = 1}$

NC always $\Sigma\omega_i = 1$

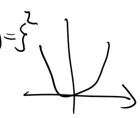
$f(\xi)=\xi$



$\int_{-1}^{1} \xi\,d\xi = 2(\omega_1 f(\underset{-1}{1}) + \omega_2 f(\underset{0}{0}) + \omega_3 f(\underset{1}{1})) = 2(\omega_3 - \omega_1) = 0$

(ii) $\boxed{\omega_3 - \omega_1 = 0}$

$f(\xi)=\xi^2$



$\int_{-1}^{1} \xi^2\,d\xi = 2(\omega_1 f(\underset{(-1)^2}{-1}) + \omega_2 f(\underset{(0)^2}{0}) + \omega_3 f(\underset{(1)^2}{1})) = 2(\omega_3 + \omega_1) = 2\int_0^1 \xi^2\,d\xi = \frac{2}{3}\xi^3\Big|_0^1 = \frac{2}{3}$

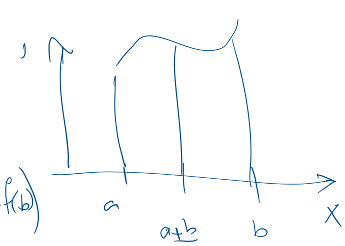
(iii) $\boxed{\omega_3 + \omega_1 = \frac{1}{3}}$

ii & iii $\longrightarrow$ $\omega_1 = \omega_3 = \frac{1}{6}$

i $\longrightarrow$ $\omega_2 = \frac{4}{6}$

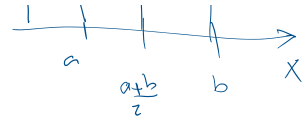
Simpson's rule

(1) $$\int_a^b f(x)\,dx = (b-a)\left(\frac{1}{6}f(a) + \frac{4}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right)$$



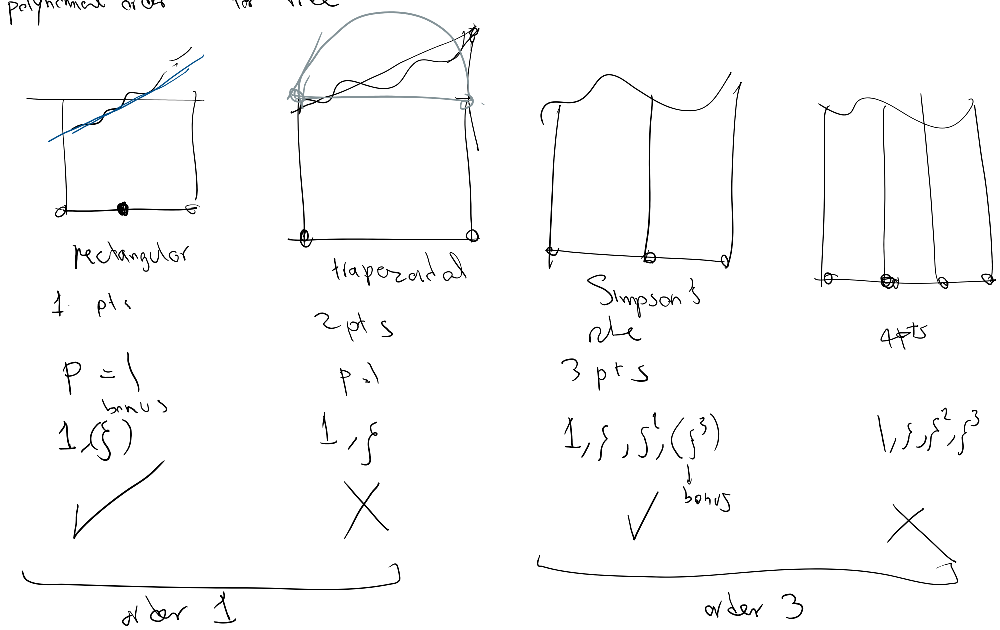
(1)

$$\int_a^b f(x)\,dx = (b-a)\left(\frac{1}{6}f(a) + \frac{4}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right)$$



The maximum polynomial that this rule can integrate exactly is:
2+ 1 = 3

in NC methods that with odd # pts give one extra polynomial order for free



polynomial order	Intervals, i	No. of Points, n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	1	2	1/2	1/2			(trapezoid rule)		
3	2	3	1/6	4/6	1/6		(Simpson s 1/3 rule)		
3	3	4	1/8	3/8	3/8	1/8	(Simpson s 3/8 rule)		
5	4	5	7/90	32/90	12/90	32/90	7/90		
5	5	6	19/288	75/288	50/288	50/288	75/288	19/288	
7	6	7	41/840	216/840	27/840	272/840	27/840	216/840	41/840

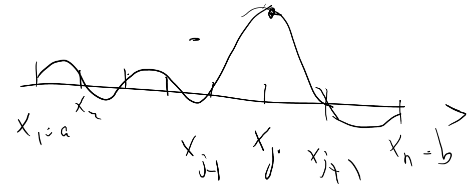
TABLE 5.5 *Newton-Cotes numbers and error estimates*

Number of intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{II}(r)$
2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{IV}(r)$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{IV}(r)$
4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{VI}(r)$
5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{VI}(r)$
6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{VIII}(r)$

Is there a faster way to obtain the weights?

$$\int_a^b f(x)\,dx = (b-a)\sum_{i=1}^{n} \omega_i f(x_i)$$

$$= (b-a)\left(\omega_1 f(x_1) + \dots + \omega_j f(x_j) + \dots + \omega_n f(x_n)\right)$$



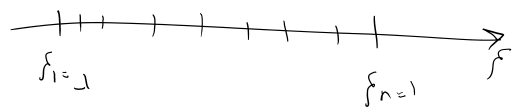
$$L_j(x) = \begin{cases} 1 & @\ x_j \\ 0 & @\ x_i \quad i \neq j \end{cases}$$

$$\int_a^b L_j(x)\,dx = (b-a)\,\omega_j \rightarrow \quad \omega_j = \frac{\int_a^b L_j(x)\,dx}{b-a}$$

better to do this is ξ:

(2)

$$\boxed{\omega_j = \frac{1}{2}\int_{-1}^{1} L_j(\xi)\,d\xi}$$



Gauss quadrature:

In Newton-Cotes every point is counted as once (just give one extra polynomial order improvement)

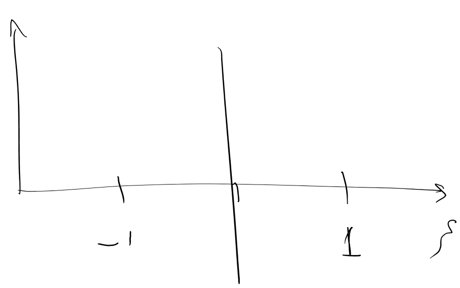
In Gauss quadrature each point gives 2 extra polynomial order improvement

Gauss quadrature

$$\int_{-1}^{1} f(\xi)\,d\xi = \sum_{i=1}^{n} \omega_i f(\xi_i)$$

Let's do the two point scheme:

$$\int_{-1}^{1} f(\xi)\,d\xi = \omega_1 f(\xi_1) + \omega_2 f(\xi_2)$$



$\omega_1, \omega_2, \xi_1, \xi_2$ are to be determined

$$f(\xi) = \alpha_0 1 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 + \alpha_4 \xi^4 - -$$

$\checkmark \quad \checkmark \quad \checkmark \quad \checkmark$

we should be able to integrate up to ξ^3 exactly we should be able to integrate up to ξ^3 exactly

$f(\xi)=1$: $\int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1} 1\, d\xi = 2 = \omega_1 \overset{1}{f(\xi_1)} + \omega_2 \overset{1}{f(\xi_2)} \rightarrow$

$f(\xi)=\xi$: $\int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1}\xi d\xi = 0 = \omega_1 f(\xi_1) + \omega_2 f(\xi_2) = \omega_1\xi_1 + \omega_2\xi_2$

$f(\xi)=\xi^2$: $\int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1}\xi^2 d\xi = \frac{2}{3} = \omega_1 f(\xi_1) + \omega_2 f(\xi_2) = \omega_1\xi_1^2 + \omega_2\xi_2^2$

$f(\xi)=\xi^3$: $\int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1}\xi^3 d\xi = 0 = \text{''} \quad \text{''} = \omega_1\xi_1^3 + \omega_2\xi_2^3$

$$\boxed{\begin{aligned} &\omega_1 + \omega_2 = 2 && (e1)\\ &\omega_1\xi_1 + \omega_2\xi_2 = 0 && (e2)\\ &\omega_1\xi_1^2 + \omega_2\xi_2^2 = \frac{2}{3} && (e3)\\ &\omega_1\xi_1^3 + \omega_2\xi_2^3 = 0 && (e4)\end{aligned}}$$

Multiply (e2) by ξ_1^2: $\omega_1\xi_1^3 + \omega_2\xi_1^2\xi_2 = 0$

e4: $\omega_1\xi_1^3 + \omega_2\xi_2^3 = 0$

$\rightarrow \omega_2\xi_2\underbrace{(\xi_1-\xi_2)(\xi_1+\xi_2)}_{\xi_1^2-\xi_2^2} = 0$

$\omega_2 = 0 \longrightarrow \cancel{\ }$ 1 pt scheme

$\xi_2 = 0$

$\xi_1 = \xi_2 \longrightarrow \cancel{\ }$ 1 pt scheme

$\boxed{\xi_1 = -\xi_2}$ this is the only valid solution

eq3: $(\omega_1+\omega_2)\xi_1^2 = \frac{2}{3}$

eq1: $\omega_1 + \omega_2 = 2$

$\longrightarrow \xi_1^2 = \frac{1}{3}$

$\rightarrow \xi_1 = \pm\frac{1}{\sqrt{3}} \qquad \xi_2 = -\xi_1 = \mp\frac{1}{\sqrt{3}}$ we want $\xi_1 < \xi_2$

$$\boxed{\xi_1 = \frac{-1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}}}$$

eq1: $\omega_1 + \omega_2 = 2$

eq2: $\omega_1\xi_1 + \omega_2\xi_2 = 0 \rightarrow \frac{1}{\sqrt{3}}(\omega_2 - \omega_1) = 0$

$\longrightarrow \omega_1 = \omega_2 = 1$

$$\int_{-1}^{1} f(\xi)\,d\xi = \omega_1 f(\xi_1) + \omega_2 f(\xi_2)$$

$$\omega_1 = \omega_2 = 1$$

$$\xi_1 = \frac{-1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}}$$

2 pt Gauss quadrature

It integrates up to O (order) 3 polynomials

3

From HW5

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$

roots of this $3x^2-1=0$ $x = \pm\frac{1}{\sqrt{3}}$

matches what we have above

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

above

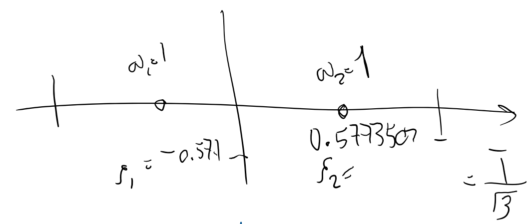
roots of this $3x^2-1=0$ $x = \pm\frac{1}{\sqrt{3}}$

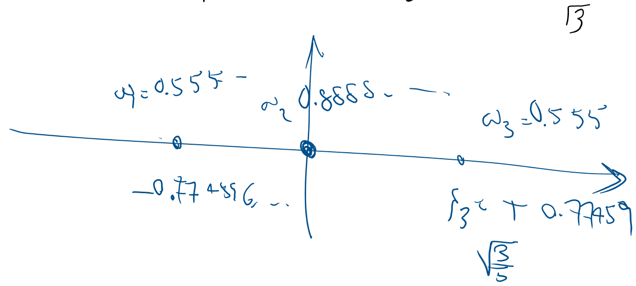
$x=0$, $x = \mp\sqrt{\frac{3}{5}}$

Figure 4: Legendre polynomials (Source: http://en.wikipedia.org/wiki/Legendre_polynomials

$$\int_{-1}^{1} P_m(\xi)P_n(\xi)\,\mathrm{d}\xi = \frac{2}{2n+1}\delta_{mn} \quad \text{no sum on } n \tag{4}$$

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
$\mp$ 0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
$\mp$ 0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189



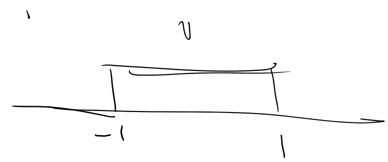


- The global nature of trial functions ϕ in spectral method results in full $\mathbf{K}$ matrices that are expensive to solve.
- To circumvent this problem we employ trial functions that make $\mathbf{K}$ diagonal.
- In weak statement $K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_{\mathcal{D}} L_m^w(\phi_i) L_m(\phi_j)\,\mathrm{d}v$.
- If the problem is self-adjoint $\mathcal{A}(.,.)$ is an inner product and we can construct an orthogonal trial function basis ϕ_i for example using Gram Schmidt method.
- Given the particular form of $\mathcal{A}$ (from L_m^w and L_m) and domain of integration $\mathcal{D}$ ($[0\ 1]$, $[-1\ 1]$, semi-infinite, infinite, *etc.*) we employ various trigonometric and orthogonal polynomial spaces. Some examples are:
  - $\phi_k(x) = e^{ikx}$ Fourier spectral method.
  - $\phi_k(x) = T_k(x)$ Chebyshev spectral method.
  - $\phi_k(x) = L_k(x)$ or $P_k(x)$ Legendre spectral method.
  - $\phi_k(x) = \mathcal{L}_k(x)$ Laguerre spectral method.
  - $\phi_k(x) = H_k(x)$ Hermite spectral method.

  where $T_k(x)$, $L_k(x)(P_k(x))$, $\mathcal{L}_k(x)$, and $H_k(x)$ are the Chebyshev, Legendre, Laguerre, and Hermite polynomials of degree k, respectively.
- The orthogonal property of these functions is for simple geometries. That is why spectral methods are more popular for simple geometries where we can take advantage of their exponential convergence property while keeping computational costs low by using orthogonal trial functions.

$$\int_{-\infty}^{\infty} \phi_i(x)\ \phi_j(x)\ \rho(x)\ dx = 0 \quad \text{for } i \neq j$$

Legendre $\rho =$



Summary

n ~~pts~~ $\longrightarrow$ o

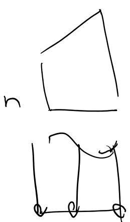
order of integration

(polynomial order o that can be integrated exactly)

Newton-Cotes

$\underline{n \longrightarrow o}$

n $\qquad$ $o = \begin{cases} n-1 & \text{even } n \\ n & \text{odd } n \end{cases}$



Gauss Quadrature

n $\qquad$ $o = 2n - 1 \rightarrow$

$o \longrightarrow n$

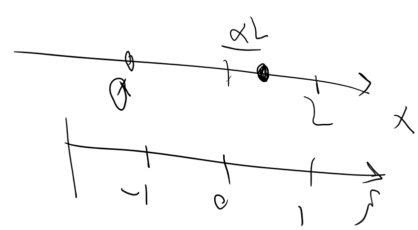
NC $\qquad$ $n = \begin{cases} o & o \text{ odd} \\ o+1 & o \text{ even} \end{cases}$

case often encountered in FEM

Gauss $\qquad$ $n = \operatorname{ceil}\left(\frac{o+1}{2}\right)$

From last class

$$K^e = \int_{-1}^{1} \overbrace{\frac{EA(\xi)}{\underbrace{2\xi(x_{arc}-x_3)+\frac{L_e}{2}}_{J(\xi)}} \begin{bmatrix} \xi-\frac{1}{2} \\ \xi+\frac{1}{2} \\ -2\xi \end{bmatrix} \left[\xi-\frac{1}{2}, \ \xi+\frac{1}{2}, \ -2\xi\right]}^{I(\xi)} d\xi$$



We want to integrate K^e

If

1. Material is heterogeneous (E and A not constant) OR
2. Distorted (skewed) (J not constant)

We cannot integrate K with ANY number of NC or G points in general.
However, this is something called FULL INTEGRATION order in that, JUST in deciding the number of points needed, we ignore material part (EA above) and Jacobian term (J above) temporarily.

$$\begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \left[\xi - \frac{1}{2}, \ \xi + \frac{1}{2}, \ -2\xi \right] \qquad O = 2$$

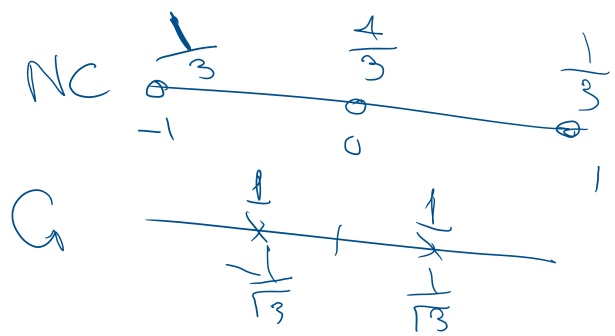
NC $\quad n = O + 1 \quad$ even O

$= 3 \quad$ Simpson's rule

G $\quad n = \text{ceil}\left(\frac{O+1}{2}\right) = \text{ceil}\left(\frac{3}{2}\right) = 2$

NC $\quad K = \underbrace{(1 - -1)}_{\text{length}} \left(\frac{1}{6} I(-1) + \frac{4}{6} I(0) + \frac{1}{6} I(1) \right)$

Gauss $\quad K = \left(1 \, I\left(\frac{-1}{\sqrt{3}}\right) + 1 \, I\left(\frac{1}{\sqrt{3}}\right) \right)$



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1. $\boxed{\textbf{50 Points}}$ Use a 3 point Gauss and 5 point Newton-Cotes quadrature rule to evaluate the following integral and obtain their respective errors with respect to exact value of the integral $I_e = \tan^{-1}(2) - \tan^{-1}(-1)$. Quadrature points and weights are given in fig. 1.

$$I = \int_{-1}^{2} \frac{dx}{1 + x^2}$$

5 points NCs

$h = \frac{2 - (-1)}{\ldots} = \frac{3}{\ldots}$

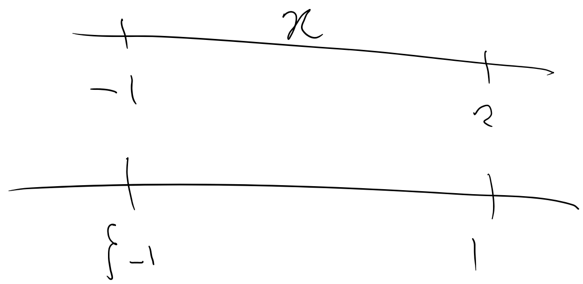
-1 $\quad 1 \quad 2$

$$h = \frac{2-(-1)}{5-1} = \frac{3}{4}$$

Intervals, i	No. of Points, n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	2	1/2	1/2			(trapezoid rule)		
2	3	1/6	4/6	1/6		(Simpson s 1/3 rule)		
3	4	1/8	3/8	3/8	1/8	(Simpson s 3/8 rule)		
4	5	7/90	32/90	12/90	32/90	7/90		
5	6	19/288	75/288	50/288	50/288	75/288	19/288	
6	7	41/840	216/840	27/840	272/840	27/840	216/840	41/840

Gauss 3pts

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189



$$x = (-1)\left(\frac{1-\xi}{2}\right) + 2\left(\frac{1+\xi}{2}\right)$$

$$dx = \frac{3}{2}\,d\xi$$

$$I = \int_{-1}^{2} \frac{1}{1+x^2}\,dx = \int_{-1}^{1} \frac{1}{1+x(\xi)^2}\left(\frac{3}{2}\,d\xi\right)$$

then use Gauss Quad