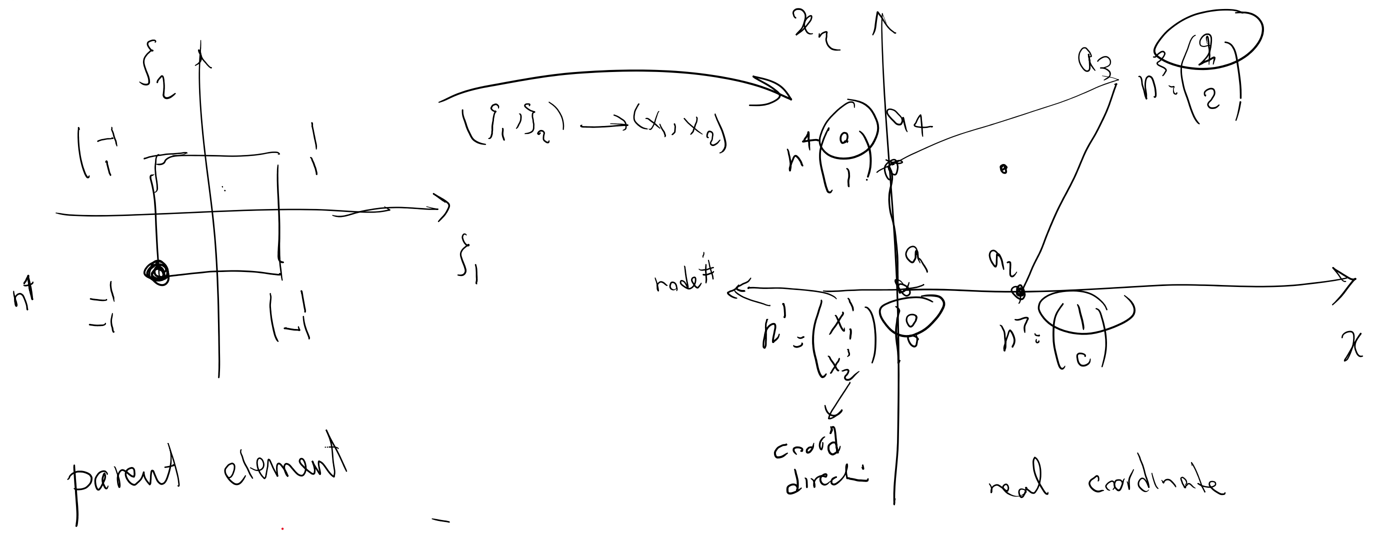
2023/12/06

Wednesday, December 6, 2023 11:14 AM

From last time



$$X_1(\xi_1, \xi_2) = \overset{\text{node\#}}{x_1^1} N_1(\xi_1, \xi_2) + \overset{\text{node 2}}{x_1^2} N_2(\xi_1, \xi_2) + x_1^3 N_3(\xi_1, \xi_2) + x_1^4 N_4(\xi_1, \xi_2)$$

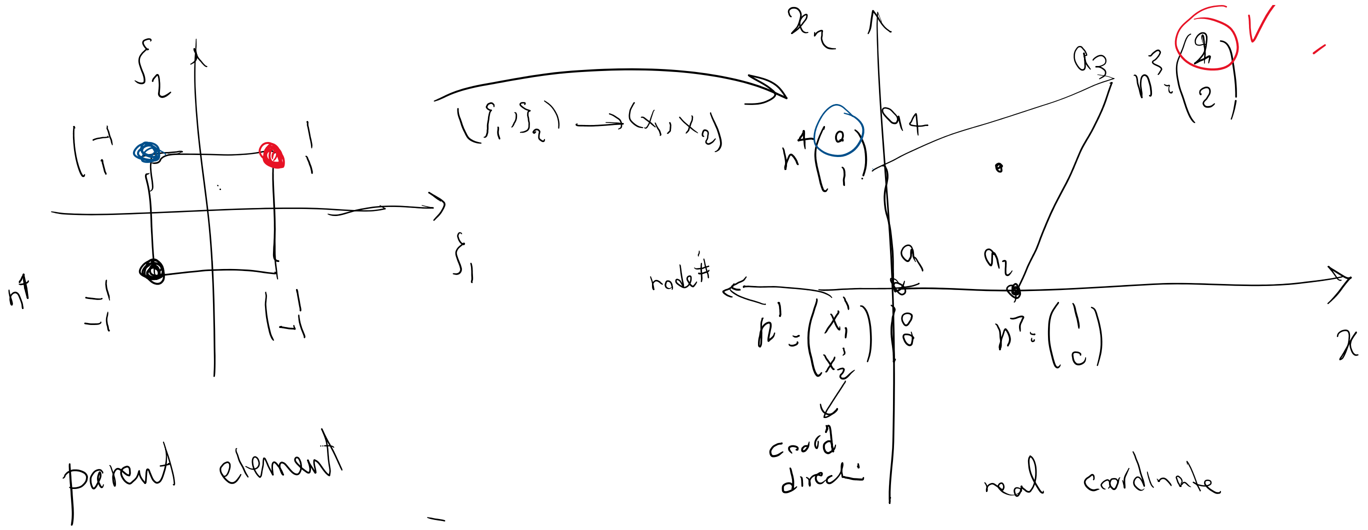
Same for x_2

hint: $\underset{\text{Soln}}{T}(\xi_1, \xi_2) = N \underset{\text{nodal temperatures}}{a} = N_1(\xi_1, \xi_2)\, a_1 + \cdots + N_4(\xi_1, \xi_2)\, a_4$

$$X_1(\xi_1, \xi_2) = \underbrace{0}_{x_1^1} N_1(\xi_1, \xi_2) + \underbrace{1}_{x_1^2} N_2(\xi_1, \xi_2) + \underbrace{2}_{x_1^3} N_3(\xi_1, \xi_2) + \underbrace{0}_{x_1^4} N_4(\xi_1, \xi_2)$$

$$= 1 \left(\frac{(1 + \xi_1)(1 - \xi_2)}{4} \right) + 2 \left(\frac{(1 + \xi_1)(1 + \xi_2)}{4} \right)$$

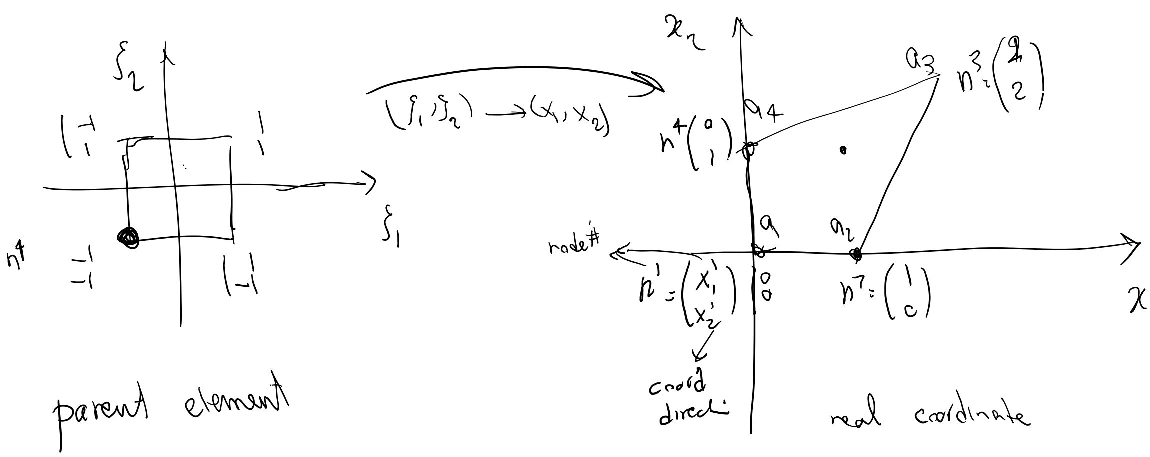
$$\Rightarrow \quad * \; X_1(\xi_1, \xi_2) = \frac{1}{4}\left(3 + 3\xi_1 + \xi_2 + \xi_1 \xi_2\right)$$



$\xi_1 = 1,\ \xi_2 = 1$ $\qquad x_1 = \frac{1}{4}(3+3+1+1) = 2$

$\xi_1 = -1,\ \xi_2 = 1$ $\qquad x_2 = \frac{1}{4}(3+3(-1)+(1)+(1)(-1)) = 0$ ✓

We can do the same to interpolate x2 in terms of xi1 and xi2:



$$x_2(\xi_1,\xi_2) = X_2^1 N^1(\xi_1,\xi_2) + X_2^2 N^2 + X_2^3 N^3 + X_2^4 N^4$$

$$= 0 \quad '' \quad + 0\, N^2 + 2\left(\frac{(1+\xi_1)(1+\xi_2)}{4}\right) + 1\left(\frac{(1-\xi_1)(1+\xi_2)}{4}\right)$$

$$= \frac{1}{4}(3+\xi_1+3\xi_2+\xi_1\xi_2)$$

(1)

$$x_1(\xi_1,\xi_2) = \frac{1}{4}\left(3 + 3\xi_1 + \xi_2 + \xi_1\xi_2\right)$$

$$x_2(\xi_1,\xi_2) = \frac{1}{4}\left(3 + \xi_1 + 3\xi_2 + \xi_1\xi_2\right)$$

only true for

$$\frac{\partial N_i}{\partial x_1} \qquad i=1,\dots,4$$

$$\frac{\partial N_i(\xi_1,\xi_2)}{\partial x_1} = \frac{\partial N_i(\xi_1,\xi_2)}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1} + \frac{\partial N_i(\xi_1,\xi_2)}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_1}$$

we have x_1 & x_2 as functions of ξ_1, ξ_2, not the other way around. The highlighted terms cannot be computed.

we have this

we don't have these

$$\frac{\partial N_i}{\partial \xi_1} = \frac{\partial N_i}{\partial x_1}\frac{\partial x_1}{\partial \xi_1} + \frac{\partial N_i}{\partial x_2}\frac{\partial x_2}{\partial \xi_1}$$

$$\frac{\partial N_i}{\partial \xi_2} = \frac{\partial N_i}{\partial x_1}\frac{\partial x_1}{\partial \xi_2} + \frac{\partial N_i}{\partial x_2}\frac{\partial x_2}{\partial \xi_2}$$

(3)

$$\underbrace{\begin{bmatrix} \frac{\partial N_i}{\partial \xi_1} \\ \frac{\partial N_i}{\partial \xi_2} \end{bmatrix}}_{\text{we have this}} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} \frac{\partial N_i}{\partial x_1} \\ \frac{\partial N_i}{\partial x_2} \end{bmatrix}}_{\text{we need this}}$$

$$\rightarrow \begin{bmatrix} \frac{\partial N_i}{\partial x_1} \\ \frac{\partial N_i}{\partial x_2} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi_1} \\ \frac{\partial N_i}{\partial \xi_2} \end{bmatrix} \qquad i=1\dots 4$$

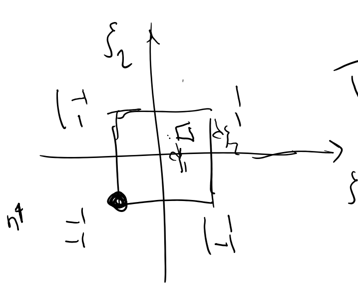
$$\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_2}{\partial x_1} & \frac{\partial N_3}{\partial x_1} & \frac{\partial N_4}{\partial x_1} \end{bmatrix} \quad J^{-1} \begin{bmatrix} \frac{\partial N_1}{\partial \xi_1} & \frac{\partial N_2}{\partial \xi_1} & \frac{\partial N_3}{\partial \xi_1} & \frac{\partial N_4}{\partial \xi_1} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_2}{\partial x_1} & \frac{\partial N_3}{\partial x_1} & \frac{\partial N_4}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_3}{\partial x_2} & \frac{\partial N_4}{\partial x_2} \end{bmatrix}}_{B} = J^{-1} \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \xi_1} & \frac{\partial N_2}{\partial \xi_1} & \frac{\partial N_3}{\partial \xi_1} & \frac{\partial N_4}{\partial \xi_1} \\ \frac{\partial N_1}{\partial \xi_2} & \frac{\partial N_2}{\partial \xi_2} & \frac{\partial N_3}{\partial \xi_2} & \frac{\partial N_4}{\partial \xi_2} \end{bmatrix}}$$

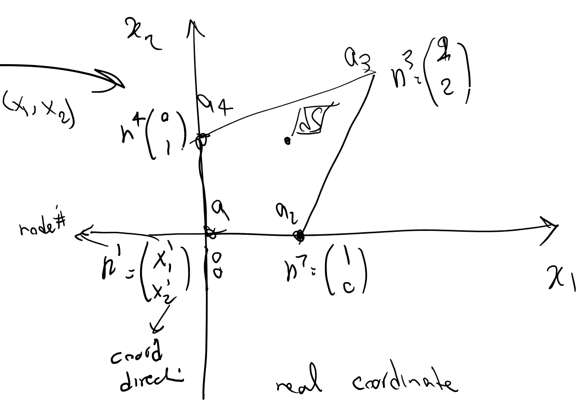
$$L_m = \nabla, \quad B = \nabla_x N = \nabla [N_1, N_2, N_3, N_4] \qquad B_\xi \equiv \nabla_\xi N \qquad (4)$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \cdots & \frac{\partial N_4}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \cdots & \frac{\partial N_4}{\partial x_2} \end{bmatrix}$$

$$B = J^{-1} B_\xi$$



parent element



real coordinate

$$k^e = \int_e B^t \underset{k}{D} B \, dS = \int_{\xi_1=-1}^{1} \int_{\xi_2=-1}^{1} B^t D B \underbrace{(\det J)}_{\text{we'll evaluate this shortly}} \overbrace{(d\xi_1 \, d\xi_2)}^{dS_\xi}$$

$$= \int_{\xi_1=-1}^{1} \int_{\xi_2=-1}^{1} \underbrace{(J^{-1} B_\xi)^t}_{B_\xi^t J^{-t}} k \, (J^{-1} B_\xi) \det J \; d\xi_1 \, d\xi_2$$

$$\Longrightarrow \quad \boxed{k^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} B_\xi^t \left(J^{-t}\kappa J^{-1}\right)\det J \, B_\xi \; d\xi_1\, d\xi_2} \qquad (5)$$

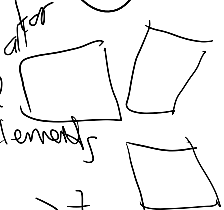
we just need to calculate (a) B_ξ (b) J $\left(\rightarrow J^{-1},\ J^{-t},\ \det J \right)$

(a)
$$B_\xi = \nabla_\xi \left[N_1 \ N_2 \ N_3 \ N_4 \right] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi_1} & \frac{\partial N_2}{\partial \xi_1} & \frac{\partial N_3}{\partial \xi_1} & \frac{\partial N_4}{\partial \xi_1} \\ \frac{\partial N_1}{\partial \xi_2} & \frac{\partial N_2}{\partial \xi_2} & \frac{\partial N_3}{\partial \xi_2} & \frac{\partial N_4}{\partial \xi_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial \xi_1}\left(\frac{(1-\xi_1)(1-\xi_2)}{4}\right) & \frac{\partial}{\partial \xi_1}\left(\frac{(1+\xi_1)(1-\xi_2)}{4}\right) & \frac{\partial}{\partial \xi_1}\left(\frac{(1+\xi_1)(1+\xi_2)}{4}\right) & \frac{\partial \left((1-\xi_1)(1+\xi_2)/4\right)}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2}\left(\frac{(1-\xi_1)(1-\xi_2)}{4}\right) & \frac{\partial}{\partial \xi_2}\left(\frac{(1+\xi_1)(1-\xi_2)}{4}\right) & \frac{\partial}{\partial \xi_2}\left(\frac{(1+\xi_1)(1+\xi_2)}{4}\right) & \frac{\partial}{\partial \xi_2}\left(\frac{(1-\xi_1)(1+\xi_2)}{4}\right) \end{bmatrix}$$

$$\boxed{B_\xi = \begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & \left(\frac{1-\xi_2}{4}\right) & \left(\frac{1+\xi_2}{4}\right) & -\left(\frac{1+\xi_2}{4}\right) \\ -\left(\frac{1-\xi_1}{4}\right) & -\left(\frac{1+\xi_1}{4}\right) & \left(\frac{1+\xi_1}{4}\right) & \left(\frac{1-\xi_1}{4}\right) \end{bmatrix}} \qquad (6)$$

for Q elements



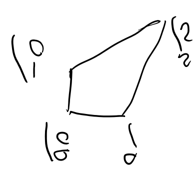
(b) J:

$$x_1(\xi_1,\xi_2) = \frac{1}{4}\left(3 + 3\xi_1 + \xi_2 + \xi_1\xi_2\right)$$
$$x_2(\xi_1,\xi_2) = \frac{1}{4}\left(3 + \xi_1 + 3\xi_2 + \xi_1\xi_2\right)$$

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} = \left(\nabla_\xi x\right)^T$$

$$J_{ij} = \frac{\partial x_j}{\partial \xi_i}$$

$$\boxed{J = \begin{bmatrix} \frac{3+\xi_2}{4} & \frac{1+\xi_2}{4} \\ \frac{1+\xi_1}{4} & \frac{3+\xi_1}{4} \end{bmatrix}} \quad \text{only for} \qquad (7)$$



just to see how $\det J$ looks like

just to see how det J looks like

$$\det J = \frac{(3+\xi_1)(3+\xi_2) - (1+\xi_1)(1+\xi_2)}{16}$$

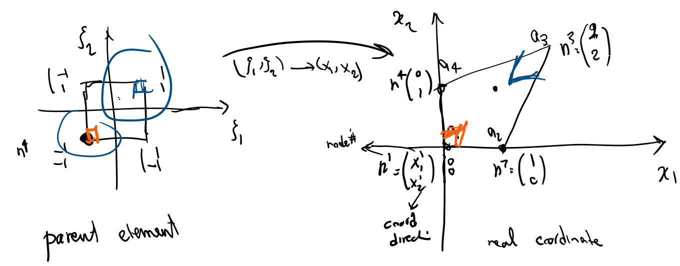
Side note to understand the meaning of det J

$\xi_1 = \xi_2 = -1$

$$\det J = \frac{2\times 2}{16} = \frac{1}{4}$$

$\xi_1 = \xi_2 = 1$

$$\det J = \frac{16-4}{16} = \frac{12}{16}$$



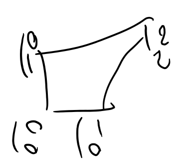
Plug equation (6) in equation (5):

$$K^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & -\left(\frac{1-\xi_1}{4}\right) \\ \left(\frac{1-\xi_2}{4}\right) & -\left(\frac{1+\xi_1}{4}\right) \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\left(\frac{1+\xi_2}{4}\right) & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi^t} J^{t}\,\kappa\, J^{-1} (\det J) \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & \left(\frac{1-\xi_2}{4}\right) & \frac{1+\xi_2}{4} & -\left(\frac{1+\xi_2}{4}\right) \\ -\left(\frac{1-\xi_1}{4}\right) & -\left(\frac{1+\xi_1}{4}\right) & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi} d\xi_1 d\xi_2$$

$I(\xi_1,\xi_2)$

For any Q4 element

for



get J from eqn (7) (8)

Determine the full integration order:

- We ignore J and kappa **only** for this order calculation.

K_{11} has $\left[-\left(\frac{1-\xi_2}{4}\right)\right]^2 + \left[\left(\frac{1-\xi_1}{4}\right)\right]^2$

$O_{\xi_1} = 2$

$O_{\xi_2} = 2$

Use NC to integrate this

$O_{\xi_1} = 2 \longrightarrow n_{\xi_1} = O_{\xi_1} + 1 = 3$ Simpson's rule

$\xi_1 = -1$ $\xi_1 = 0$ $\xi_1 = 1$

$2*\left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right)$

$= \left(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right)$

$O_{\xi_2} = 2 \longrightarrow n_{\xi_2} = 3$

+1

0

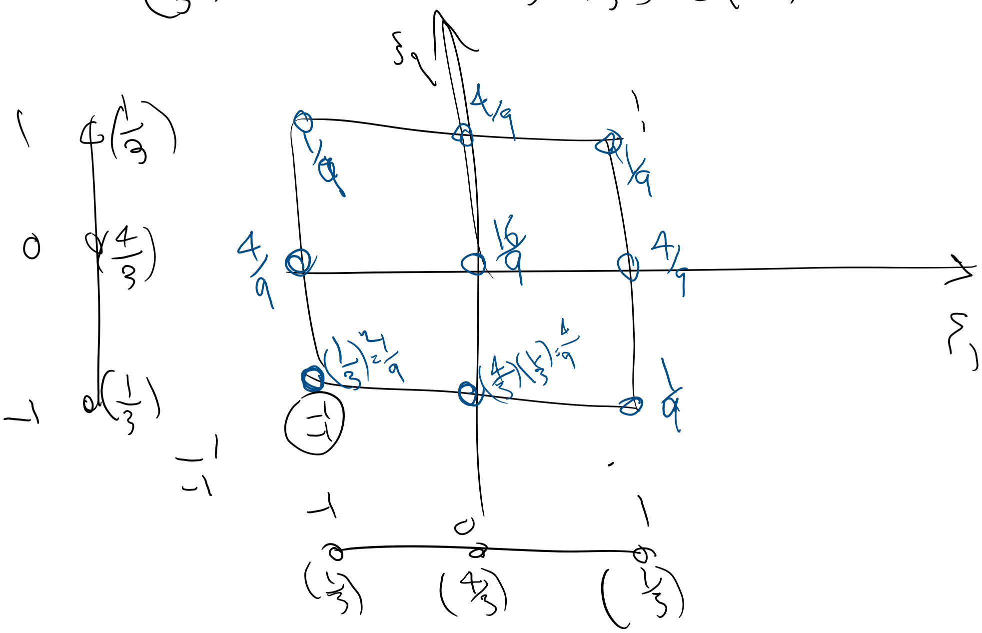
$o_{\xi_2} = 2 \longrightarrow n_{\xi} = 3$ ... same pts $= \left(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right)$

$$k = \int_{\xi_2} d\xi_2 \left(\int_{-1}^{1} I(\xi_1, \xi_2)\, d\xi_1 \right) = \int_{\xi_2} \left(\left(\frac{1}{3}\right) I(-1, \xi_2) + \frac{4}{3} I(0, \xi_2) + \frac{1}{3} I(1, \xi_2) \right) d\xi_2$$

$I'(\xi_2)$

$$= \frac{1}{3} I'(-1) + \frac{4}{3} I'(0) + \frac{1}{3} I'(1)$$

$$= \left(\frac{1}{3}\right)^2 I(-1,-1) + \left(\frac{1}{3}\right)\left(\frac{4}{3}\right) I(-1, 0) + \cdots$$

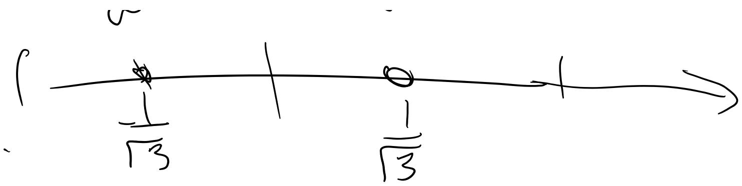


$$k = \frac{1}{9} I(-1,-1) + \frac{4}{9} I(0,-1) + \frac{1}{9} I(1,-1) + \frac{4}{9} I(-1, 0) + \frac{16}{9} I(0,0) + \cdots + \frac{1}{9} I(1,1)$$

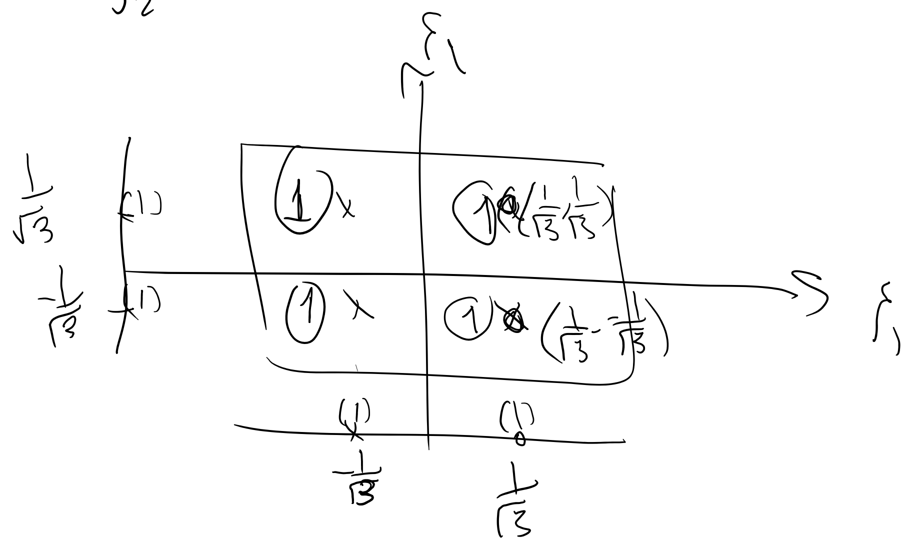
I is given in eqn (8)

Gauss Quadrature

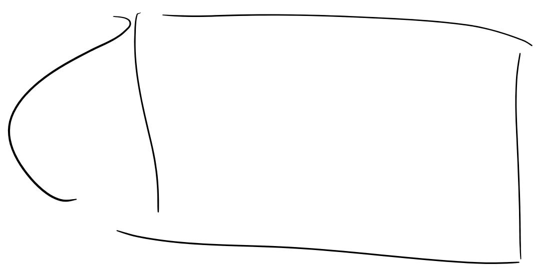
$o_{\xi_1} = 2 \qquad n_{\xi_1} = \text{Ceil}\left(\frac{o_{\xi_1} + 1}{2}\right) = 2$



$^o\xi_2 = 2$ $\qquad$ $^n\xi_2 = \operatorname{ceil}\left(\frac{2+1}{2}\right) = 2$



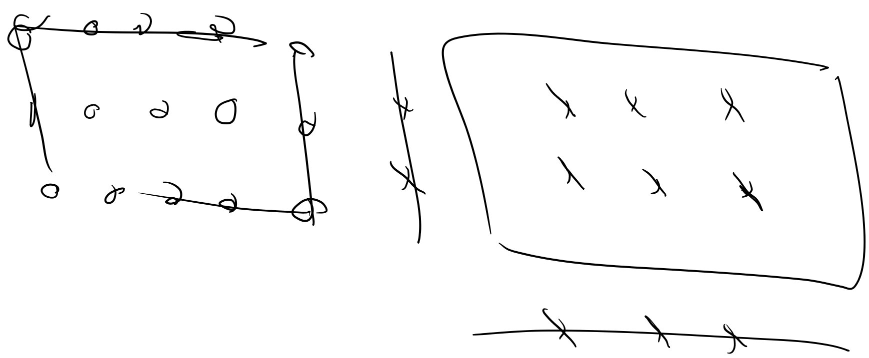
Final



$^o\xi_1 = 4 \to 4+1$

$^o\xi_2 = 2 \to 2+1$

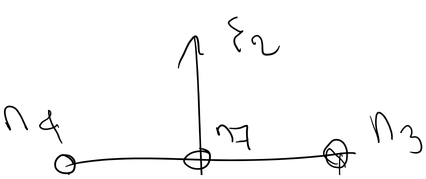
NC

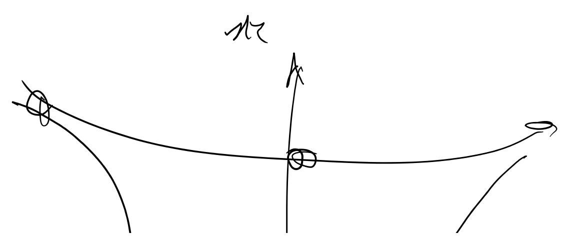


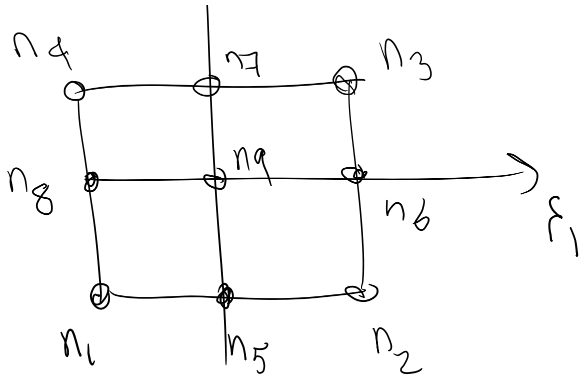
$\frac{4+1}{2} = 3$

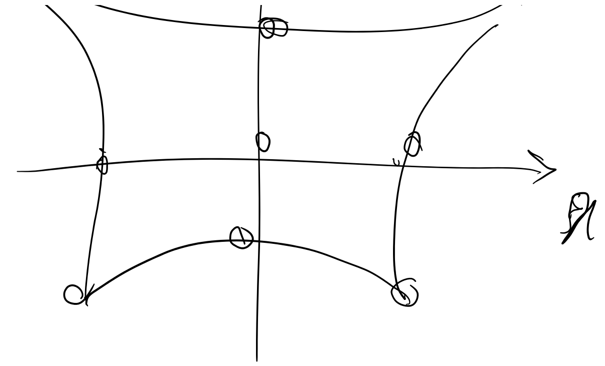
$\frac{2+1}{2} = 2$

Higher order elements in 2D and 3D

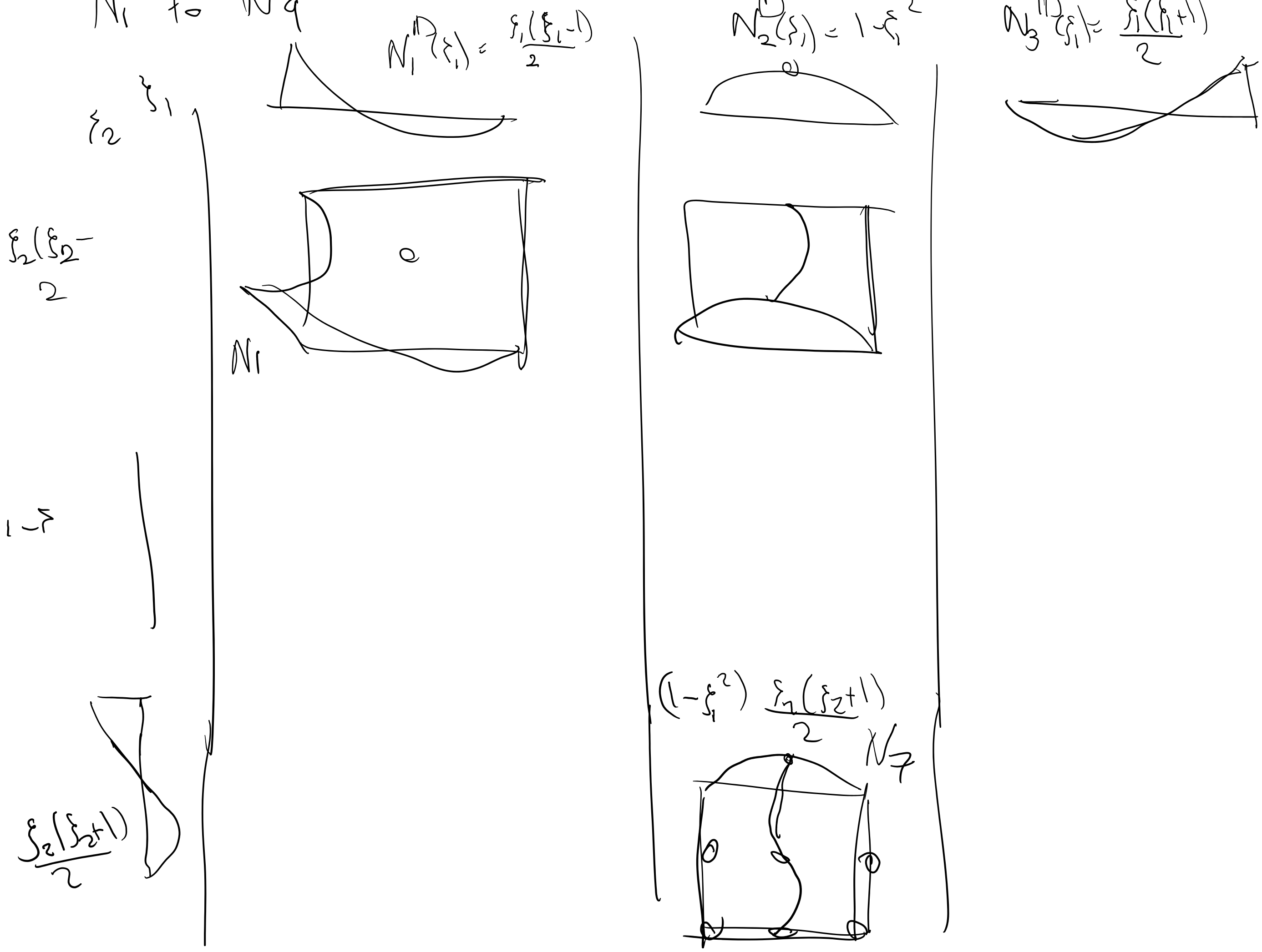








N_1 to N_9



node #

$$x_1(\xi_1,\xi_2) = \sum_{i=1}^{9} x_1^i N^i(\xi_1,\xi_2)$$

$$x_2(\xi_1,\xi_2) = \sum_{i=1}^{9} x_2^i N^i(\xi_1,\xi_2)$$

$$x_2(\xi_1,\xi_2) = \sum_{i=1} x^i N^i(\xi_1,\xi_2)$$

Isoparametric element

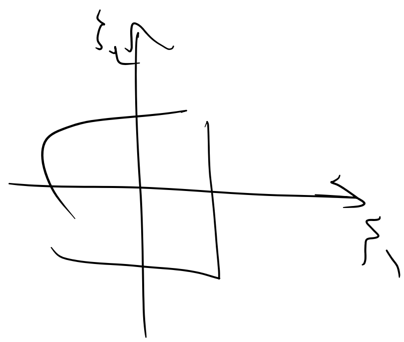
geometry is as accurate as solution

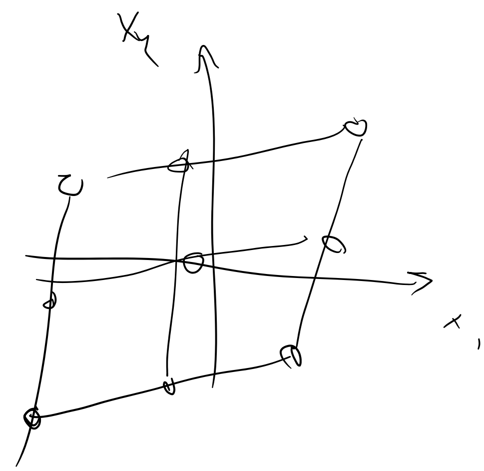
Subparametric

$$T(\xi_1,\xi_2) = \sum_{i=1}^{9} N^i(\xi_1,\xi_2)\, T^i$$

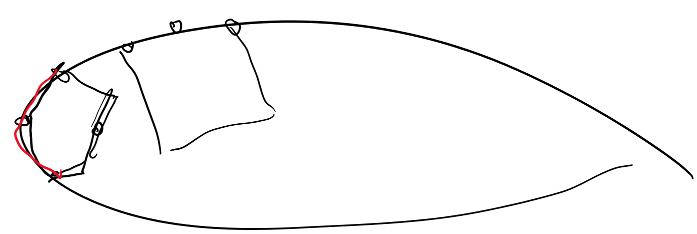
T is 2nd order

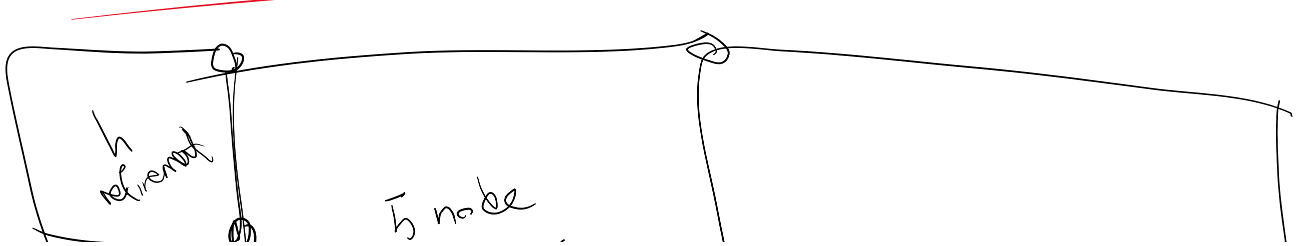
geometry is 1st order

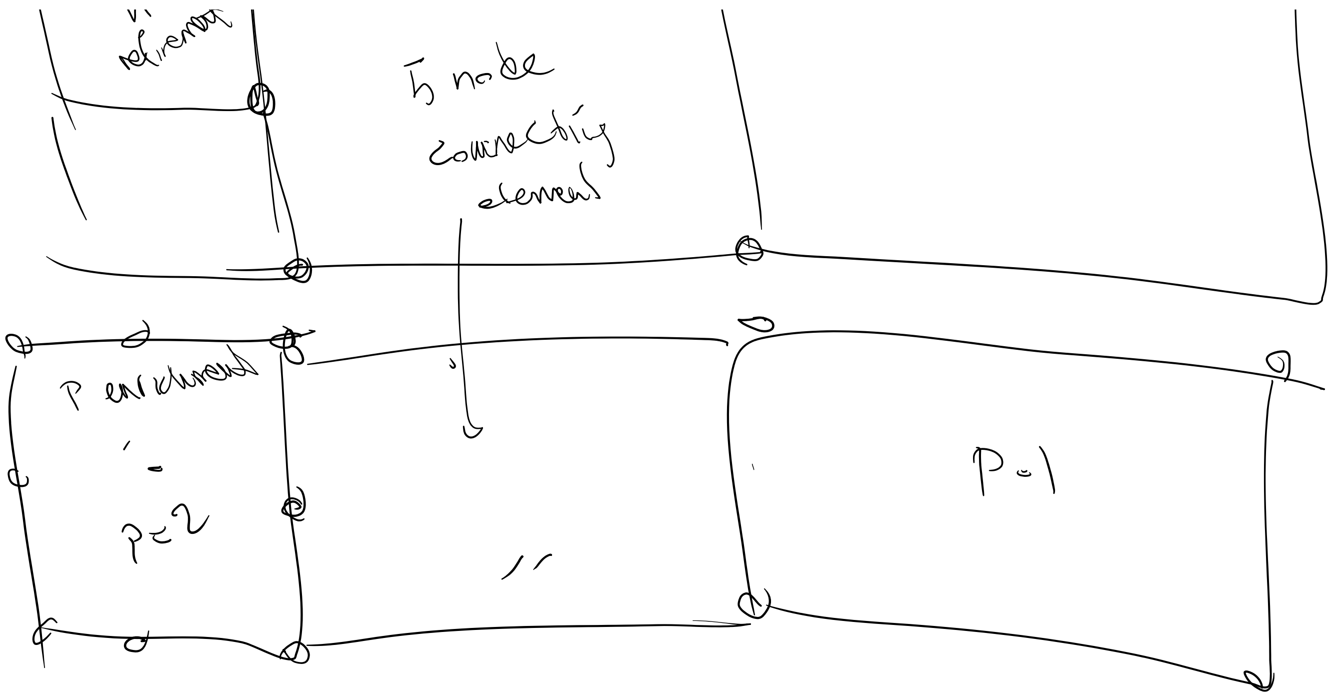




Isoparametric



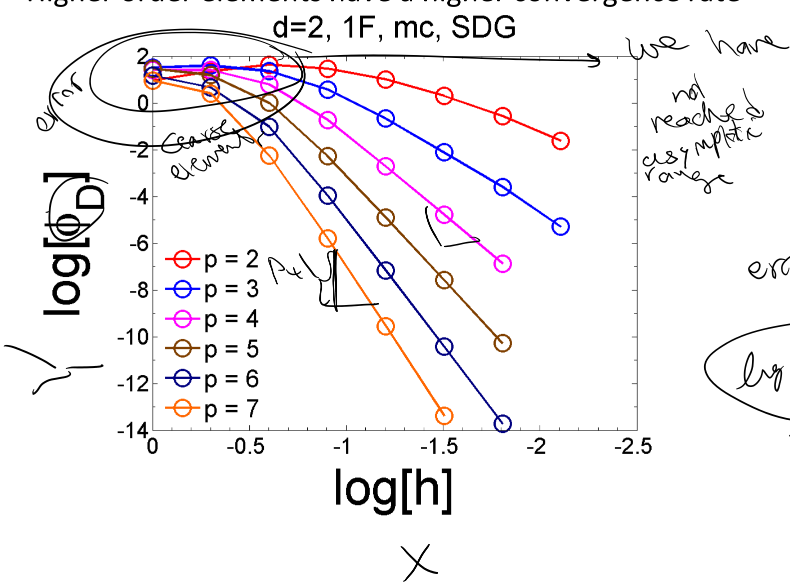




Higher order elements in 2D and 3D

Motivation

Higher order elements have a higher convergence rate



element order

more general

error $= Ch^{p+1}$

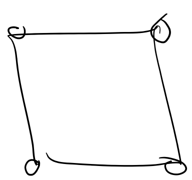
Ch^{alpha}

$\log \text{error} = \log C + (p+1)\log h$

Y X

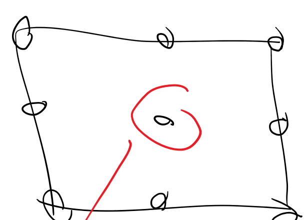
Higher order element is better because convergence rate is p+1

FEM



p=1

Q4



Q9

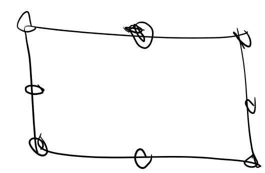
p=2

Ch^2

in Final example you'll see how we get rid of the internal node

Ch^2

in Final example you'll see how we get rid of the internal node



Q8 serendipity

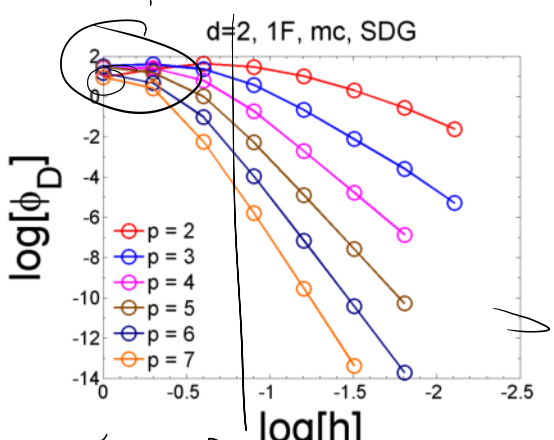
Ch^3

Efficiency: for the same resource (Wall clock time, memory, etc) we'll get a lower error

If the solution is smooth enough (not dealing with strong and weak discontinuities such as crack tip, sharp wave fronts, shocks in fluid mechanics, …) it is often beneficiary to use higher order elements as we benefit from higher convergence rate.

BUT the solution should be accurate enough (already using small elements) so we are in the asymptotic convergence rate

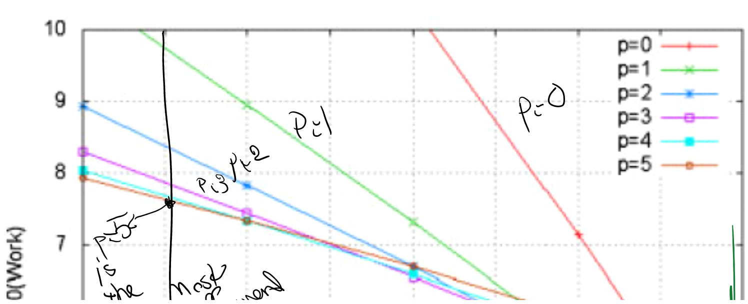
Bad for high order elements



Finer element

Coarse element

Sample efficiency plot



Solution smoothness

p=5 is the most efficient

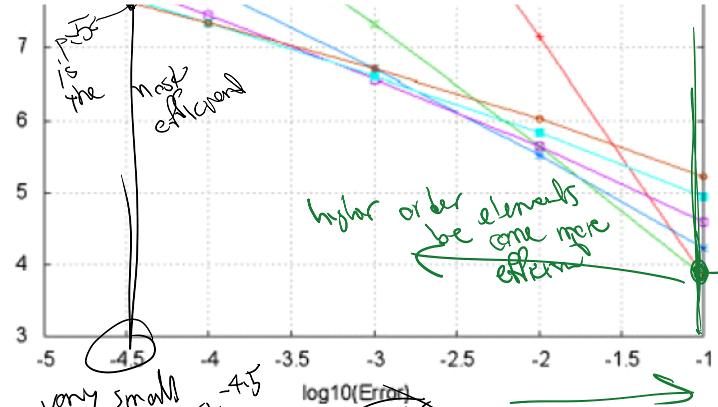
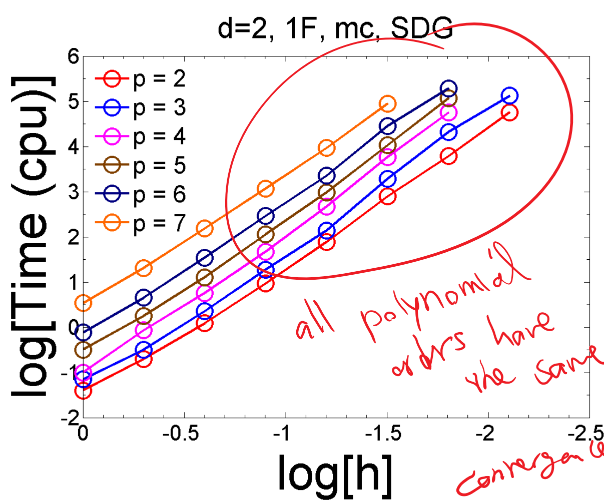
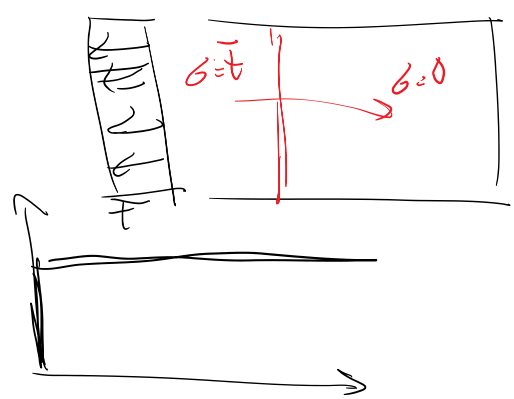


Figure 1. Asymptotic work estimates for 3-D linear problems.

if the problem is not smooth enough, wouldn't get to optimal convergence rate

problem with nonsmooth solution

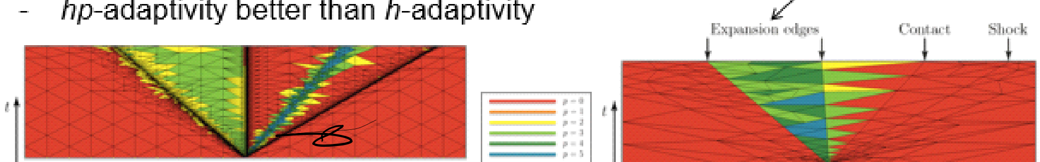




- Adaptive operations in spacetime:
  - Front-tracking better than shock capturing
  - *hp*-adaptivity better than *h*-adaptivity

Sod's shock tube problem

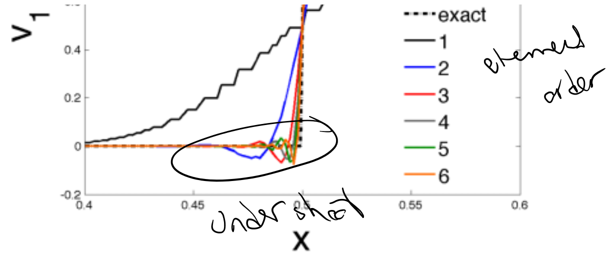
Results by Scott Miller



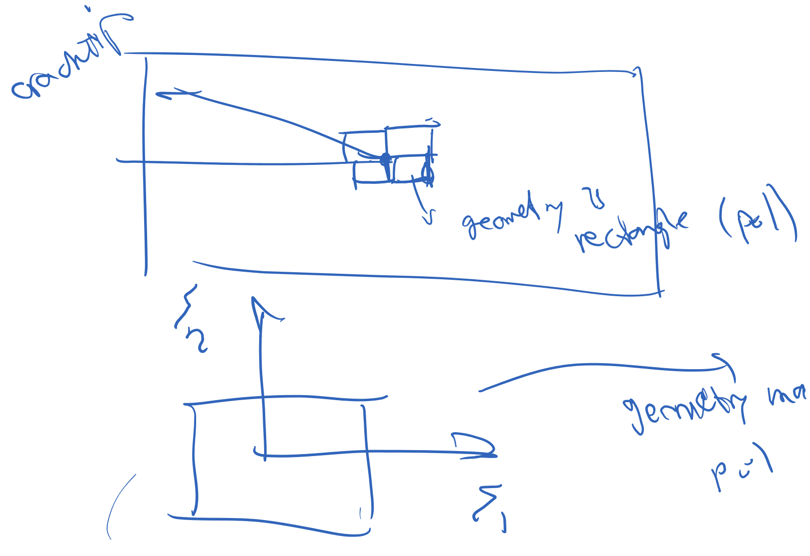
Shock capturing: 473K elements

Shock tracking: 446 elements

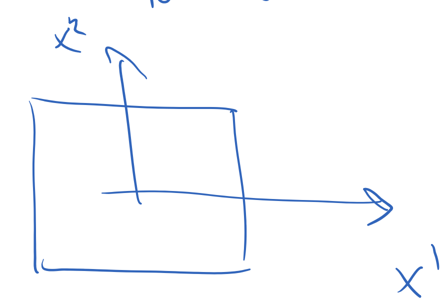
overshoot

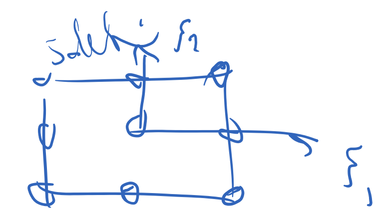


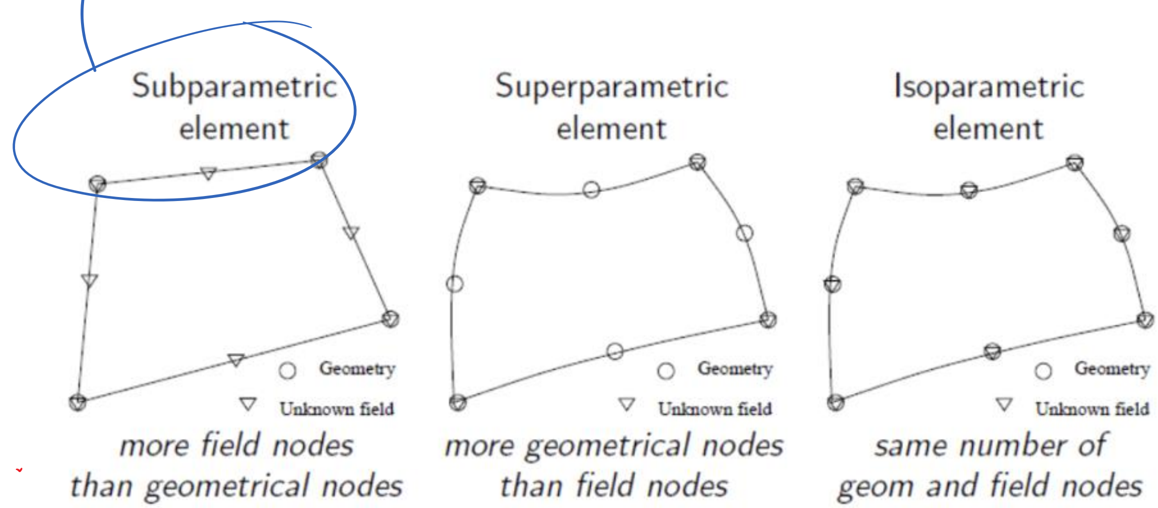
There are cases that we only deal with simple geometries
but the solution should be accurate



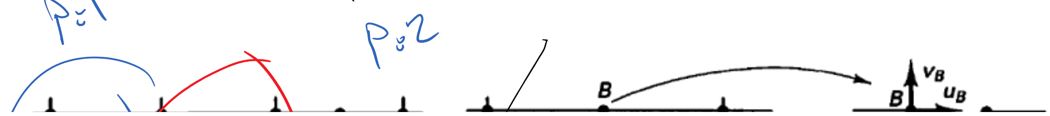
but the soln can be more accurate

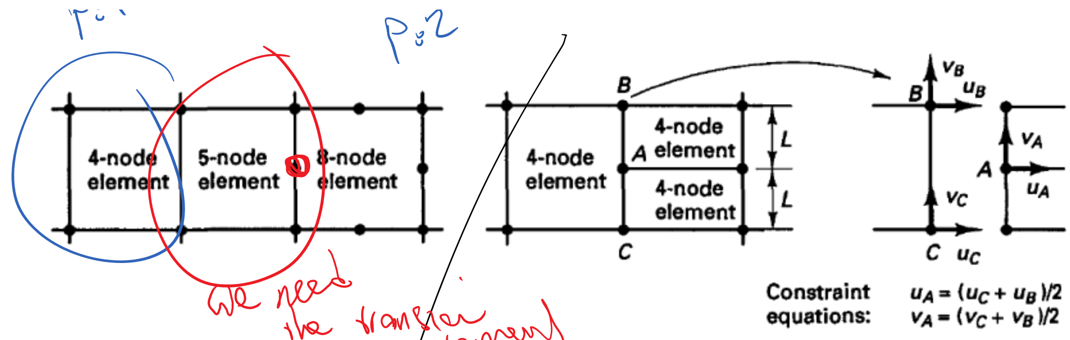






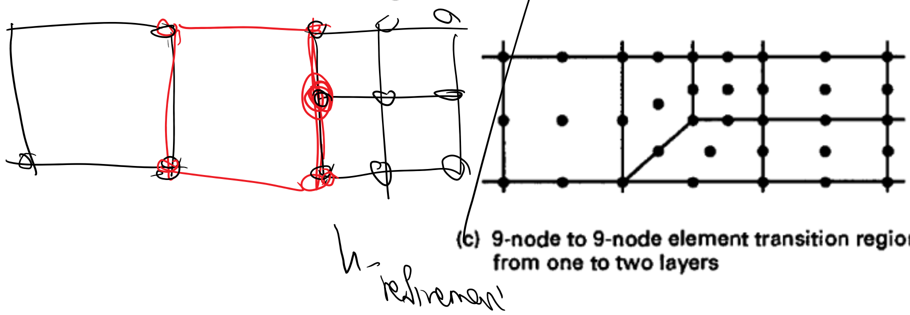
p enrichment





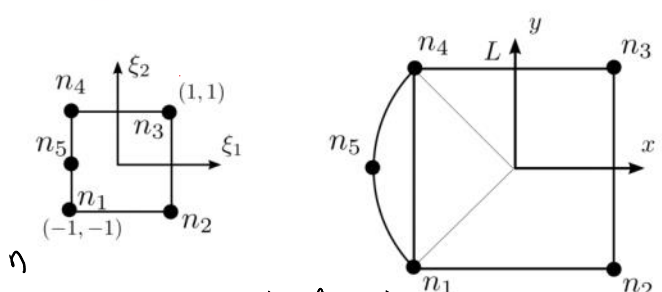
(a) 4-node to 8-node element transition region

(b) 4-node to 4-node element transition; from one to two layers

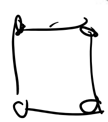


(c) 9-node to 9-node element transition region; from one to two layers

The questions for this element are:



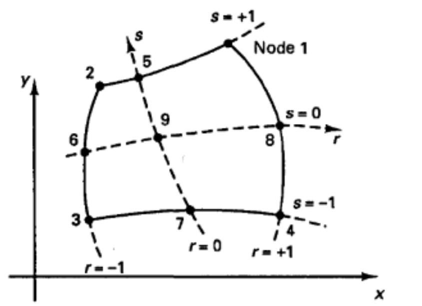
connecting element

start with  basis functions & modify it

N^1	$\frac{(1-\xi_1)(1-\xi_2)}{4}$	$-\frac{1}{2}N^5$
	$N^1 = \frac{(1-\xi_1)(1-\xi_2)}{4} - \frac{1}{2}N^5$	
N^2	$\frac{(1+\xi_1)(1-\xi_2)}{4}$	
N^3	$\frac{(1+\xi_1)(1+\xi_2)}{4}$	
N^4	$\frac{(1-\xi_1)(1+\xi_2)}{4}$	$-\frac{1}{2}N^5$

N^4 $\quad \frac{(1-\xi_1)(1+\xi_2)}{4}$ $\quad -\frac{1}{2}N^5$

N^5 $\quad \frac{(1-\xi_1)}{2}$ $\quad (1-\xi_2^2)$ $\quad \frac{(1-\xi_1)(1-\xi_2^2)}{2}$



(a) 4 to 9 variable-number-nodes two-dimensional element

Include only if node i is defined

		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					

(b) Interpolation functions