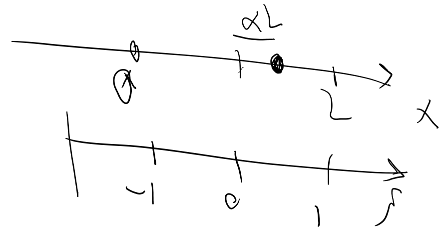
2023/12/04

Monday, December 4, 2023 11:13 AM

$$K^e = \int_{-1}^{1} \overbrace{\frac{EA(\xi)}{\underbrace{2\xi(x_{arc} - x_3) + \frac{L_e}{2}}_{J(\xi)}} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \left[\xi - \frac{1}{2}, \ \xi + \frac{1}{2}, \ -2\xi \right]}^{I(\xi)} d\xi$$



we want to integrate K^e

If

1. Material is heterogeneous (E and A not constant) OR
2. Distorted (skewed) (J not constant)

We cannot integrate K with ANY number of NC or G points in general.
However, this is something called FULL INTEGRATION order in that, JUST in deciding the number of points needed, we ignore material part (EA above) and Jacobian term (J above) temporarily.

$$\begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \left[\xi - \frac{1}{2}, \ \xi + \frac{1}{2}, \ -2\xi \right] \qquad O = 2$$

NC $\quad n = O + 1 \qquad$ even O

$= 3 \qquad$ Simpson's rule

G $\quad n = \text{ceil}\left(\frac{O+1}{2}\right) = \text{ceil}\left(\frac{3}{2}\right) = 2$

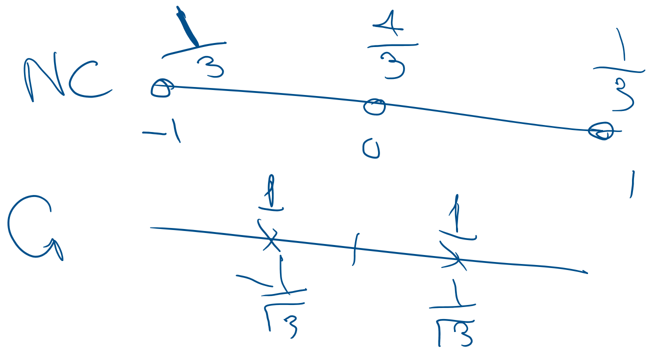
<u>NC</u>

$$K = \underbrace{(1 - -1)}_{\text{length } L} \left(\frac{1}{6} I(-1) + \frac{4}{6} I(0) + \frac{1}{6} I(1) \right)$$

Gauss

$$K = \left(1 \ I\left(\frac{-1}{\sqrt{3}}\right) + 1 \ I\left(\frac{1}{\sqrt{3}}\right) \right)$$

$$K = \left(1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + 1 \cdot f\left(\frac{1}{\sqrt{3}}\right) \right)$$

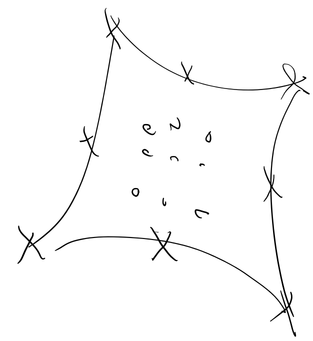


Are these values (stiffness matrices computed by NC or Gauss quadrature) exact in general?

Yes: For homogeneous material and unskewed elements, the integrals are exact.

Else: how many quadrature points are needed to integrate k exactly?
Not in general, no order would do this.

Do we need to increase the number of quadrature points beyond full integration?

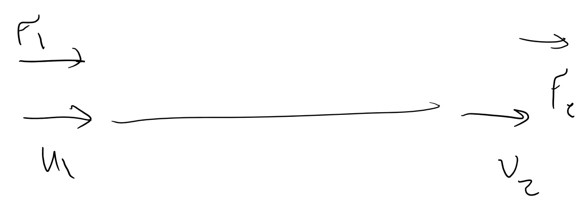


> In general, we don't need to go beyond full integration, as the error from quadrature is equal or smaller order than discretization error (going to a finite number of unknowns)
>
> Exceptions: nonlinear equations (plasticity, Navier-Stokes equations for fluids, ...) we may use a higher order than full integration.

Idea:
Can we use reduced order integration, meaning that we use even fewer than full integration order points?

Background for reduced order integration:

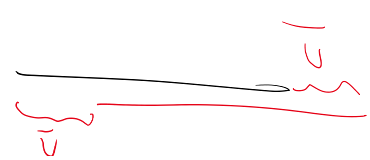
$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



Are there any u = [u1, u2] such that F = 0?

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq 0 \ni \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} u_1 - u_2 \\ -u_1 + u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow u_2 = u_1 = \bar{u}$$



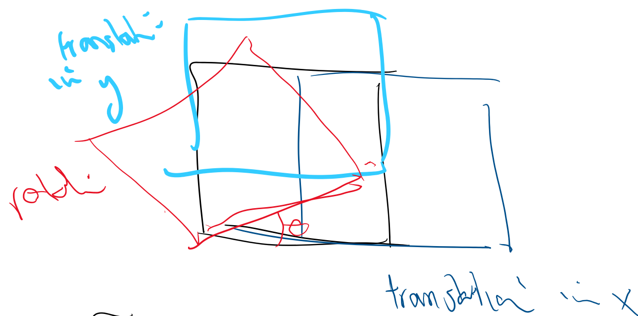
This is a rigid motion

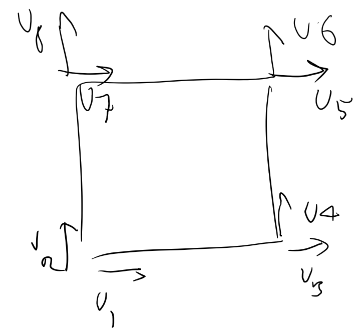
This is a rigid motion

Rigid motion does not generate force

So, it is good that for rigid motion we get a zero force

2D elasticity:





$$K_{8\times 8}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \end{bmatrix} = 0$$

How many nontrivial solutions we have?

$$\vec{U} = \begin{bmatrix} \bar{U} \\ 0 \\ \bar{U} \\ 0 \\ \bar{U} \\ 0 \\ \bar{U} \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \bar{U} \\ 0 \\ \bar{U} \\ 0 \\ \bar{U} \\ 0 \\ \bar{U} \end{bmatrix}$$

& rotation

$$K_{8\times 8} U = 0$$

$$Ku = 0 = 0u$$

Eigenvalue & eigenvector of K

$$\underbrace{K u^*}_{\text{eigenvector}} = \underbrace{\lambda^* u^*}_{\text{eigenvalue}}$$

$\Rightarrow$ For a zero mode u

$\rightarrow$ 0 is an eigenvalue

$\rightarrow$ u is a eigenvector

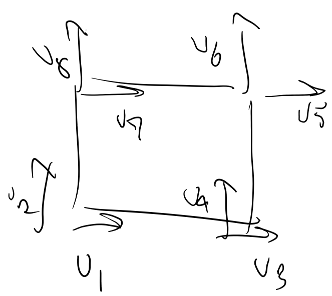
Rank of a matrix $K_{n\times n}$

$$\text{rank}(K) = n - \#\text{ zero modes}$$

$\#$ zero modes $= \#$ of zero eigenvalues

$$Ku = 0$$

1



$$\begin{bmatrix} F_1 \\ \vdots \\ F_8 \end{bmatrix} = K_{8\times 8} \begin{bmatrix} u_1 \\ \vdots \\ u_8 \end{bmatrix}$$

$$\text{rank}(K) = 8 - \underbrace{3}_{\#\text{ zero modes}} = 5$$

is the correct rank we want to get

- If the rank of the stiffness is lower than the rank implied by the size of the matrix and # of zero models, we call this rank deficiency.
- This can happen if we use reduced order integration (using fewer than full integration points)

2

$\boxed{\times}$

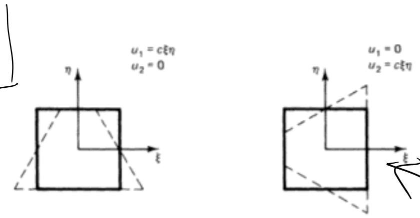


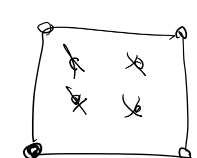
Figure 4.6.2 Hourglass modes; c is an arbitrary constant.

nonphysical zero modes

rank = 3

rigid modes

$\#$ zero modes $= 5 \neq 3 + 2$



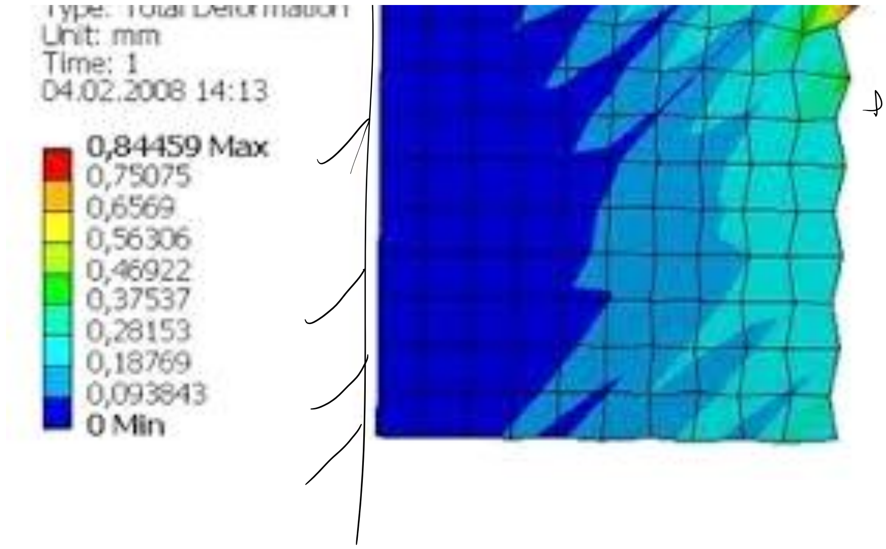
node4 node3

node1 node2

X Gauss pts

Full integration order

rank = 5



Same problem with reduced order integration of higher order elements

$$u_1 = c\xi(\eta^2 - \tfrac{1}{3})$$
$$u_2 = -c\eta(\xi^2 - \tfrac{1}{3})$$

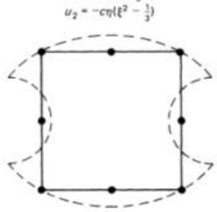
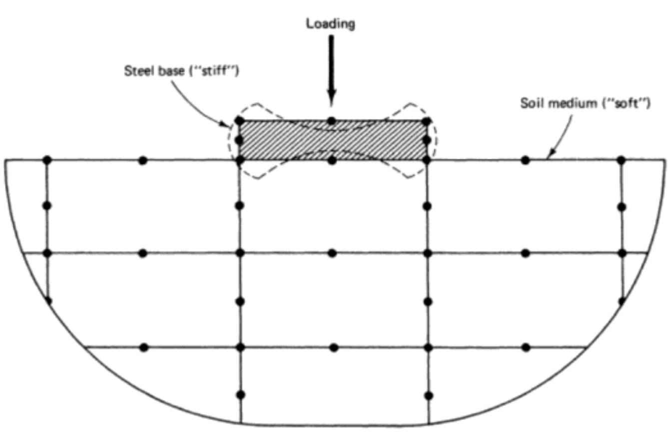


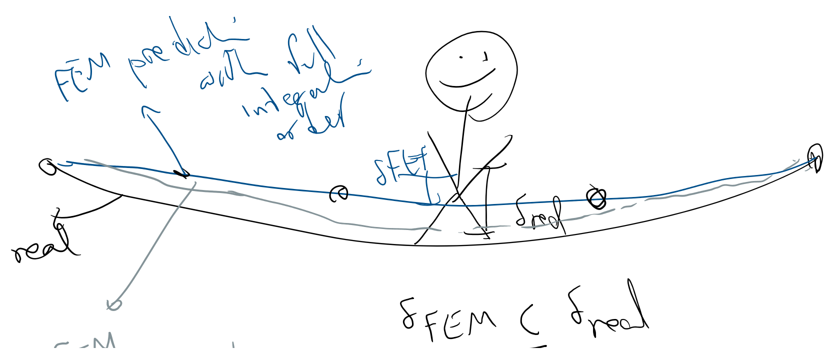
Figure 4.6.3 The spurious zero-energy mode of the reduced 2 × 2 Gaussian integration eight-node serendipity quadrilateral; *c* is an arbitrary constant.

The eight-node serendipity quadrilateral with reduced 2 × 2 quadrature possesses one spurious zero-energy mode; see Fig. 4.6.3. This mode is often described as "non-communicable" because in an assembly of two or more elements no zero-energy modes are present.



We see that reduced order integration can be dangerous by introducing zero modes. However, it also has advantages:

1. Cost (e.g. 1 versus 4 quad pts / element in the example above)
2. Finite element solutions tend to be too stiff -> reducing the integration order can remedy this to some extent.



FEM with reduced integration order

$\delta_{FEM} \subseteq \delta_{real}$

3

Reduced order integration is very good, but we need to always check that we are not introducing nonphysical zero modes

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Example, p = 2, 1D bar

(2 GPs full)

$$\underset{3\times 3}{K} = \frac{AE}{L}\begin{bmatrix} 7/3 & -8/3 & \cdots \\ & \cdots & \end{bmatrix}$$

$\text{rank}(K) = 3 - 1 = 2$

1 rigid mode

we recover it :)

$\text{rank}(K) \leq 1 < 2$

bad

2D elements

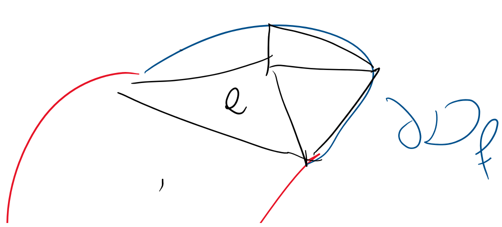
We are going to form the stiffness for 2D heat conduction problem

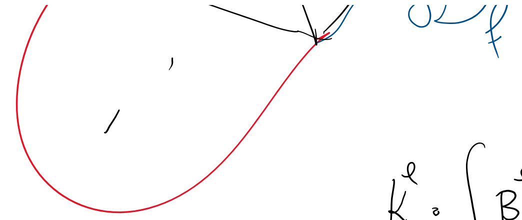
weak statement

$$\int_{\Omega} \underbrace{\nabla w}\, k\, \nabla T\, dV = \int_{\Omega} w Q\, dV - \int_{\partial\Omega_f} w \bar{q}\, d\Gamma$$

$L_m = \nabla$

$D = k$ conductivity matrix

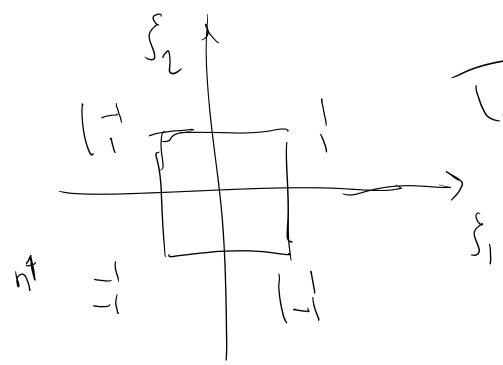




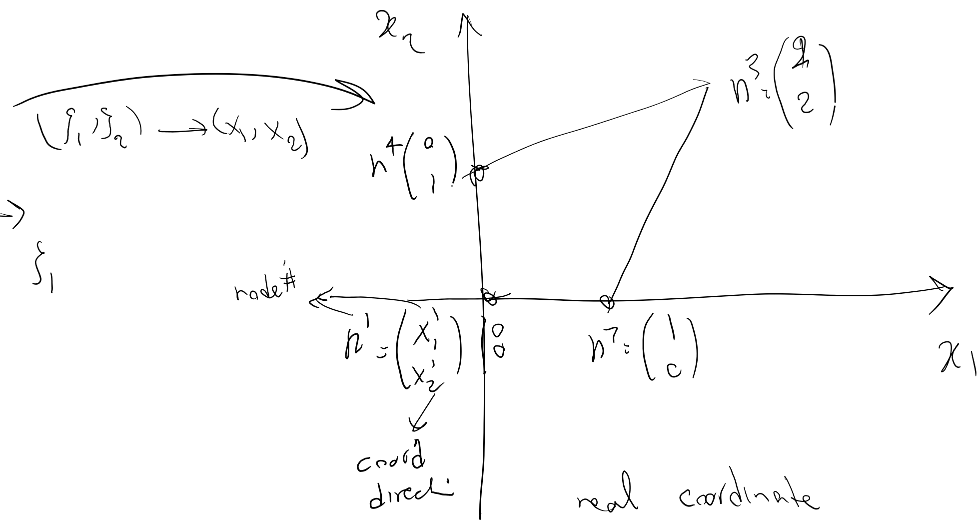
$$K^e = \int_e B^{e^t} D^e B^e \, dV, \quad B^e = L_m(N^e) = \nabla N^e$$

$$D^e = k$$

We'll focus on quad elements:

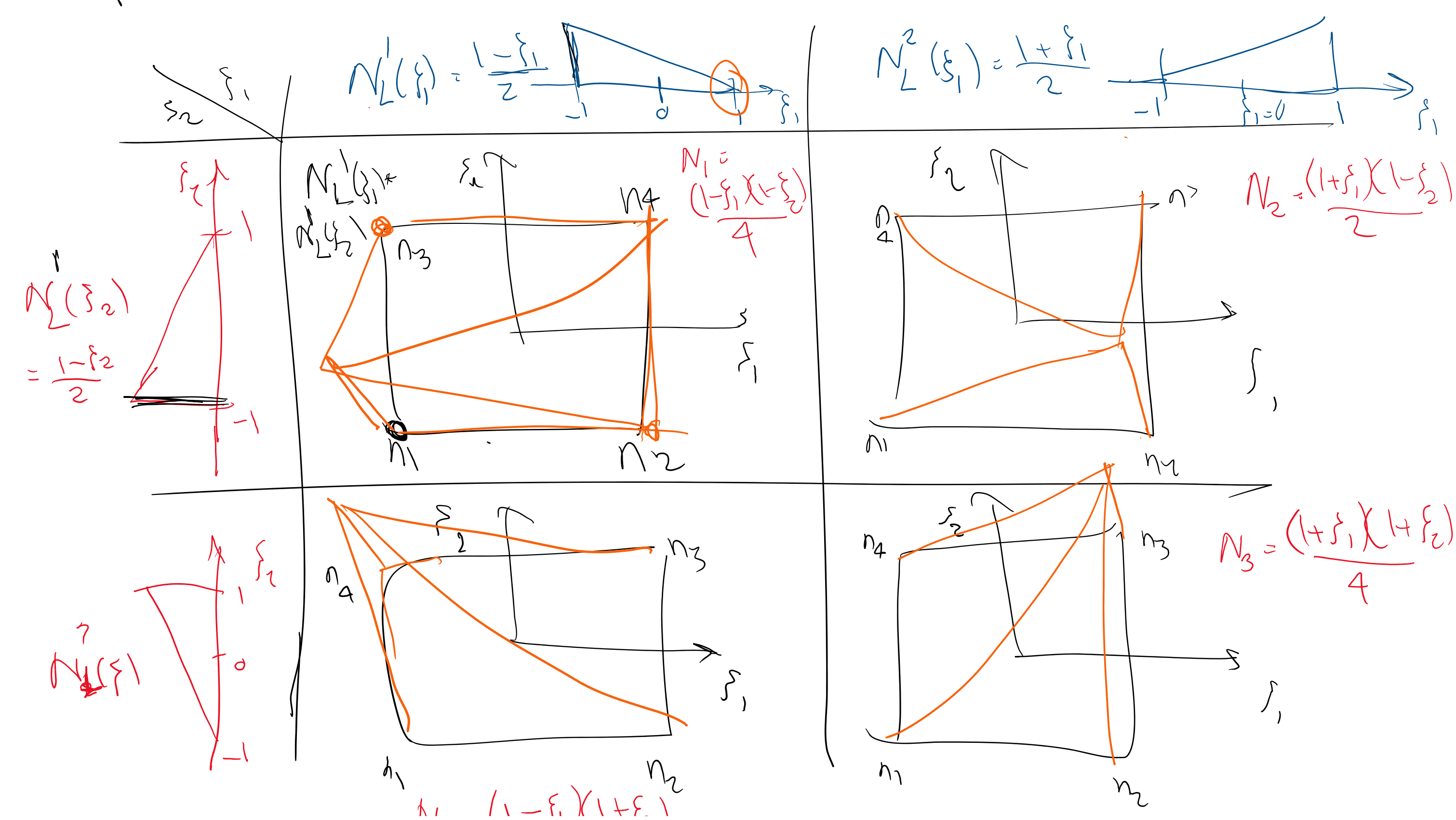


parent element



real coordinate

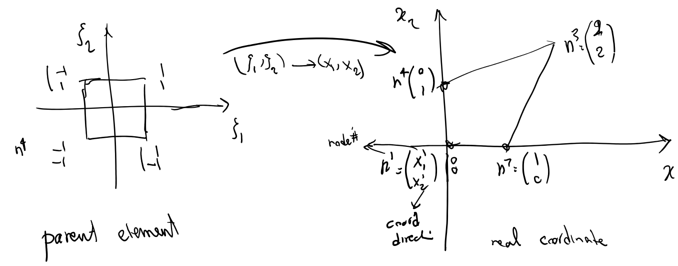
Step 1 Form the shape functions



$N_4 = \frac{(1-\xi_1)(1+\xi_2)}{4}$

(5) $$\boxed{\begin{aligned} N(\xi_1,\xi_2) &= [N_1, N_2, N_3, N_4] = \\ &\left[\frac{(1-\xi_1)(1-\xi_2)}{4}, \frac{(1+\xi_1)(1-\xi_2)}{4}, \frac{(1+\xi_1)(1+\xi_2)}{4}, \frac{(1-\xi_1)(1+\xi_2)}{4}\right]\end{aligned}}$$

$$K^e = \int_e (\underbrace{B_m(N)}_{B^T})^T D B_m(N)\, dV$$



i) $$B^e = \nabla N = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_2}{\partial x_1} & \frac{\partial N_3}{\partial x_1} & \frac{\partial N_4}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_3}{\partial x_2} & \frac{\partial N_4}{\partial x_2} \end{bmatrix}$$

use chain rule here

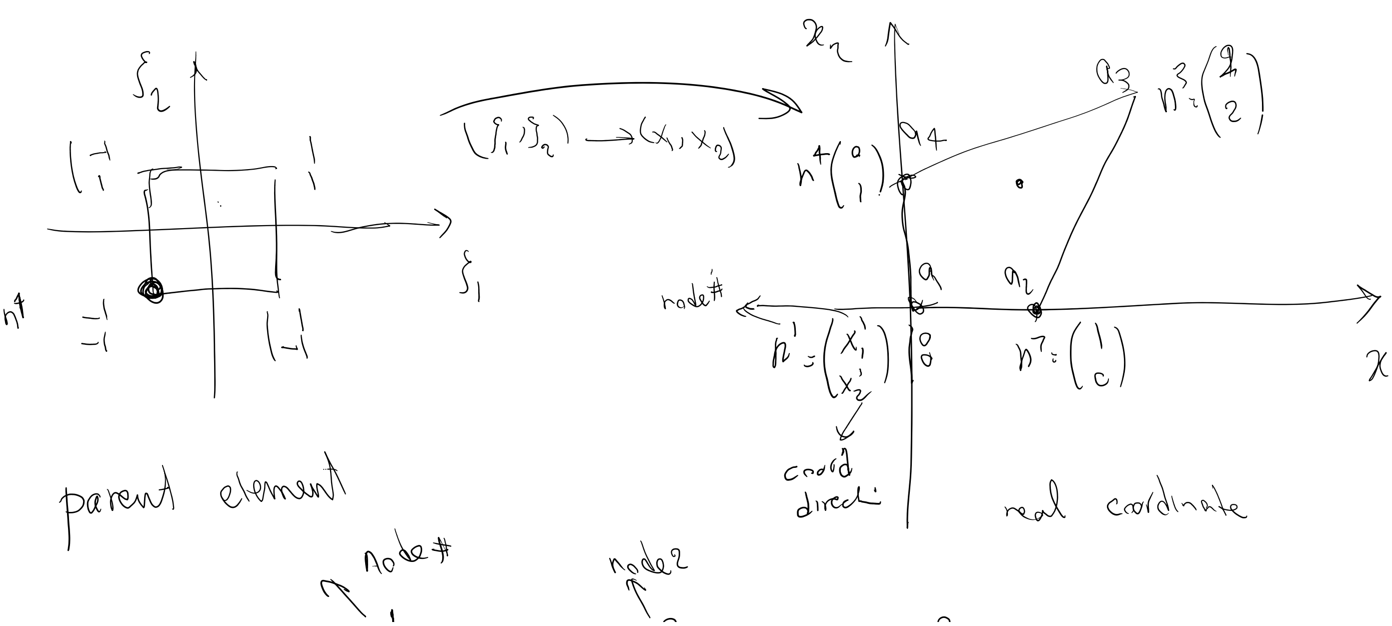
$$\frac{\partial N^{(I)}}{\partial x_i} = \frac{\partial N^{(I)}}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} \quad \text{or} \quad \frac{\partial N^I}{\partial \xi_j} = \frac{\partial N^I}{\partial x_i}\frac{\partial x_i}{\partial \xi_j}$$

we need $\vec{x}(\vec{\xi})$

ii) $$dV = (\det J)\, d\xi_1 d\xi_2$$

$$J = \nabla_{\xi} \vec{x}$$

Because of the coordinate transformation, we need to express x as a function of ξ so that we can evaluate the integral.



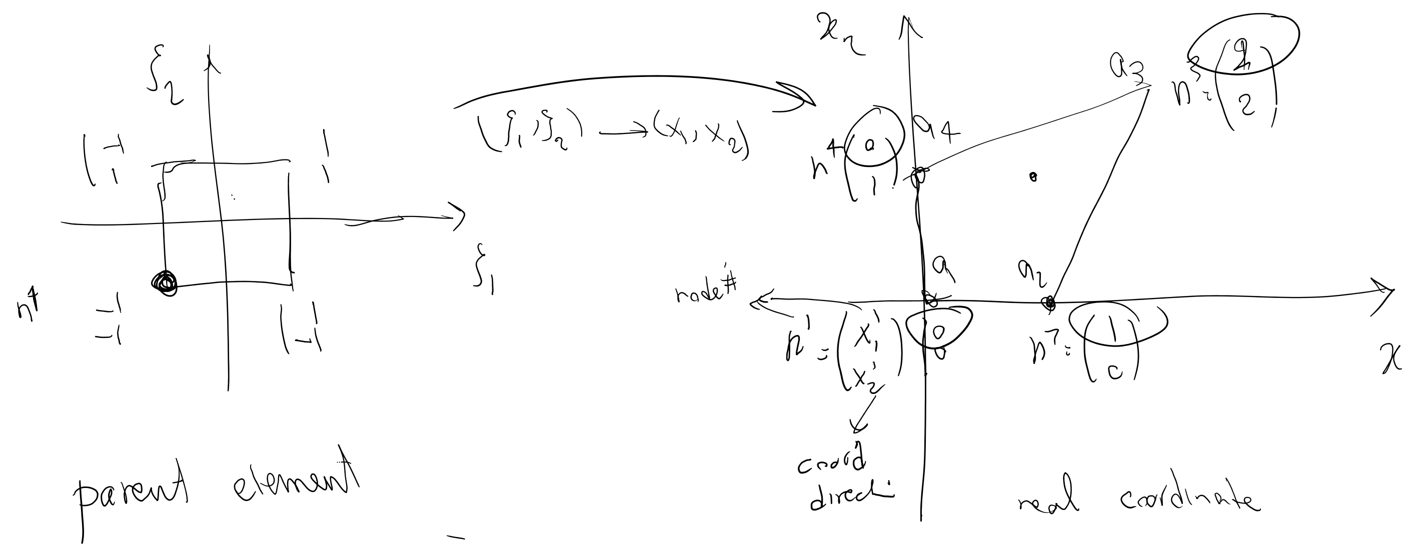
1

$$X_1(\xi_1,\xi_2) = x_1^1 N_1(\xi_1,\xi_2) + x_1^2 N_2(\xi_1,\xi_2) + x_1^3 N_3(\xi_1,\xi_2) + x_1^4 N_4(\xi_1,\xi_2)$$

node # node 2

Same for x_2

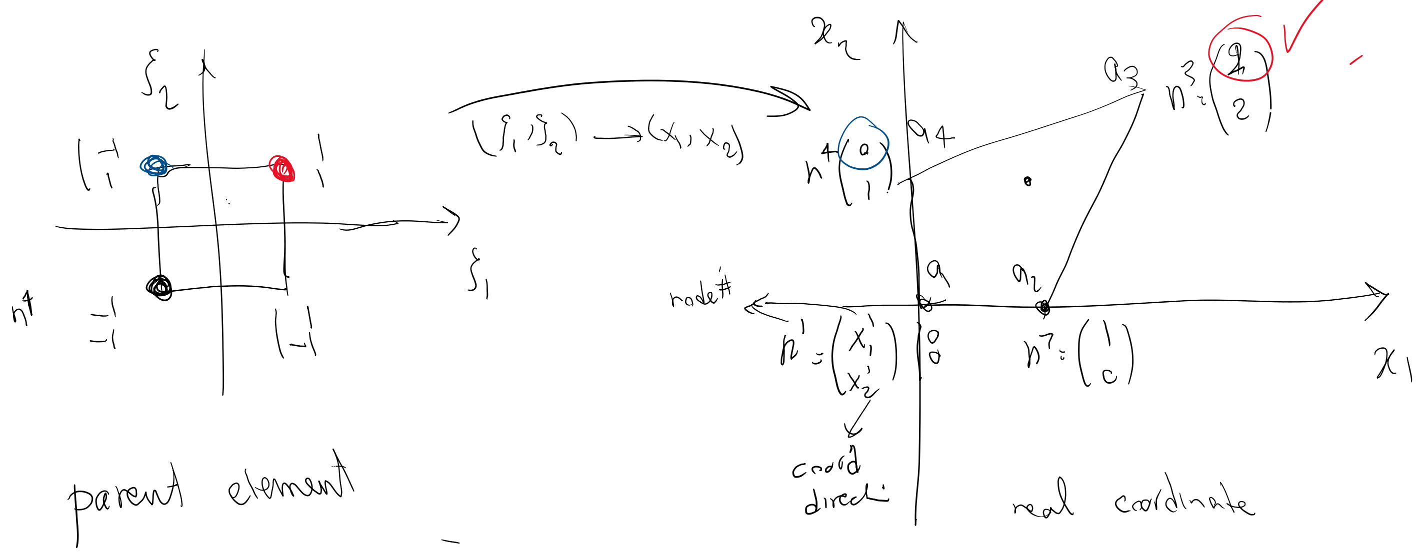
hint: $\underbrace{T(\xi_1,\xi_2)}_{\text{Solution}} = N\underbrace{a}_{\text{nodal temperatures}} = N_1(\xi_1,\xi_2)\,a_1 + \cdots + N_4(\xi_1,\xi_1)\,a_4$



$$X_1(\xi_1,\xi_2) = \underbrace{0}_{x_1^1} N_1(\xi_1,\xi_2) + \underbrace{1}_{x_1^2} N_2(\xi_1,\xi_2) + \underbrace{2}_{x_1^3} N_3(\xi_1,\xi_2) + \underbrace{0}_{x_1^4} N_4(\xi_1,\xi_2)$$

$$= 1\left(\frac{(1+\xi_1)(1-\xi_2)}{4}\right) + 2\left(\frac{(1+\xi_1)(1+\xi_2)}{4}\right)$$

$$\Longrightarrow \quad \star \quad X_1(\xi_1,\xi_2) = \frac{1}{4}\left(3 + 3\xi_1 + \xi_2 + \xi_1\xi_2\right)$$



$\xi_1 = 1, \ \xi_2 = 1$ $\qquad x_1 = \frac{1}{4}(3 + 3 + 1 + 1) = 2$

$\xi_1 = -1, \ \xi_2 = 1$ $\qquad x_2 = \frac{1}{4}(3 + 3(-1) + (1) + (1)(-1)) = 0$