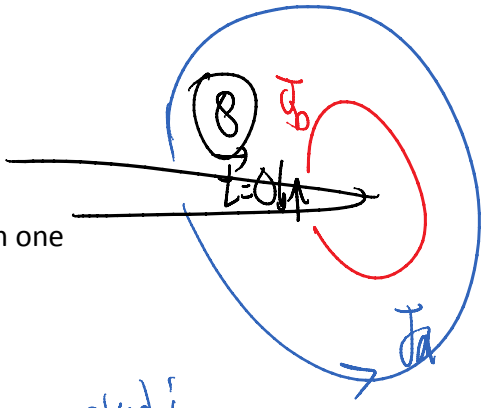
2015/10/22

Thursday, October 22, 2015
1:03 PM

$J_a = J_b$



Assumptions that we used to show G = J for **any** curve from one side of the crack to the other side:

1. Homogeneous body
2. Linear or non-linear elastic solid (Or elastic plastic material when there is no unloading)
3. No inertia, or body forces; no initial stresses
4. No thermal loading
5. 2-D stress and deformation field
6. Plane stress or plane strain
7. Mode I loading
8. Stress free crack

Static

$\sigma_{ij,j} = 0$

$\nabla \cdot \sigma = 0$

$\nabla \cdot \sigma + \rho b = \rho a$

Generalization of J integral

- Dynamic loading
- Surface tractions on crack surfaces
- Body force
- Initial strains (e.g. thermal loading)
- Initial stress from pore pressures

cf. Saouma 13.11 & 13.12 for details

Importance of J integral:

1. Extend the applicability of LEFM
2. A Much more robust method to evaluate G, K, …:
   a. We can even calculate all KI, KII, … with J integral

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$K_I, K_{II}, K_{III} \Rightarrow G$

$G \not\Rightarrow K_I, K_{II}, K_{III}$ (problem)

we know different methods to calculate G

$G = \frac{P^2}{2B}\frac{dC}{da}$

J integral fixes this problem!
Having J integrals we can evaluate KI, KII (in-plane loading)

$$J_1 = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$$J_k = \int_\Gamma \left(W n_k - t_i \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$k=1 \Rightarrow J_1 \quad (J)$

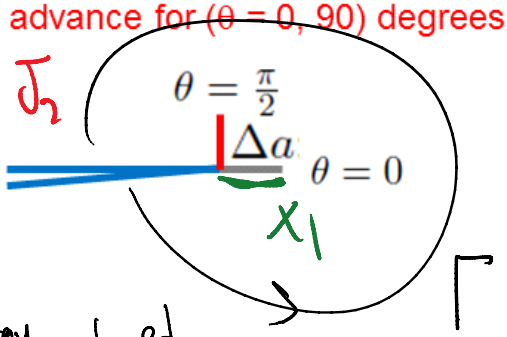
$k=2 \quad J_2$

In fact both J1 (J) and J2 are related to SIFs:

$$G \simeq J_1 = \int_\Gamma \left(w dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$

$$J_2 = \int_\Gamma \left(-w dx - \mathbf{t} \frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

J1 & J2: crack advance for (θ = 0, 90) degrees



energy released per unit length of crack

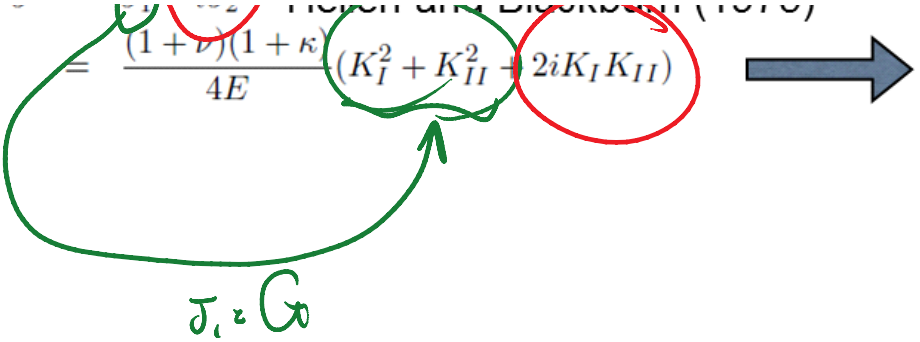
energy released per unit length of the crack in X_1 direction

Normal to current crack direction

$$J = J_1 - iJ_2$$

Hellen and Blackburn (1975)

$$= \frac{(1+\nu)(1+\kappa)}{4E}(K_I^2 + K_{II}^2 + 2iK_I K_{II})$$



$$J_1 = \frac{K_I^2 + K_{II}^2}{E'}$$
$$J_2 = \frac{-2K_IK_{II}}{E'}$$

2 equations
&
2 unknows K_I, K_{II} ⇒

$K_2 = \frac{J_2}{2K_1}$ (=0)

evaluate K_I, K_{II}

2nd order equation

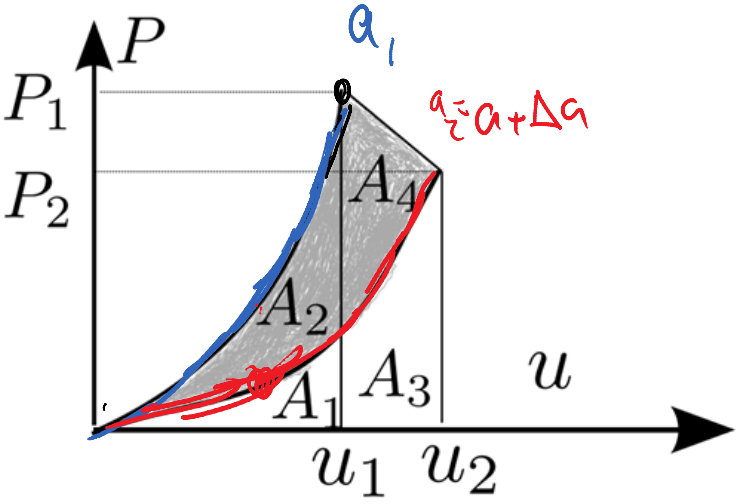
$J_1 = K_1^2 + a^2 \frac{1}{K_1^2}$ ⇒ $K_I^4 + \bigcirc K_I^2 + \bigcirc = 0$

2nd order equation for K_I^2

$K_I = \pm \bigcirc$
$K_2 = \pm \bigcirc$

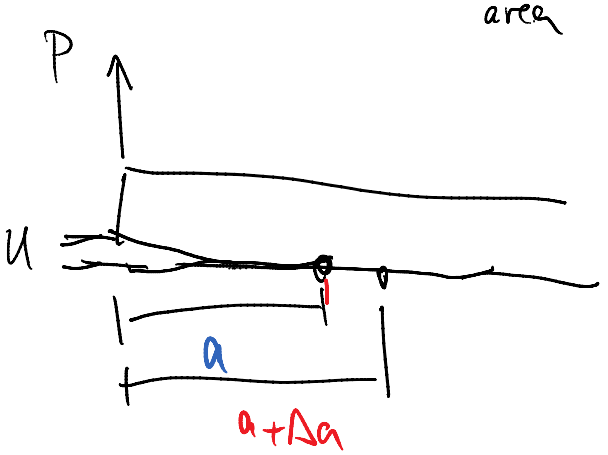
Note that if $K_I = a, K_{II} = b$ is a solution the general solution is:
$K_I = \pm a, K_{II} = \pm b$ and $K_I = \pm b, K_{II} = \pm a$

5.3. 4. Energy Release Rate, crack growth and R curves



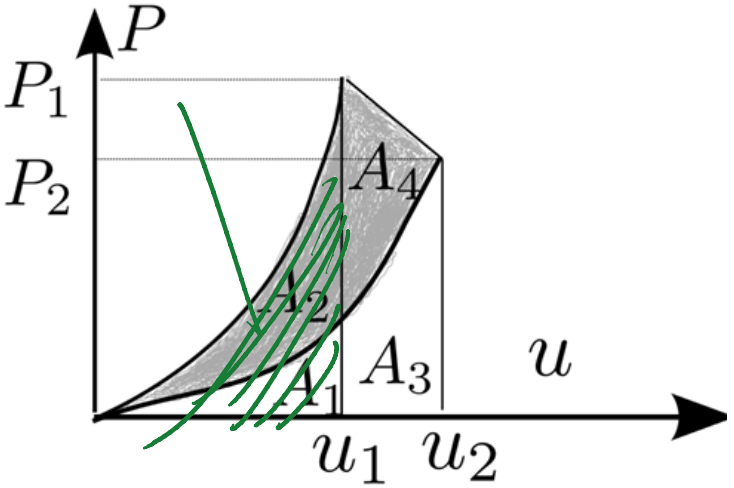
$G = -\frac{\Delta \Pi}{\Delta A}$ → potential energy

→ change in crack area

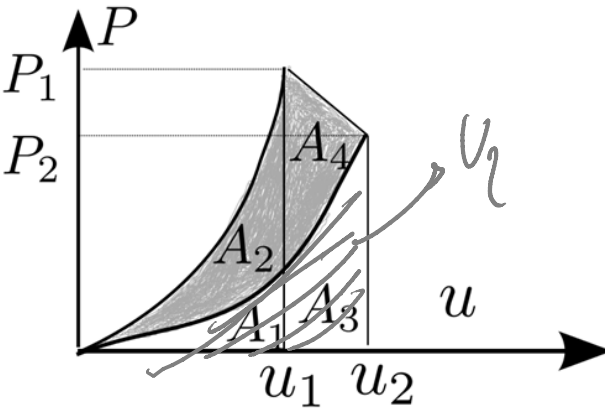


$\Delta \Pi = \Delta U - W$

$\Delta U = U_2 - U_1$

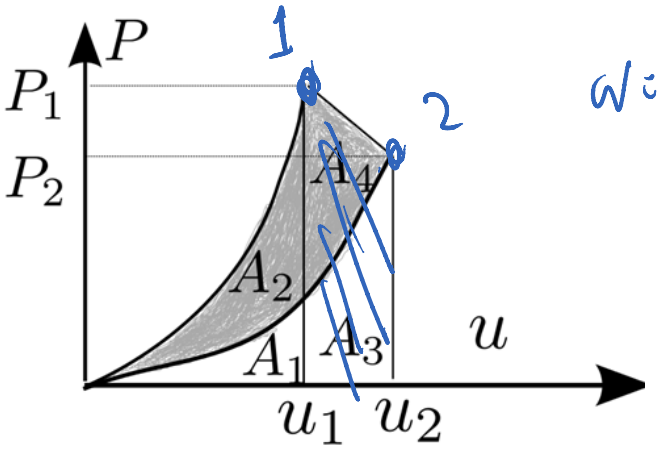


$U_1 = A_1 + A_2$



$U_2 = A_3 + A_4$

W_{12} work done from configuration 1 to 2

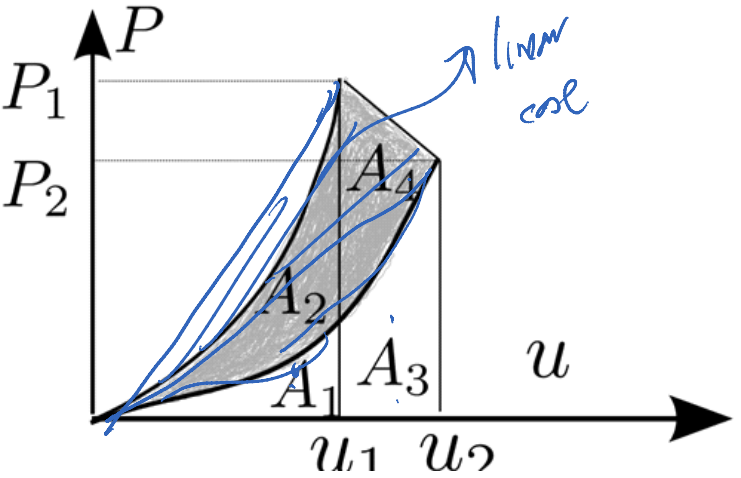


$$W = \int_{u_1}^{u_2} P\,du = A_3 + A_4$$

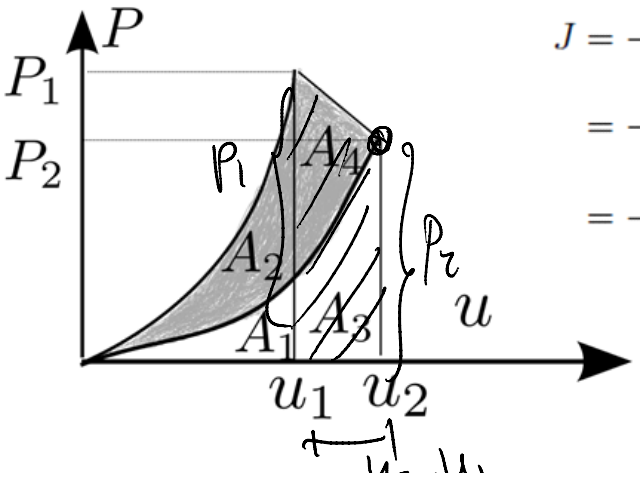
$$\left.\begin{aligned} U_{e1} &= A_1 + A_2 \\ U_{e2} &= A_1 + A_3 \\ W_{12} &= A_3 + A_4 \end{aligned}\right\} \Rightarrow$$

$$\Delta\Pi = \{U_{e2} - U_{e1}\} - W_{12} = A_2 + A_4$$

$$\boxed{J \approx \frac{1}{B}\frac{\Delta\Pi}{\Delta a} = \frac{1}{B\Delta a}\,(\text{shaded area})}$$



G: $\frac{\text{shaded area}}{B\,\Delta a}$ ... Similar to linear case



$$J = -\lim_{\Delta a \to 0} \frac{\Delta\Pi}{B\Delta a}$$

$$= -\lim_{\Delta a \to 0} \frac{U_{e2} - U_{e1} - W_{12}}{B\Delta a}$$

$$= -\lim_{\Delta a \to 0} \frac{1}{B}\left\{\frac{U_{e2} - U_{e1}}{a_2 - a_1} - \frac{P_1 + P_2}{2}\frac{u_2 - u_1}{a_2 - a_1}\right\}$$

$a_2 \to a_1$, P

$a_1 \to a_2$

$$= \frac{1}{B}\left(\frac{dU}{da} + P\frac{du}{da}\right)$$ displacement

u_1 u_2 → $u_2 - u_1$

$= \frac{1}{B}\left(\frac{\quad}{da} + 1 \frac{}{da} \right)$ displacement

internal energy

$W_{12} = (u_2 - u_1)\left(\frac{P_1 + P_2}{2} \right)$

$$J = \frac{1}{B}\left\{ -\frac{dU_e}{da} + P\frac{du}{da} \right\} = \frac{1}{B}\left\{ -\frac{dU_e}{du} + P \right\}\frac{du}{da}$$

$\frac{dU_e}{da} = \frac{dU_e}{du}\frac{du}{da}$

$U_e(u)$ not $U_e(a)$

$\frac{d\,U_e(u)}{du}$ is easier to compute

Fixed Grip

u = constant

fixed (dead load)

u increases as a = $a(a)$

P is fixed

P is fixed

$$J = -\frac{1}{B}\frac{dU_e}{da}$$

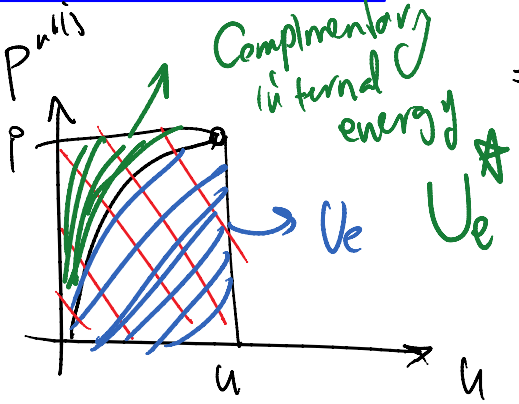
fixed grip

$J = \frac{1}{B}\left\{ -\frac{dU_e}{da} + P\frac{du}{da} \right\}$ P is constant

$= \frac{1}{B}\left\{ -\frac{dU_e}{da} + \frac{d\,Pu}{da} \right\}$

$= \frac{1}{B}\left\{ \frac{d(Pu - U_e)}{da} \right\}$

$= \frac{1}{B}\frac{dU_e^*}{da}$



$$J = \frac{1}{B}\left\{ -\frac{dU_e}{da} + P\frac{du}{da} \right\} = \frac{1}{B}\left\{ -\frac{dU_e}{du} + P \right\}\frac{du}{da}$$

Fixed grip

General

Fixed grip

$G = J = -\frac{1}{B}\frac{dU_e}{da}$

Dead load

$G = J = \frac{1}{B}\frac{dU_e}{da}$

Show that the equation

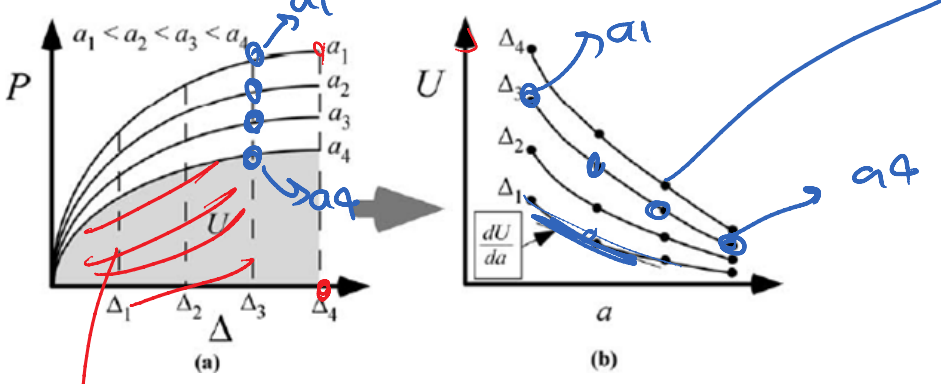
$$J = \frac{1}{B}\left\{-\frac{\mathrm{d}U_e}{\mathrm{d}a} + P\frac{\mathrm{d}u}{\mathrm{d}a}\right\} = \frac{1}{B}\left\{-\frac{\mathrm{d}U_e}{\mathrm{d}u} + P\right\}\frac{\mathrm{d}u}{\mathrm{d}a}$$

Reduces to what obtained for linear solid:

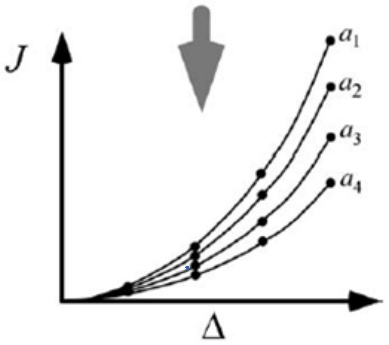
- Compare this equation with $G = -\frac{u^2}{2B}\frac{\mathrm{d}K}{\mathrm{d}a} = \frac{P^2}{2B}\frac{\mathrm{d}C}{\mathrm{d}a}$
- For linear case $U_e = \frac{1}{2}Ku^2$ and $P = Ku$. Show that equation for J agrees for the two for G

Experimental estimate of J

Landes and Begley, ASTM 1972



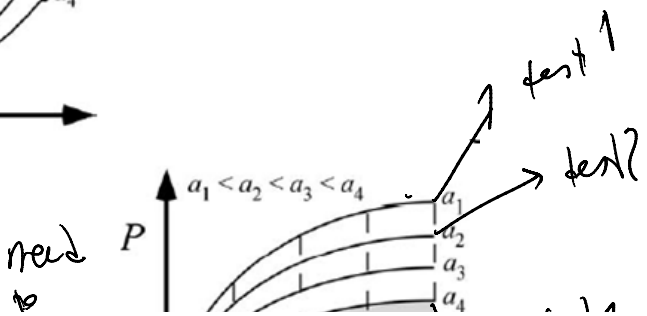
$U_e(a_4, \Delta_3)$



U(a) for a fixed grip (Δ)

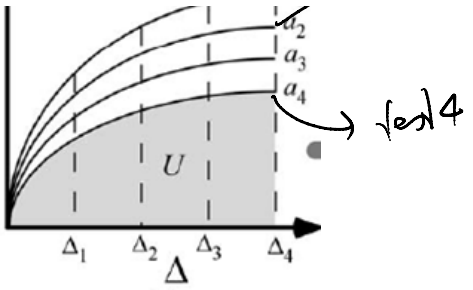
$G = J = -\frac{1}{B}\frac{dU}{da}$

- Rice proposes a method to obtain J with



- Rice proposes a method to obtain J with only one test for certain geometries

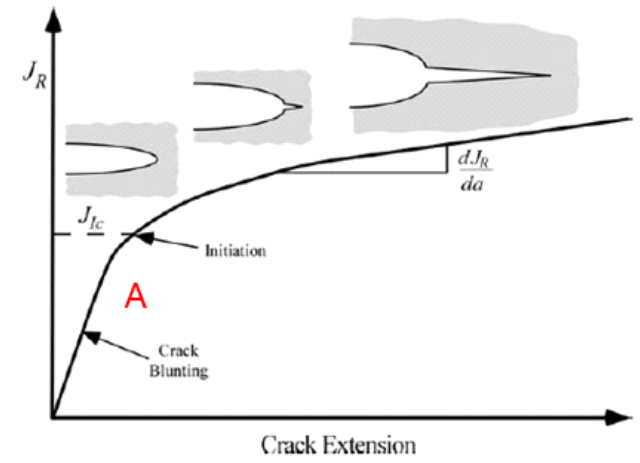
need to do several test

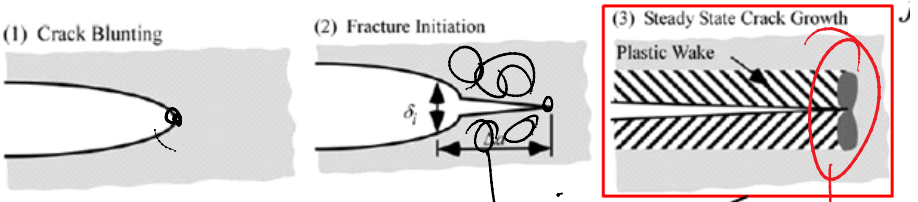


cf. Anderson 3.2.5 for details

Crack growth resistance curve

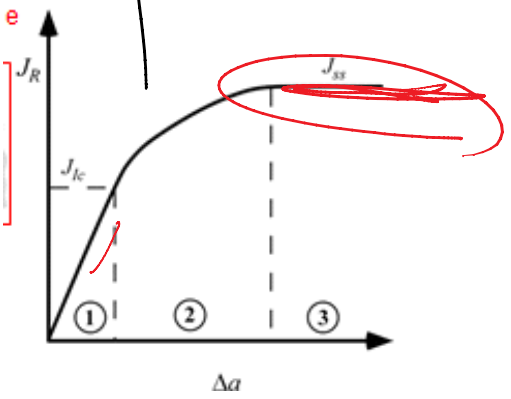
J, R curve extended to nonlinear material





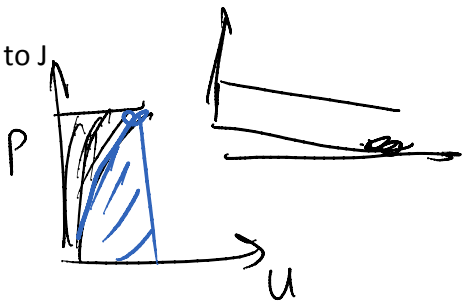
plastic deformation

more resistive mechanisms are activated ahead of the crack

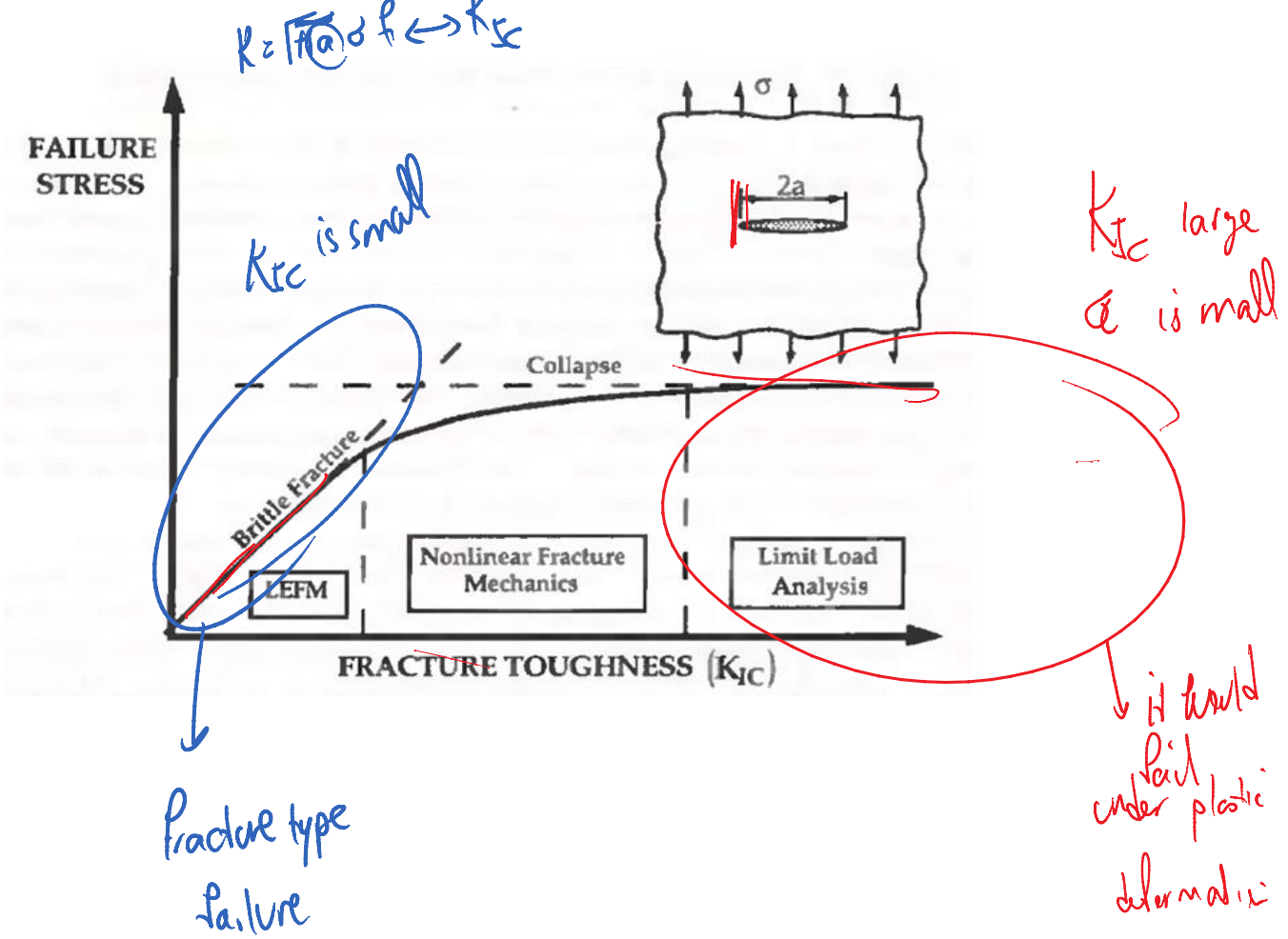


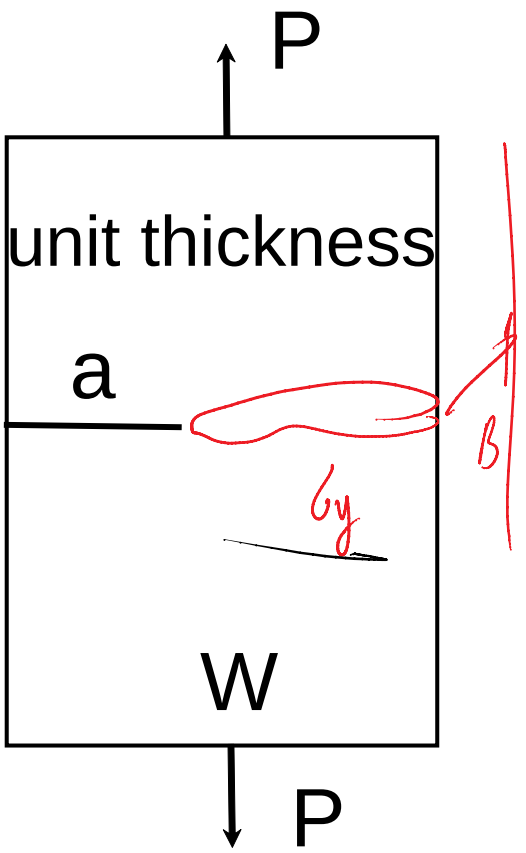
plastic zone fully developed

Plane stress loading develops larger plastic zone so, initial increase to J critical is more pronounced for plane stress



5.3. 7. Fracture mechanics versus material (plastic) strength





$$P_{ultimate\ plastic} = (W-a)\,B\,\sigma_y$$

area

$$P_{ultimate\ fracture} = \sigma_{ultimate\ fracture}(BW)$$

area

$$\sqrt{a}\,\pi\,\sigma_u\,f\left(\frac{a}{W}\right) = K_{Ic}$$

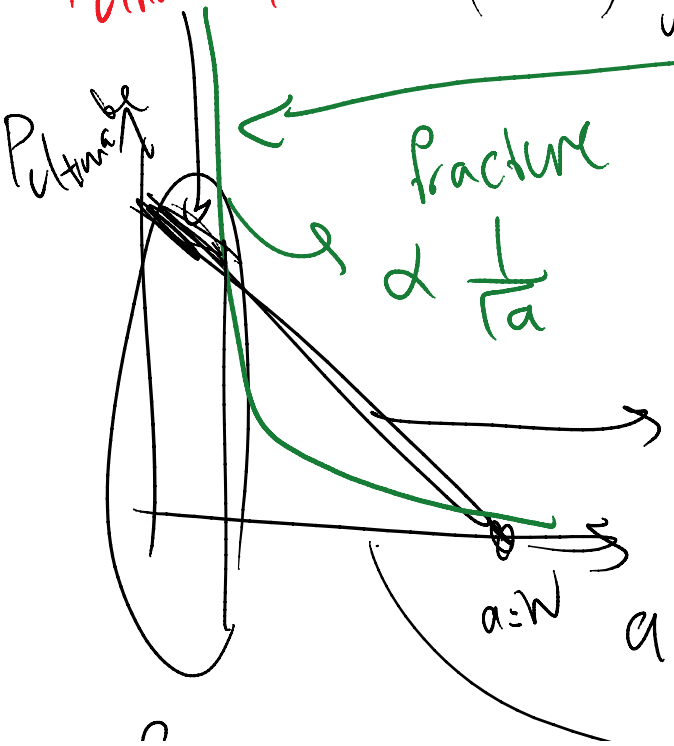
$$\sigma_{ultimate} = \frac{K}{\sqrt{\pi a}\,f\left(\frac{a}{W}\right)}$$

$P_{ultimate\ plastic} = (W-a)\,\sigma_y B$

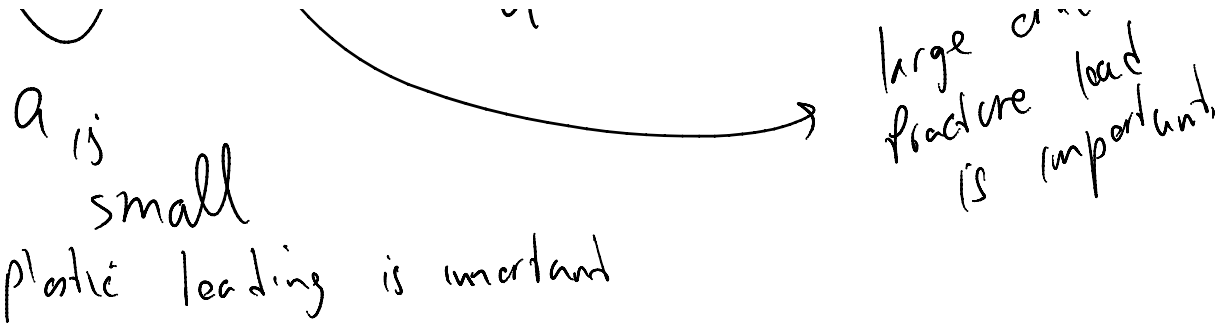
$P_{ultimate\ fracture}$

fracture $\propto \frac{1}{\sqrt{a}}$

$\frac{K_{Ic}}{\sqrt{\pi a}}\,BW$



large crack


a is small
plastic loading is important
large
fracture load
is important