

Note that the midterm will be graded out of 250 with 50 extra points on the question sheet.

1. **(90 Points)**. A cylindrical pressure vessel with closed ends has a radius  $R = 1$  m and thickness  $t = 40$  mm and is subjected to internal pressure  $p$ . The vessel must be designed safely against failure by yielding (according to the von Mises yield criterion) and fracture. The vessel is made of steel with yield stress  $\sigma_y = 860$  MPa and fracture toughness  $K_{Ic} = 100$  MPa $\sqrt{\text{m}}$ .

- (a) For von Mises yield stress, yielding occurs when,

$$\sigma_v = \sigma_y \quad \text{for} \quad \sigma_v = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}} \quad (1)$$

where  $\sigma_1, \sigma_2, \sigma_3$  are principal stresses.

By using the values of  $\sigma_{zz}, \sigma_{\theta\theta}$ , and  $\sigma_{rr} = 0$  (exterior surface of the vessel, producing largest  $\sigma_v$ ), obtain  $p_p$  the maximum allowable  $p$  from plastic yielding perspective.

- (b) What direction of the crack between axial to circumferential direction experiences the highest stress intensity factor?
- (c) Plot the maximum permissible pressure  $p_c$  versus crack depth  $a_c$  considering both plastic yielding and fracture. Employ LEFM model for fracture analysis.
- (d) What is the crack length  $a_{\text{tran}}$  corresponding to the transition between plastic and fracture failure mechanisms?
- (e) Calculate the maximum permissible crack length  $a_c$  for an operating pressure  $p = 12$  MPa.
- (f) Calculate the failure pressure  $p_c$  for a minimum detectable crack depth  $a = 1$  cm.
- (g) Calculate the failure pressure  $p_c$  for a minimum detectable crack depth  $a = 1$  mm.
2. **(60 Points)**. For the notch problem shown in (1) we obtain the power of singularity for strain and strain ( $\lambda_1 - 1 = -\frac{1}{2}$ ) from the equation  $\sin(2\pi\lambda_n) = 0 \Rightarrow \lambda_n = \frac{n}{2}, n > 1$ . Using the equation,

$$\sin(2\lambda\alpha) + \lambda\sin(2\alpha) = 0 \quad \text{mode I} \quad (2a)$$

$$\sin(2\lambda\alpha) - \lambda\sin(2\alpha) = 0 \quad \text{mode II} \quad (2b)$$

obtain the power of singularity of stress and strain ( $\lambda - 1$ ) for modes I and II. To ensure that internal energy is finite around the crack tip  $\lambda - 1 \geq -\frac{1}{2}$  ( $UdA = \sigma\epsilon r dr d\theta$  bounded for  $r \rightarrow 0$ ). Also, for the singular response  $\lambda - 1 < 0$ . So the acceptable range for the first term  $\lambda$  is  $\frac{1}{2} \leq \lambda < 1$  for a singular response. For more information refer to the course presentation pages 135-138.

- Find the stress and strain singularity power of mode I and II for  $90^\circ$  notch ( $\alpha = \frac{3}{4}\pi$ . You need to obtain the  $\lambda_1$  the minimum root of equations (2) for  $\lambda \in [\frac{1}{2}, 1)$ ).
  - Noting that  $\sigma = K_I r^{\lambda^I - 1} + K_{II} r^{\lambda^{II} - 1} + \dots$  discuss which mode will dominate the stress field near the crack tip. How is this compared to sharp crack,  $\alpha = \pi$ , where  $\lambda^I = \lambda^{II} = \frac{1}{2}$ .
  - **For your interest, no need to submit.** Plot radius of singularity (when applicable) for modes I and II for  $\alpha = \pi/2$  to  $\pi$ .
3. **(150 Points)**. Figure2 shows a point force displacement system with crack length A, force P, and beam width and height B and 2H, respectively. The moment at the end of the crack due to the

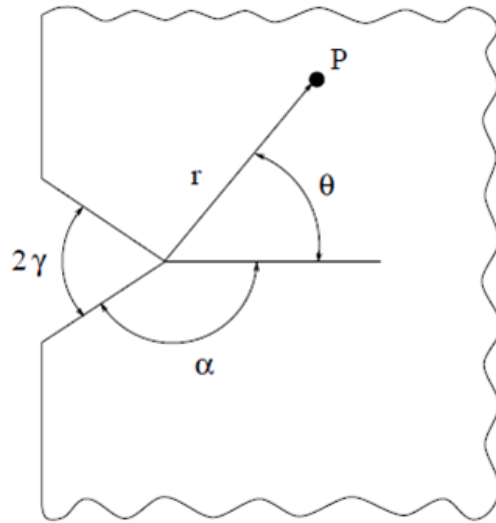


Figure 1: Schematic of notch geometry

force is  $M = PA$ . To distinguish  $A$  from area of the crack surface we use  $\mathcal{A} = AB$  for the latter. We employ the following nondimensional parameters to facilitate the analysis of this problem,

$$a = \frac{A}{H} \quad \text{normalized crack length} \quad (3a)$$

$$p = \frac{P}{\sigma_y BH} \quad \text{normalized force} \quad (3b)$$

$$m = pa = \frac{PA}{\sigma_y BH^2} \quad \text{normalized moment} \quad (3c)$$

$$\delta = \frac{\Delta E}{H \sigma_y} \quad \text{normalized displacement (crack opening)} \quad (3d)$$

where  $\sigma_y$  is the yield stress.

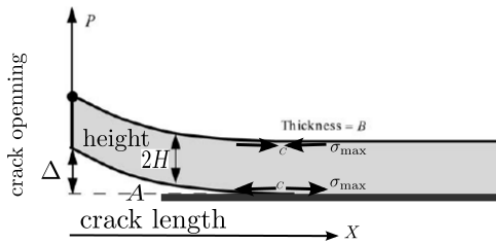


Figure 2: Force displacement relation for a point force system.

The purpose of this problem is plastic fracture mechanics analysis of this crack and comparison with LEFM. We adopt an elastic-perfectly plastic material behavior. From linear analysis we know that the maximum moment  $M$  that this beam can withstand without plastic deformation is when  $\sigma_{\max}$  at points  $C$  in the figure reach  $\sigma_y$ . If  $M$  further increases (through increasing  $P$  or crack length  $A$ ) we will have plastic yielding in points  $C$  and the plastic region further penetrates inside the domain, until  $M$  at crack tip eventually reaches maximum possible moment that the section

can withstand. The limit for initiation of plastic deformation and maximum value moments are,

$$M_{\text{lmax}} = \frac{I}{\sigma_y} z_{\text{max}} = \frac{2}{3} BH^2 \sigma_y \quad \text{maximum moment for linear response} \quad (4a)$$

$$M_{\text{max}} = BH^2 \sigma_y \quad (\text{all interface is yielded}) \quad \text{maximum moment of the interface} \quad (4b)$$

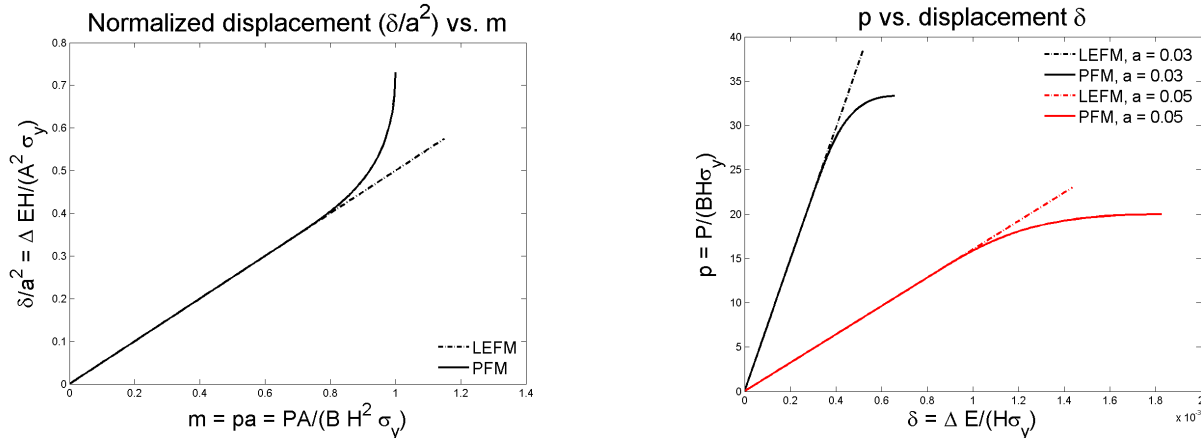
To determine the deflection  $\Delta$  at the tip of the crack we employ relations between  $M(x)$  and  $\frac{d^2 \Delta(x)}{dX^2}$  as follows:

$$\frac{d^2 \Delta(x)}{dX^2} = \begin{cases} \frac{M(x)}{EI} & M(x) < M_{\text{lmax}} \\ \frac{\sigma_y}{HE} \frac{1}{\sqrt{3} \sqrt{1 - M(x)/M_{\text{max}}}} & M_{\text{lmax}} < M(x) < M_{\text{max}} \end{cases} \quad (5)$$

Note that  $\Delta(x)$ ,  $M(x)$  denote displacement and moment values along the beam while undecorated  $\Delta$  and  $M$  denote their maximum values at the two end points of the crack. By locating the initiation position of plastic deformation in the beam and integrating (5) we obtain,

$$\delta = \frac{\Delta}{H} \frac{E}{\sigma_y} = a^2 f(m), \quad f(m) = \begin{cases} \frac{1}{2} m & m < \frac{2}{3} \\ \frac{20}{27m^2} - \frac{2}{3\sqrt{3}m^2} \sqrt{1 - m(2 + m)} & \frac{2}{3} < m < 1 \end{cases} \quad (6)$$

Equation (6) implies that when the applied moment  $m = pa$  is small ( $< \frac{2}{3}$  corresponding to  $M_{\text{lmax}}$ ) the linear response holds between load and displacement. However, as  $m$  increases either through increasing load  $P$  or crack length  $A$ , the  $P - \Delta$  relation is no longer linear.



(a) Relation between normalized displacement and moment, equation (6).

(b)  $P - \Delta$  plots for sample crack lengths  $A = aH$  based on (6).

Figure 3: Linear and nonlinear  $P - \Delta$  relations for the crack problem in figure 2. The dash line LEFM curves show that for small “loading” ( $m$ ,  $P$ ), the actual  $P - \Delta$  relation is linear.

- (a) **Energy release rate  $J$ :** To characterize plastic fracture response of this crack, we need to evaluate energy release rate  $J = G$ . Since equation (6) is the  $P - \Delta$  relation (in normalized form), we should be able to evaluate internal (strain) energy  $U(\Delta, A) = \int_0^\Delta P(\bar{\Delta}) d\bar{\Delta}|_{\text{fixed } A}$  or complimentary internal energy  $U^*(P, A) = \int_0^P \Delta(\bar{P}) d\bar{P}|_{\text{fixed } A}$ . Subsequently, using one of the following equations  $J = G = -\frac{1}{B} \frac{dU(\Delta, A)}{dA}|_{\text{fixed } \Delta}$  or  $J = G = \frac{1}{B} \frac{dU^*(P, A)}{dA}|_{\text{fixed } P}$  we can evaluate  $J$ . Note that  $J$  is taken as the energy release rate per unit area of crack advance  $A = AB$  rather than crack length  $A$ .

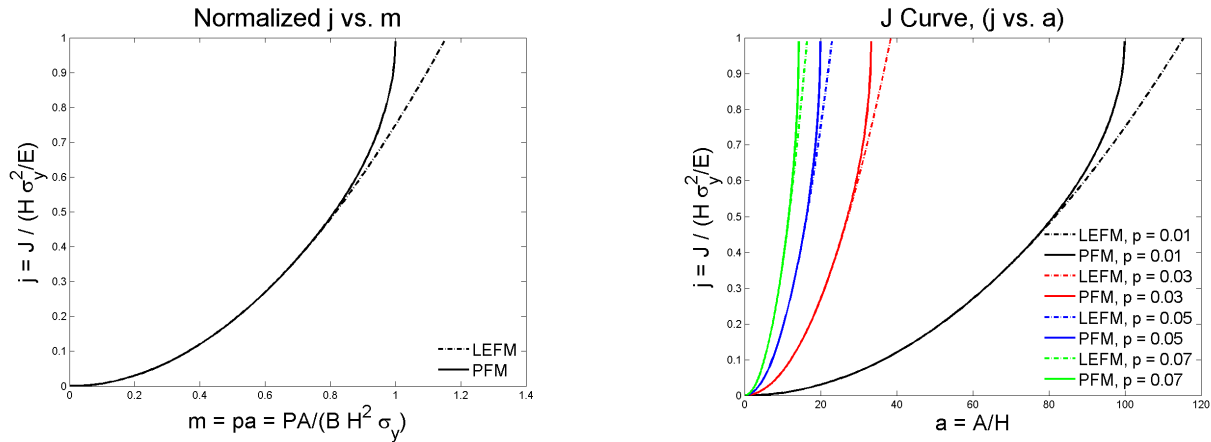
Choosing the appropriate form of  $J$  in terms of  $U$  or  $U^*$  for this problem show that,

$$J(m) = \frac{\sigma_y^2 H}{E} \left\{ \int_0^m f(\bar{m}) d\bar{m} + mf(m) \right\} \tag{7}$$

(b) **LEFM vs. PFM, Small Scale Yielding (SSY):** After evaluating (7) we can show (no need to prove (7) yields (8)) that normalized energy release rate  $j$  is equal to,

$$j = \frac{J}{\frac{\sigma_y^2 H}{E}} = \begin{cases} \frac{3}{4}m^2 & m < \frac{2}{3} \\ 1 - \frac{2}{\sqrt{3}}\sqrt{1-m} & \frac{2}{3} < m < 1 \end{cases} \tag{8}$$

This  $J - m$  relation and its realization as J curve for specific load samples  $p$  are shown in (3b).



(a) Energy release rate as a function of normalized moment  $m = pa$ , (8)

(b) J curve ( $J$  vs.  $A$ ) plots for sample applied loads  $P = p\sigma_y HB$  based on (8).

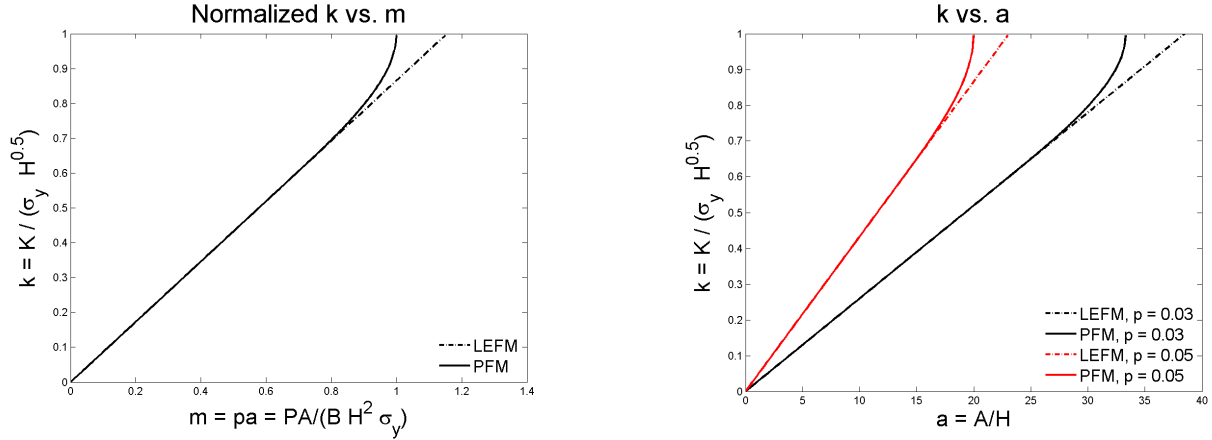
Figure 4: Energy release rate  $J$  as a function of normalized moment  $m = pm$  and its realization for specific load values  $p$ . The LEFM solution does not take material yielding into account.

- i. What is the limiting  $m$  value,  $m_{\text{tran}}$ , below which LEFM solution can be used? For the geometry shown in 2, what is the transition load  $P_{\text{tran}}(A)$  for a given crack length  $A$  for which LEFM solution can be employed?
- ii. Briefly (less than 2-4 sentences) explain why for  $j > j_{\text{tran}}$  ( $P > P_{\text{tran}}(A)$ ) plastic solution has a larger energy release rate?
- iii. Since for LEFM  $K = \sqrt{GE}$  (plane stress), the “effective” normalized  $K$  for this problem is,

$$k = \frac{K}{\sigma_y \sqrt{H}} = \begin{cases} \frac{\sqrt{3}}{2}m & m < \frac{2}{3} \\ \sqrt{1 - \frac{2}{\sqrt{3}}\sqrt{1-m}} & \frac{2}{3} < m < 1 \end{cases} \tag{9}$$

Consider a loading  $P_1 = 0.05\sigma_y BH$ ,  $A = 10H$ . What is the stress intensity factor  $K_1$  corresponding to this load? What is the stress intensity factor  $K_2$  for  $P_2 = 2P_1$  and same  $A$ ? What is the relation between  $2K_1$  and  $K_2$ ? Using figure 5(a) explain why the superposition principle (*e.g.*,  $K$  of  $2P$  is  $2K$ ) does not hold here.

- iv. Principle of SSY states that LEFM solutions are acceptable when *fracture process zone size* is much smaller than *other relevant length scales of the problem* such as domain

(a) Normalized stress intensity factor  $k = \frac{K}{\sigma_y \sqrt{H}}$ , (9)(b) Stress intensity factor plots for sample applied loads  $P = p\sigma_y HB$  based on (9).Figure 5: “Effective” stress intensity factor computed from  $J$ .

dimensions and radius of singularity. Show that plastic zone size  $r_p \propto \left(\frac{K}{\sigma_y}\right)^2$  as the representative of fracture process zone size satisfies  $r_p \propto jH$ . By comparing  $r_p$  with relevant length scales of the problem show that SSY condition is directly related to the condition  $j < j_{\text{tran}}$  we obtained for the validity of LEFM solution.

- (c) **Critical load  $P_{cr}$  and displacement  $\Delta_{cr}$**  correspond to load and displacement values that the crack can start propagating for a given fracture resistance  $J_c$ . To determine  $P_{cr}$ , as done before, in R plot (e.g., 4(b)) we find the smallest load (for load control) or displacement (displacement control) value whose J curve intersect R curve for the initial crack length  $A_0$ . If  $R(A)$  is constant  $R(A) = J_c$ , for linear regime ( $j_c = \frac{J_c}{\frac{\sigma_y^2 H}{E}} < \frac{1}{3}$ , cf. (8)), we obtain,

$$J = J_c \Rightarrow j = \frac{3}{4}m^2 = j_c \left(m < \frac{2}{3} \text{ linear branch of } j\right) \Rightarrow m_{cr} = p_{cr}a_0 = \frac{2}{\sqrt{3}}\sqrt{j_c} \Rightarrow$$

$$P_{cr} = BH\sigma_y p_{cr} = BH\sigma_y \frac{m_{cr}}{a_0} = BH\sigma_y \frac{2}{\sqrt{3}}\sqrt{j_c} \frac{H}{A_0} = \frac{2}{\sqrt{3}}\sqrt{j_c} \frac{BH^2}{A_0} \sigma_y \quad (10)$$

Note that  $P_{cr}$  depends on the initial crack length  $A_0$ . Similarly, by plugging  $m_{cr}$  in (6) we obtain  $\Delta_{cr}$ , the critical displacement for crack propagation initiation.  $\Delta_{cr}$  is either directly applied in displacement control loading or is the displacement corresponding to  $P_{cr}$  for load control setting. These values are summaries as follows,

$$P_{cr} = \frac{2}{\sqrt{3}}\sqrt{j_c} \frac{BH^2}{A_0} \sigma_y \quad (11a)$$

$$\Delta_{cr} = \sqrt{\frac{j_c}{3}} \frac{\sigma_y A_0^2}{E H} \quad (11b)$$

for  $m < \frac{2}{3}$  ( $j_c < \frac{1}{3}$ ).

- Evaluate  $P_{cr}$ ,  $\Delta_{cr}$  for the nonlinear range  $1 > m > \frac{2}{3}$  ( $1 > j_c > \frac{1}{3}$ ) in terms of  $j_c$  and  $m_{cr}$ .
- Combining the solution from (11a) and your solution for  $1 > m > \frac{2}{3}$ , plot  $P_{cr}$  in the form  $p_{cr}a_0 = P \frac{A_0}{BH^2\sigma_y}$  versus  $j_c = \frac{J_c}{H\sigma_y^2/E}$  for the entire range  $j_c = 0$  to 1. In addition to  $P_{cr}$  from PFM, add the  $P_{cr}$  that you would have obtained from LEFM analysis for the entire  $j_c \in [0, 1]$  using (11a).

- iii. For what ranges of  $j_c$ ,  $P_{cr}$  from LEFM and PFM analysis are different and in that range is PFM  $P_{cr}$  smaller or larger than that of LEFM analysis. Explain (less than 2-3 sentences) why  $P_{cr}$  of PFM is smaller or larger than that of LEFM.
- iv. Similarly, plot  $\Delta_{cr}$  in the form  $\frac{\delta_{cr}}{a_0} = \Delta \frac{HE}{A_0^2 \sigma_y}$  versus  $j_c = \frac{J_c}{H \sigma_y^2 / E}$  for  $j_c \in [0, 1]$  for both PFM and LEFM solutions using (11b) and your solution.
- v. Compare  $\Delta_{cr}$  from LEFM and PFM and comment on in which range they are different and briefly explain the cause of difference. You can refer to figure 6 for the explanation of your results.

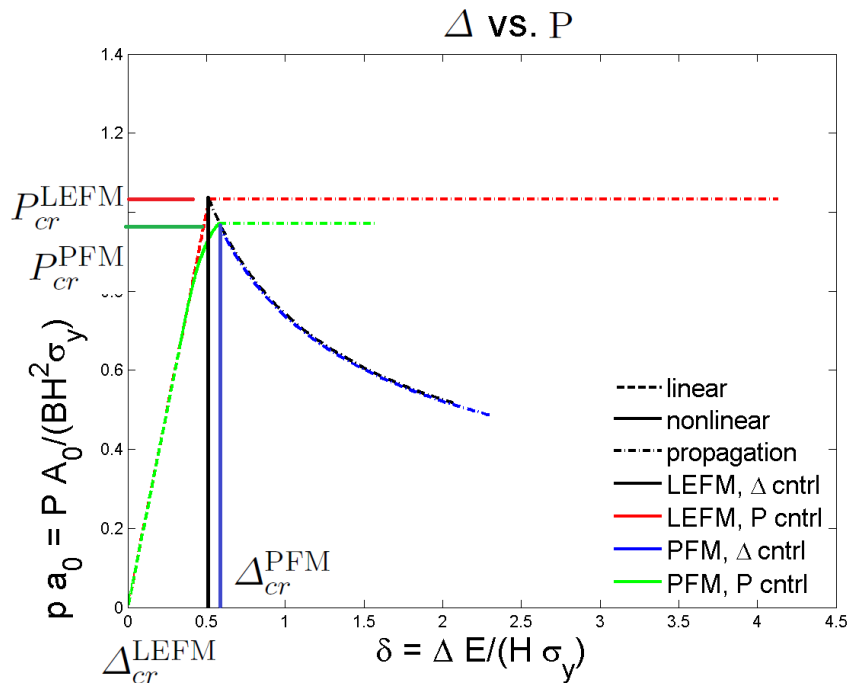


Figure 6:  $P_{cr}$  and  $\Delta_{cr}$  from LEFM and PFM analysis of the crack with initial length  $A_0$  for  $j_c = 0.8$ .