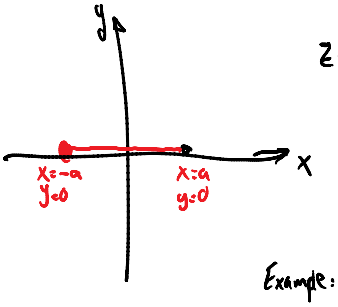
$$\sigma_{xx} = \mathrm{Re}Z - y\mathrm{Im}Z'$$
$$\sigma_{yy} = \mathrm{Re}Z + y\mathrm{Im}Z'$$
$$\tau_{xy} = -y\mathrm{Re}Z'$$

$$Z = \frac{\sigma}{\sqrt{1 - \frac{a^2}{z^2}}}$$

$z = x + yi$

$\sigma_{yy} = \mathrm{Re}\, Z + y\,\mathrm{Im}\, Z'$



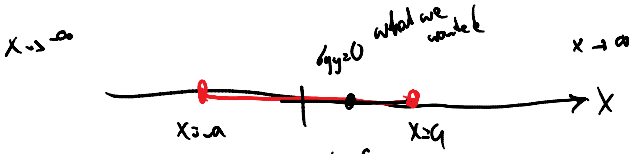
Example: If we are looking along x axis

$$\sigma_{yy} = \mathrm{Re}\left(\frac{\sigma}{\sqrt{1 - \frac{a^2}{x^2}}}\right) + y\,\mathrm{Im}\, Z'$$

$y \to 0$

$(z = x + yi,\ y = 0)$

on $(y=0)$ $\quad \sigma_{yy} = \mathrm{Re}\left(\frac{\sigma}{\sqrt{1 - \frac{a^2}{x^2}}}\right)$



$x = \frac{a}{2}$

$$\sigma_{yy} = \mathrm{Re}\left(\frac{\sigma}{\sqrt{1 - \frac{a^2}{x^2}}}\right) = \mathrm{Re}\left(\frac{\sigma}{\sqrt{-3}}\right) = 0$$

$\frac{\sigma}{\sqrt{3}i} = -\frac{\sigma}{\sqrt{3}}i$

at $x \to \infty$

$$\sigma_{yy} = \lim \mathrm{Re}\left(\frac{\sigma}{\sqrt{1 - \left(\frac{a}{x}\right)^2}}\right) = \lim \mathrm{Re}\left(\frac{\sigma}{\sqrt{1}}\right) = \sigma \quad \checkmark$$

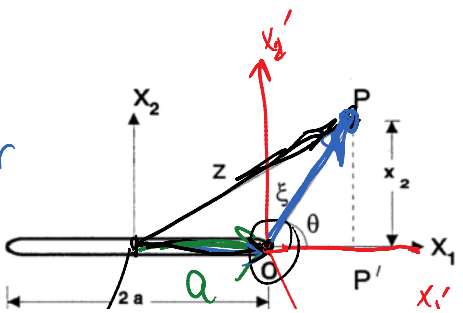
$x \to \infty$

We can also see $\sigma_{xx}(x, y) = \sigma$

$x \to \infty$, $y \to \infty$

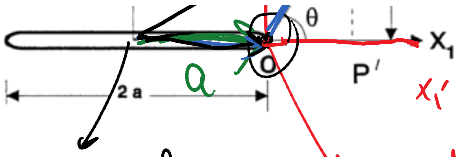
We are looking for asymptotic solutions around the crack tip

$\vec{Z} = \vec{a} + \vec{\xi}$ ← position vector w.r.t. crack tip



$z = \vec{a} + \zeta$ ' / ' w.r.t. crack tip

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}$$



original coordinate system

coordinate system centered at the crack tip

$z = a + \zeta$ → has two compnents

$\left|\frac{\zeta}{a}\right| \ll 1$ ζ is very small relative to the crack length

$$Z = \frac{\partial z}{\sqrt{z^2 - a^2}} = \frac{\partial(a+\zeta)}{\sqrt{(a+\zeta)^2 - a^2}} = \frac{\partial a\left(1+\frac{\zeta}{a}\right)}{\sqrt{2a\zeta + \zeta^2}}$$

$z = a + \zeta$

$$= \frac{\partial a}{\sqrt{2a\zeta}} \left(\frac{1+\left(\frac{\zeta}{a}\right)}{\sqrt{1+\frac{\zeta}{2a}}} \right)$$

$\left|\frac{\zeta}{a}\right| \ll 1$

ζ/a is much smaller than 1

$1 + \frac{\zeta}{a} \approx 1$

$1 + \frac{\zeta}{2a} \approx 1$ $\qquad \sqrt{1+\frac{\zeta}{2a}} \approx 1$

$$Z \approx \frac{\partial a}{\sqrt{2a\zeta}} + \text{higher orders in } \zeta$$

constant, $\zeta^{\frac{1}{2}}, \zeta^{\frac{3}{2}}, \ldots$

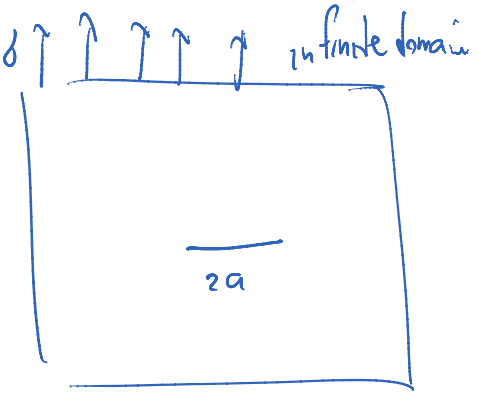
This is the first term in asymptotic expansion and dominates higher order terms

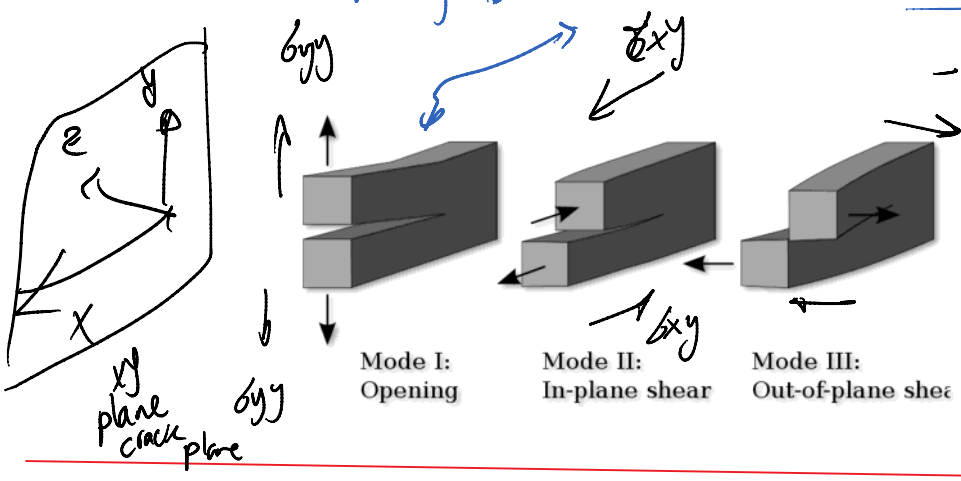
$$Z = \frac{\sigma\sqrt{a\pi}}{\sqrt{2\pi f}}$$

we call this stress intensity factor denoted by K:

$$K_I = \sigma\sqrt{\pi a}$$

referring to mode I



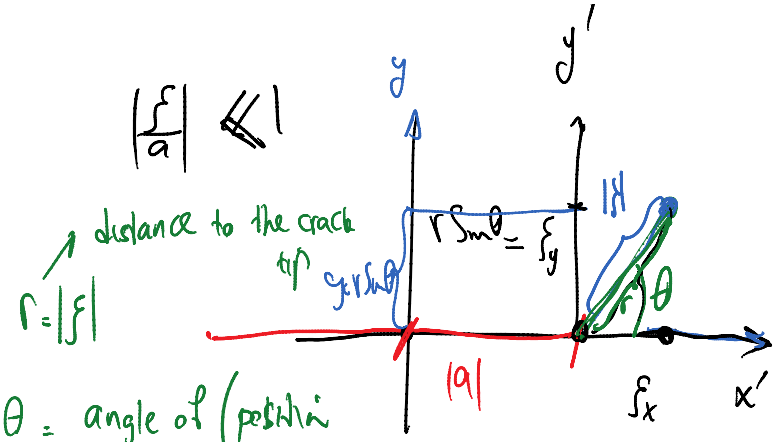


Now that we have approximate Z function, we want to evaluate stresses around the crack tip

$$Z = \frac{K_I}{\sqrt{2\pi\xi}} \qquad \left|\frac{\xi}{a}\right| \ll 1$$

$\xi = \xi_x + i\,\xi_y$

$\xi = re^{i\theta}$



θ = angle of (position w.r.t. crack tip) to x' axis

$\xi_x = r\cos\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\xi = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$= \underbrace{(r\cos\theta)}_{\xi_x} + i\,\underbrace{(r\sin\theta)}_{\xi_y}$$

$$\left.\begin{aligned} Z &= \frac{K_I}{\sqrt{2\pi\xi}} \\ \xi &= re^{i\theta} \end{aligned}\right\} \Rightarrow$$

$K_I = \sigma\sqrt{\pi a}$

(I) $$Z = \frac{K_I}{\sqrt{2\pi re^{i\theta}}} = \frac{K_I}{\sqrt{2\pi r}\,e^{i\frac{\theta}{2}}} = \frac{K_I}{\sqrt{2\pi r}}\,e^{-i\frac{\theta}{2}}$$

$$\sigma_{xx} = \mathrm{Re}Z - y\mathrm{Im}Z'$$

$$\sigma_{yy} = \mathrm{Re}Z + y\mathrm{Im}Z'$$

$$\tau_{xy} = -y\mathrm{Re}Z'$$

$Z' = ?$

$y = r\sin\theta$

we want to find σ_{yy} around the crack tip

$$Z = \frac{K_I}{\sqrt{2\pi\zeta}} = \frac{K_I}{\sqrt{2\pi}}\zeta^{-\frac{1}{2}}$$

$\zeta = z - a$

$$Z' = \frac{\partial Z}{\partial z} = \frac{\partial Z}{\partial \zeta}\left(\frac{\partial \zeta}{\partial z}\right) = 1$$

(2) $$Z' = \frac{\partial Z}{\partial \zeta} = -\frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\zeta^{-\frac{3}{2}}$$

$$\sigma_{yy} = \mathrm{Re}Z + y\,\mathrm{Im}Z'$$

$$= \mathrm{Re}\left(\frac{K_I}{\sqrt{2\pi\zeta}}\right) + (r\sin\theta)\,\mathrm{Im}\left(-\frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\zeta^{-\frac{3}{2}}\right)$$

$\zeta = re^{i\theta}$

$$= \mathrm{Re}\left(\frac{K_I}{\sqrt{2\pi r}}e^{-i\frac{\theta}{2}}\right) + r\sin\theta\,\mathrm{Im}\left(-\frac{1}{2}\frac{K_I}{\sqrt{2\pi}}r^{-\frac{3}{2}}e^{-\frac{3}{2}i\theta}\right)$$

$$= \frac{K_I}{\sqrt{2\pi r}}\left\{\mathrm{Re}\left(e^{-i\frac{\theta}{2}}\right) + -\frac{\sin\theta}{2}\,\mathrm{Im}\left(e^{-\frac{3}{2}\theta i}\right)\right\}$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

$$= \frac{K_I}{\sqrt{2\pi r}}\left\{ \mathrm{Re}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) - \frac{\sin\theta}{2}\,\mathrm{Im}\left(\cos\frac{3\theta}{2} - i\sin\frac{3\theta}{2}\right)\right\}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} + \frac{\sin\theta}{2}\sin\frac{3\theta}{2}\right)$$

checking this

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

+ HOT

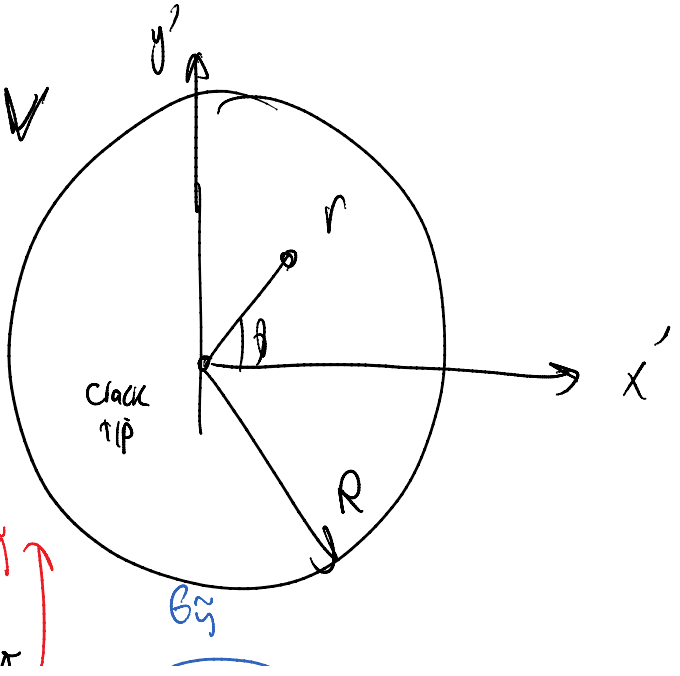
$r^0 f_0(\theta) + r^{\frac{1}{2}} f(\theta) + \cdots$

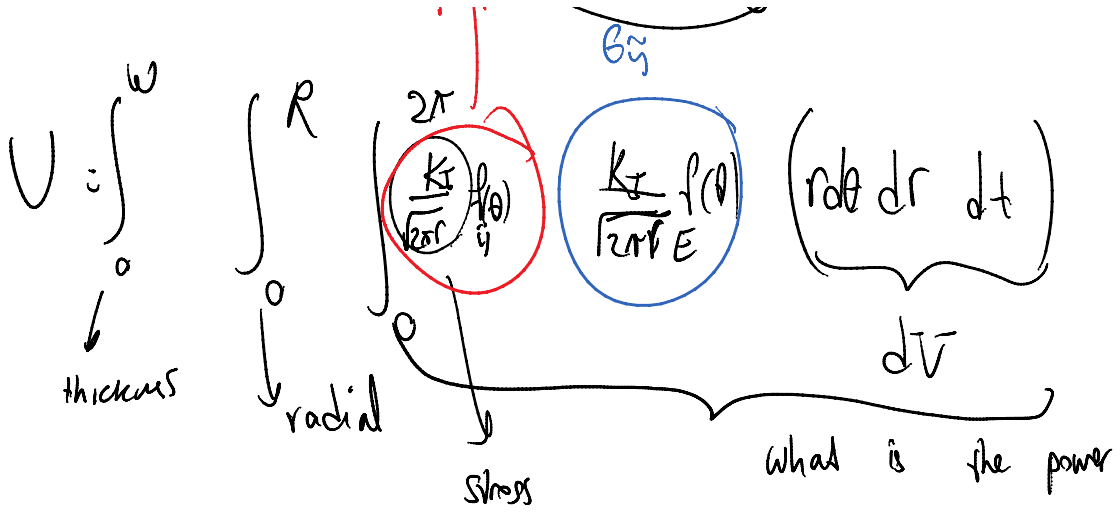
$r \to 0$

$\sigma_{xx}, \sigma_{yy}, \tau_{xy} \to \infty$ as $r \to 0$

internal energy density $= U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

internal energy $U = \int_V u\,dV$





$$U = \int_0^w \int_0^R \int_0^{2\pi} g(\theta)\, dr\, d\theta\, dt \qquad \text{or} \quad r$$

$$wR \left(\int_0^{2\pi} g(\theta)\, d\theta \right) < \infty$$

The energy within any cylindrical region of radius R, and thickness w around the crack tip is **FINITE**

The importance of 1/2 singularity =>

$$\sigma \sim \frac{1}{\sqrt{r}} \to \infty$$

$$\varepsilon \sim \frac{1}{\sqrt{r}} \to \infty$$

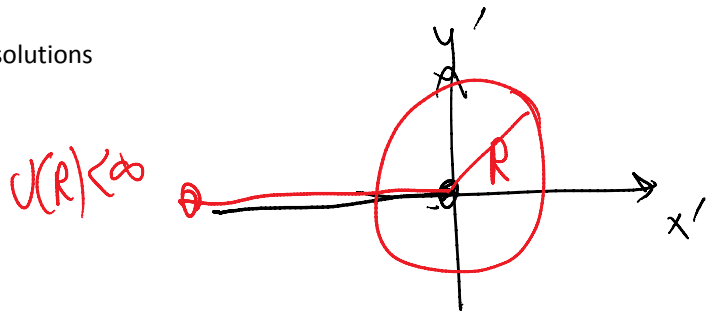
$$U(R) < \infty$$

energy within cylinder of radius R is finite

If we have stronger singularity than or equal to 1/r we would have got infinite energy around the crack tip

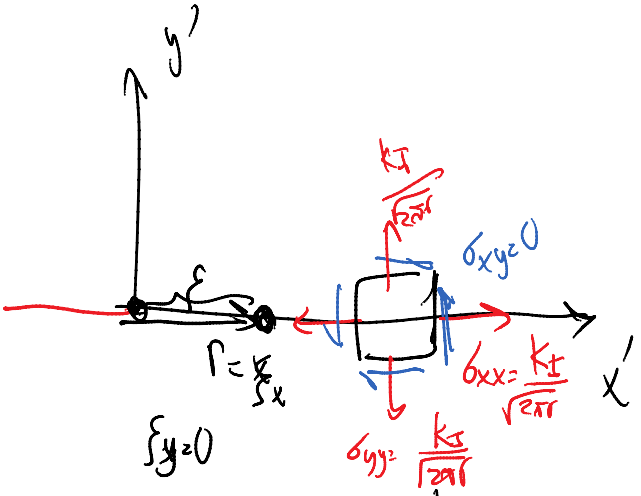
Critical review of asymptotic crack tip solutions

+ Energy is finite



- Stresses go to infinity

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2}$$



loading along the crack direction.

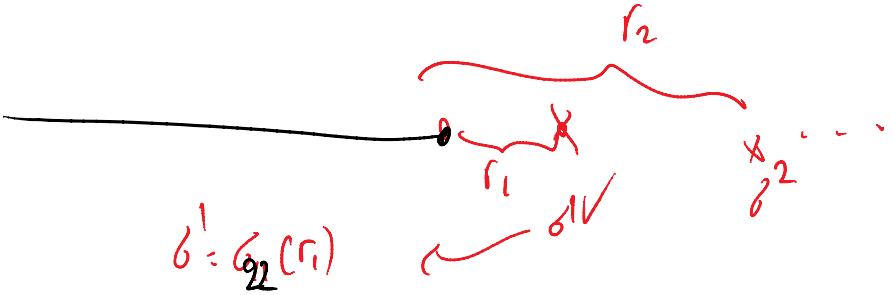
$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}$ ($\sigma_{xx} = \sigma_{yy}$ ahead of the crack $\theta = 0$) $\theta = 0$

$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$

$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cdots = 0$

even though the dominant loading is in yy direction

How to calculate stress intensity factor



$$\sigma_{22}(r) = \frac{K_I}{\sqrt{2\pi r}} \rightarrow K_I = \sqrt{2\pi r}\,\sigma_{22}$$

$$K_I = \sqrt{2\pi r_1}\,\sigma_{zz}(r_1) \simeq \sqrt{2\pi r_2}\,\sigma_{zz}(r_2)$$

$\sqrt{2\pi r}$. stress gives you K

$$K_I = \lim_{\theta=0} \sqrt{2\pi r}\,\sigma_{zz}(r,0)$$

Are these solutions general

1. There was no mode-II (along the y-direction loading)
2. We obtained the solutions for a particular geometry (center crack tension in infinite domain)

The argument for mode-II solution

Mode-II solution is obtained very much the same way?

We get a different stress function

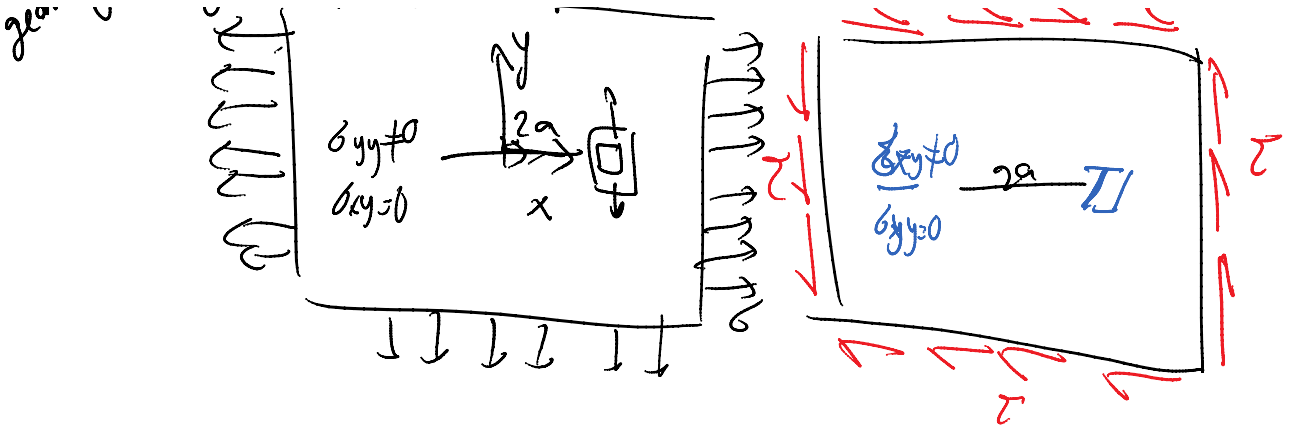
mode I

$$Z = \frac{\sigma z}{\sqrt{z^2 - a^2}}$$

infinite geometry

mode II

$$Z = -\frac{i\tau z}{\sqrt{z^2 - a^2}}$$



Z's provide analytical solutions for both cases. By asymptotic expansion around the crack tip (similar to Mode I) we obtain solutions close to crack tip for mode II as well

Westergaards, Sneddon etc.

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

(mode I)

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

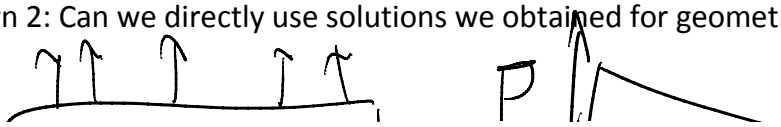
(mode II)

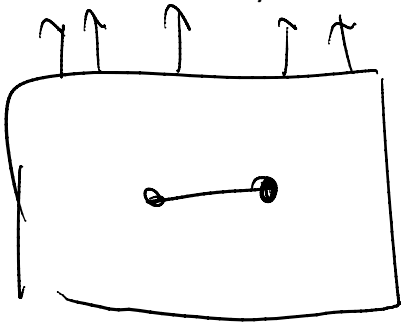
Irwin $$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \text{H.O.T}$$

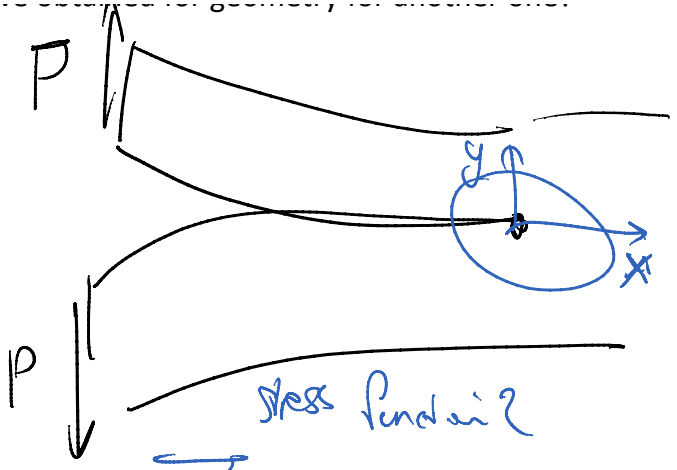
$$\sigma_{ij}^{(mode)} = \frac{K^{(mode)}}{\sqrt{2\pi r}} f_{ij}^{(mode)}(\theta)$$

$$\sigma_{II}^{(I)} = \frac{K^{I}}{\sqrt{2\pi r}} f_{II}^{(I)}(\theta) \qquad f_{II}^{(I)}(\theta) = \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

Concern 2: Can we directly use solutions we obtained for geometry for another one?







stress function ?

exact $Z = \frac{\sigma_\infty}{\sqrt{\pi}}\sqrt{\pi a}$

stress function

Z

around the crack tip

asymptotic $\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} f_{11}^{I}(\theta)$ | $\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} f_{11}^{(I)}(\theta)$