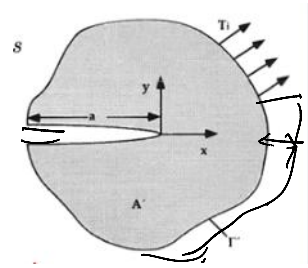
J=G

Monday, October 10, 2016
10:18 PM

Energy Release rate is:



$$G = -\frac{d\Pi}{dA} = -\frac{1}{B}\frac{d\Pi}{da} \quad ①$$

$\Pi = U_e - W_{ext}$ (nonlinear) elastic material ignoring kinetic energy

$$U_e = \int_V \omega \, dV = B \int_{A'} \omega \, dA$$

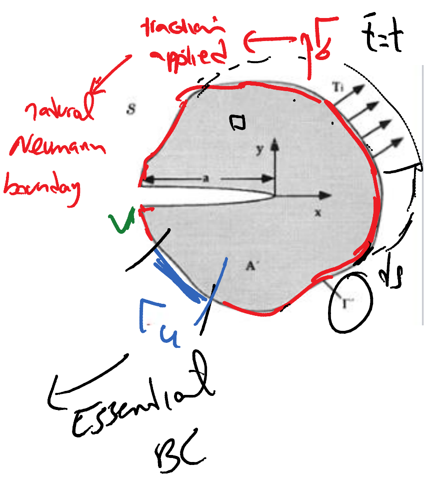
ω → Internal energy density

$$\omega(\varepsilon) = \int_0^{\varepsilon} \sigma(\varepsilon)\, d\varepsilon \quad \Longleftrightarrow \quad \sigma_{ij}(\varepsilon) = \frac{\partial \omega(\varepsilon)}{\partial \varepsilon_{ij}}$$

$$W_{ext} = \int_{\partial D_\sigma} \vec{u}\cdot \underbrace{(\vec{t}\, dA)}_{d\vec{F}} = B \int_{\partial \Gamma_\sigma} \vec{u}\cdot\vec{t}\, ds \qquad dA = B\,ds$$

$\boxed{\vec{t} \text{ applied traction on } \partial\Gamma_\sigma}$

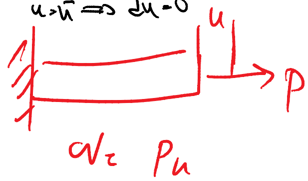
∂D_σ → natural Boundary

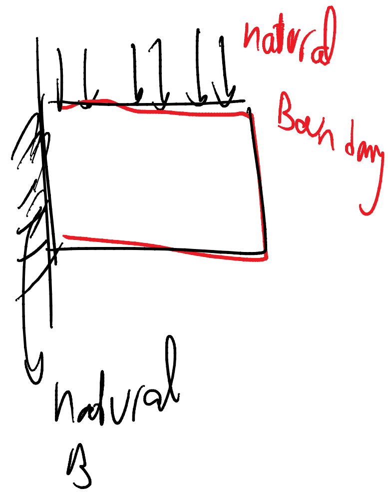


side note

No Contribution from $\partial\Gamma_u$ — $u = \bar{u}$ Essential Dirichlet BC

from essential boundary because $u = \bar{u} \Rightarrow du = 0$





So

$$\Pi = U_e - W_{ext} = B\left\{ \int_{A'} \omega(\varepsilon)\, dA - \int_{\partial\Gamma_\sigma} \vec{u}\cdot\vec{t}\, ds \right\}$$

$$\left(\int_{A'} \qquad \int_{\partial\Gamma_\sigma} \right\} \Rightarrow$$

$$G = -\frac{1}{B}\frac{d\Pi}{da}$$

only the natural boundary

$$\boxed{\begin{aligned} G &= -\frac{d}{da}\left\{ \int_{A'} w(\varepsilon)\, dA - \int_{\partial\Gamma_\sigma} \vec{u}\cdot\vec{\bar{t}}\, ds \right\} \\ &= -\left\{ \int_{A'} \frac{dW(\varepsilon)}{da}\, dA - \int_{\partial\Gamma_\sigma} \frac{d(\vec{u}\cdot\vec{\bar{t}})}{da}\, ds \right\} \end{aligned}} \quad (2)$$

$$-G = \int_{A'} \frac{dW}{da}\, dA - \int_{\partial\Gamma_\sigma} \left(\frac{d\vec{u}}{da}\cdot\vec{\bar{t}} + \vec{u}\,\underbrace{\frac{d\vec{\bar{t}}}{da}}_{0} \right) ds$$

because $\bar{t}$ is fixed & does NOT depend on crack length

$$\Rightarrow G = \int_{A'} \frac{dW}{da}\, dA - \int_{\partial\Gamma_\sigma} \frac{du}{da}\cdot\underbrace{\vec{t}}_{=\vec{\bar{t}}}\, ds \quad \underbrace{- \int \frac{du}{da}\,\vec{t}\, d}_{\text{Added zero term}}$$

because on $\partial\Gamma_\sigma$ $\vec{t} = \vec{\bar{t}}$

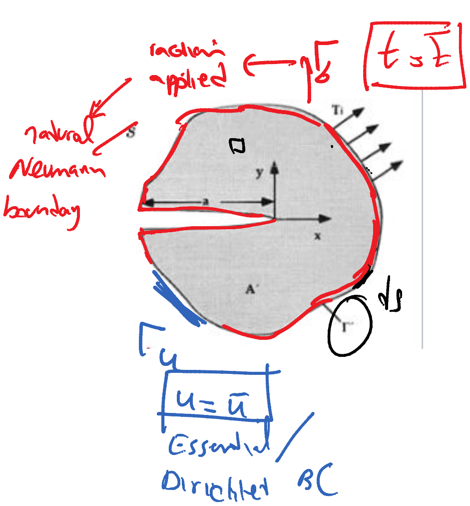
Added zero term because

on $\partial\Gamma_u$

$u = \bar{u}$ (constant)

& independent of a $\Rightarrow$

$$\frac{du}{da} = \frac{d\bar{u}}{da} = 0$$



$$\Rightarrow \quad G = \int_{A^-} \frac{dW}{da}\, dA - \left(\int_{\Gamma_\sigma} \frac{d\vec{u}}{da}\, \vec{t}\, ds + \int_{\Gamma_u} \frac{d\vec{u}}{da}\, \vec{t}\, ds \right)$$

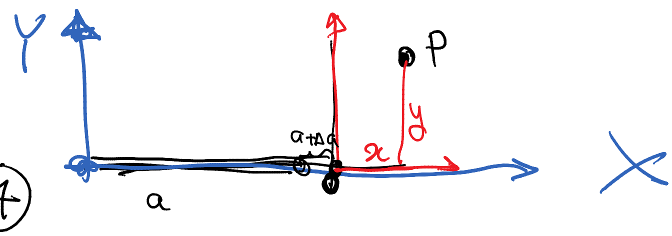
$$\Gamma_\sigma \cup \Gamma_u = \Gamma \quad / \quad \Gamma_\sigma \cap \Gamma_u = \phi$$

$$\Gamma = \partial A^-$$

boundary of A^- ⟵ ∂A^-

$$\Rightarrow \quad \boxed{G = \int_{A'} \frac{dW}{da}\, dA' - \int_{\partial A'} \frac{d\vec{u}}{da}\, \vec{t}\, ds} \qquad (3)$$

$$\boxed{\begin{cases} X = x + a \\ Y = y \end{cases}} \qquad (4)$$



X, Y Global coordinate

x, y coordinate attached to crack tip

All quantities such as $w, u, t, \ldots$ can be represented with respect to two coordinate systems (X, Y) which is a fixed coordinate system and (x, y) which moves with the crack tip

$\frac{df}{da}$ corresponds to a fixed spatial position X, Y. For example

$\frac{dw}{da}(P)$ denotes the rate of change of internal energy for this fixed point by crack advance da

$X = a + x$ depending on the origin of XY csl

$$\Rightarrow \quad \frac{df}{da} = \frac{d f(x, a)}{da}\Big|_{\text{fixed } X} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial y}\overset{0}{\frac{\partial y}{\partial a}} + \frac{\partial f}{\partial a}\frac{da}{da} \Big| X, Y \text{ fixed}$$

$$x = X - a \;(\text{keep } X \text{ fixed}) \Rightarrow \frac{\partial x}{\partial a}\Big|_{X \text{ fixed}} = \underset{0}{\cancel{\frac{\partial X}{\partial a}}}\Big|_{X \text{ fixed}} - \frac{\partial x}{\partial a}$$

$$\frac{\partial x}{\partial a} = -\frac{\partial a}{\partial a} = -1$$

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$\partial a \,|_{x \text{ fixed}}$ $\quad$ $\partial a \,|_{X \text{ fixed}}$ $\to 0$ $\quad$ da

$$\boxed{\frac{df}{da} = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial x}} \quad (4)$$

fixed coordinate X, Y

moving coordinate with crack tip x, y

plugging (4) in (3) $\Longrightarrow$

$$\frac{d\Pi}{da} = \int_{A'} \frac{dW}{da}\, dA - \int \frac{d\vec{u}}{da} \cdot \vec{t}\, ds$$

Recalling $\left(\frac{d}{da} = \frac{\partial}{\partial a} - \frac{\partial}{\partial x} \right)$

$$\boxed{\frac{d\Pi}{da} = \underbrace{\int_{A'} \left(\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x} \right) dA}_{} + \int_{\partial A'} \left(\frac{\partial \vec{u}}{\partial a} - \frac{\partial \vec{u}}{\partial x} \right) \cdot \vec{t}\, ds} \quad (5)$$

trying to change interior ($\int_{A}$) integral to boundary ($\int_{\partial A'}$) integral

to get to the form J which does not have any interior integral

$$\frac{\partial W}{\partial a} = \frac{\partial W(\varepsilon)}{\partial \varepsilon} : \frac{\partial \varepsilon}{\partial a} = \sigma : \frac{\partial \varepsilon}{\partial a} = \sigma_{ij} \frac{\partial \tfrac{1}{2}}{\partial a} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\}$$

σ

hyperelastic (nonlinear elastic)

$$= \sigma_{ij}\left\{\frac{1}{2}\frac{\partial^2 u_i}{\partial x_j \partial a} + \frac{1}{2}\frac{\partial^2 u_j}{\partial x_i \partial a}\right\} \overset{\text{symmetry of } \sigma}{=}$$

$$\sigma_{ij}\frac{\partial^2 u_i}{\partial x_j \partial a} \Longrightarrow$$ trying to form a full derivative term to use divergence theorem

$$\Rightarrow \frac{\partial W}{\partial a} = \sigma_{ij}\frac{\partial^2 u_i}{\partial x_j \partial a} = \underbrace{\frac{\partial}{\partial x_j}\left\{\sigma_{ij}\frac{\partial u_i}{\partial a}\right\}}_{\text{full derivative}} - \sigma_{ij,j}\frac{\partial u_i}{\partial a}$$

$= 0$ because

$\sigma_{ij,j} + \cancel{\rho b_i} = \rho\ddot{u}_i$

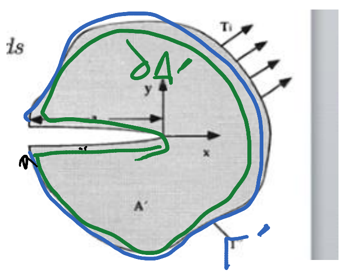
0 no body force, 0 static

$\vec{b} = 0$

$$\Rightarrow \frac{\partial W}{\partial a} = \frac{\partial}{\partial x_j}\left\{\sigma_{ij}\frac{\partial u_i}{\partial a}\right\}$$

$$\frac{1}{B}\frac{d\Pi}{da} = \int_{A'}\left\{\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x_1}\right\}dA' - \int_{\partial A'}\left(\frac{\partial u}{\partial a} - \frac{\partial u}{\partial x}\right)t\,ds \Longrightarrow$$

$$\frac{1}{B}\frac{d\Pi}{da} = \int_{A'}\left(\frac{\partial}{\partial x_j}\left\{\sigma_{ij}\frac{\partial u_i}{\partial a}\right\} - \frac{\partial W}{\partial x_1}\right)dA' - \int_{\partial A'}\left(\frac{\partial u}{\partial a} - \frac{\partial u}{\partial x}\right)\cdot t\,ds$$



$$\Rightarrow \frac{1}{B}\frac{d\Pi}{da} = \int_{\partial\Gamma'}\left\{\left(\sigma_{ij}n_j\frac{\partial u_i}{\partial a}\right) - W n_1 - \frac{\partial u}{\partial a}\cdot t + \frac{\partial u}{\partial x}\cdot t\right\} ds$$

$\sigma_{ij}n_j = t_i$

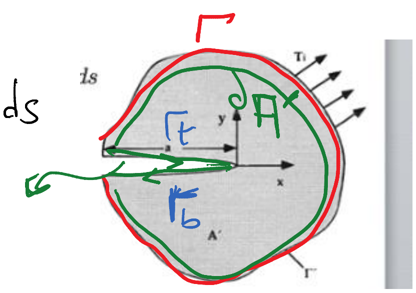
$$\int_{A'}\frac{\partial f}{\partial x_i}\,dA = \int_{\Gamma'} f n_i\,ds$$

Gauss theorem

$$\frac{1}{B}\frac{d\Pi}{da} = \int_{\partial\Gamma'}\left\{\underbrace{t\cdot\frac{\partial u}{\partial a}} - Wn_1 - \underbrace{\frac{\partial u}{\partial a}\cdot t} + \frac{\partial u}{\partial x}\cdot t\right\}ds$$

cancel out

$$\Rightarrow \frac{1}{B}\frac{d\Pi}{da} = -\int_{\partial A'}\left\{Wn_1 - \frac{\partial u}{\partial x}\cdot t\right\}ds$$



But $\displaystyle\int\left(Wn_1 - \frac{\partial \vec{u}}{\partial x}\vec{t}\right) = 0$

But $\int_{\Gamma_b} \left(W n_1 - \frac{\partial u}{\partial x} t \right) = 0$

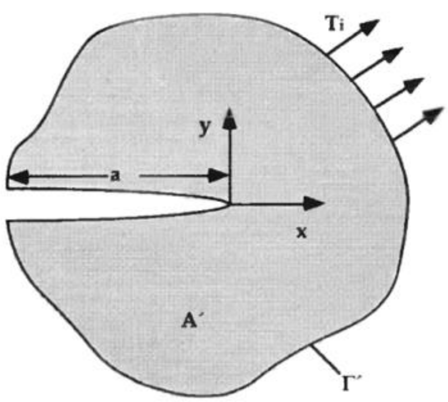
Γ_b $\quad n_1 = 0 \quad \vec{n} \quad n_1 = 0 \quad 1$

$\vec{t} = 0$ on Γ_b (traction free)

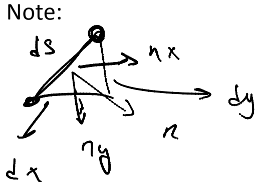
Similarly $\int_{\Gamma_t} \left(W n_1 - \frac{\partial \vec{u}}{\partial x} \vec{t} \right) = 0$

$$\Rightarrow \int_{\Gamma} \left(W n_1 - \frac{\partial \vec{u}}{\partial x} \vec{t} \right) = \overbrace{\int_{\partial A'} \left(W n_1 - \frac{\partial \vec{u}}{\partial x} \vec{t} \right) ds}^{\mathcal{G} = -\frac{d\Pi}{B\,da}} - \underbrace{\int_{\Gamma_t} \left(W n_1 - \frac{\partial u}{\partial x} \vec{t} \right) ds}_{0} - \underbrace{\int_{\Gamma_b} \left(W n_1 - \frac{\partial u}{\partial x} \vec{t} \right) ds}_{0}$$

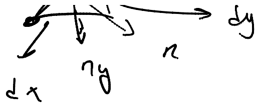
$$\Rightarrow \boxed{\mathcal{G} = J = \int_{\Gamma} \left(W n_1 - \frac{\partial \vec{u}}{\partial x} \vec{t} \right) ds}$$



Note:



$W n_1 = W dy$



$ds\, n_x = dy$

$-ds\, n_y = dx$

$\rightarrow \quad W n_1 = W dy$

$$\boxed{G = J} = \int_{\Gamma} \left(W dy - t \cdot \frac{du}{dx} ds \right)$$

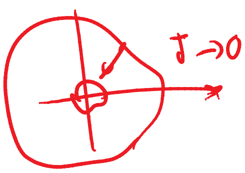
J integral provides energy release rate for both linear & nonlinear elastic material

J is path independent

Assumptions:

Crack surface is traction free $\Rightarrow$ J is path independent

— Nonlinear elastic $\quad \sigma = \dfrac{\partial W}{\partial \varepsilon}$ not true for plastic material



← Acceptable for plastic material during loading

— Static (Not dynamic loading) $\quad \sigma_{ij,j} = 0 \quad \nabla \cdot \sigma = 0$

— Static (not dynamic loading) $\nabla\cdot\sigma = 0$

— No body force $\nabla\cdot\sigma + \rho b_i = \rho \ddot{u}_i$, $b_i = 0$, $\ddot{u}_i = 0$

— No thermal strains

Relation between J & K in LEFM

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \quad \Rightarrow \quad J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$J = G$

How can we evaluate K_I, K_{II} (assume $K_{III} = 0$) using J integral?

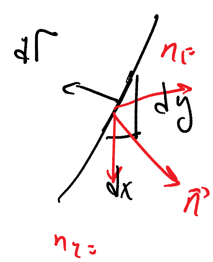
$$J = \frac{K_I^2 + K_{II}^2}{E'}$$

just using J integral we cannot evalue K_I & K_{II}

We have more than one J integral.

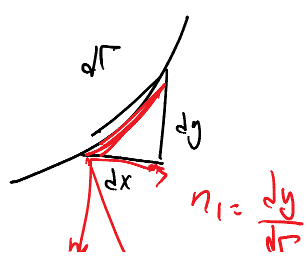
$$J_k = \int_\Gamma \left(W n_k - t_i \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$$J = J_1 = \int_\Gamma \left(W n_1 - \vec{t}\,\frac{\partial \vec{u}}{\partial x_1} \right) d\Gamma$$

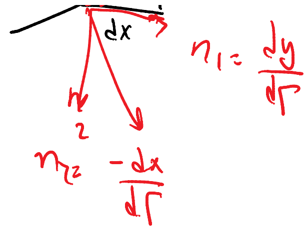


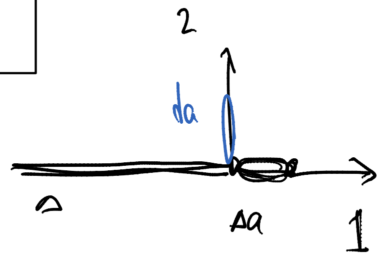
$$\boxed{J = J_1 = \int_\Gamma \left(W\,dy - \vec{t}\cdot\frac{\partial u}{\partial x} \right) d\Gamma}$$

physical meaning: ...



physical meaning: energy release rate for crack moving along the crack direction





$$J_1 = -\frac{d\Pi}{da}$$

$$J_2 = \int \left(W n_2 - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_2} \right) d\Gamma$$

Energy release rate for 90° crack extension