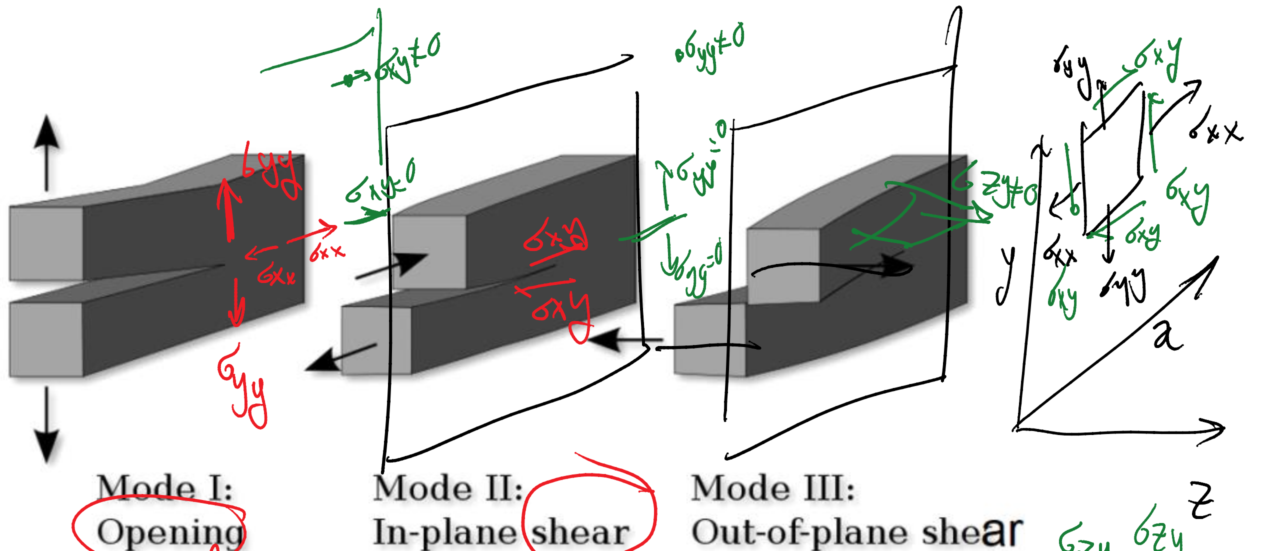
2016/09/15

Thursday, September 15, 2016
8:34 AM

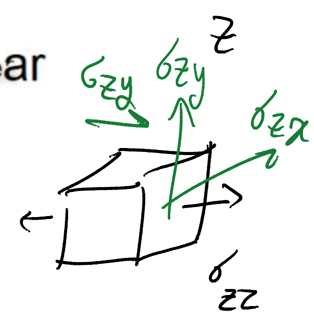
4.2. Stress solutions, Stress Intensity Factor K (SIF)



normal stress dominant

in plane fracture modes

mode III



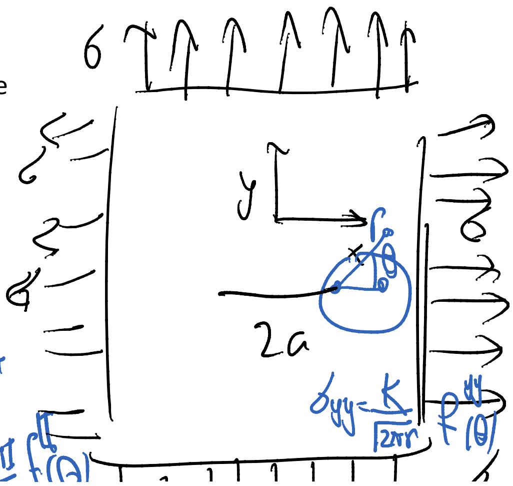
out of plane stresses

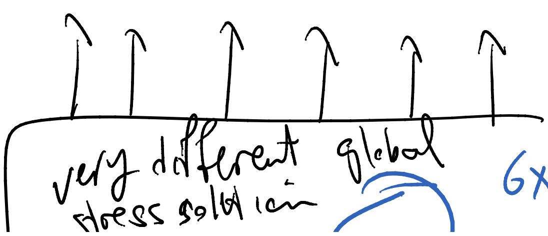
$\sigma_{zz}, \sigma_{zx}, \sigma_{zy} = 0$

More difficult to find exact solution

Plan for solving mode I and mode II solutions:

- Cover the elasticity theory needed for the solution of these problems
- Find exact solution to mid-crack case in an infinite domain
- Find the asymptotic crack tip solution fields which hold for any general loading

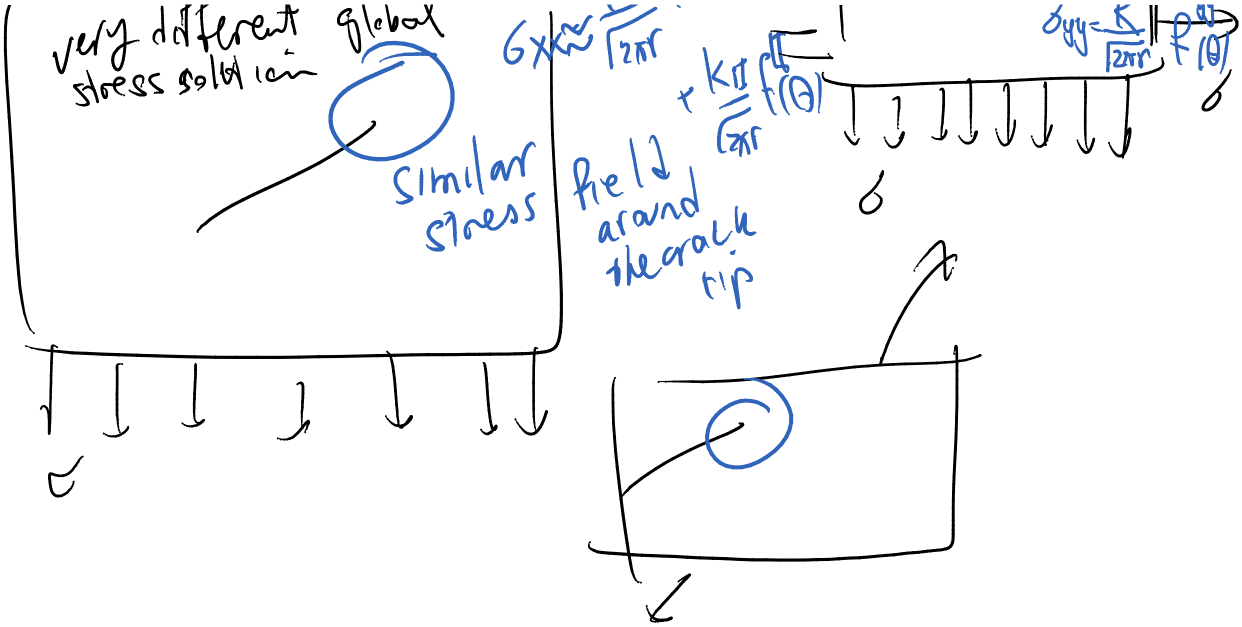




very different global stress solution

$\sigma_{xx} \approx \frac{K_I}{\sqrt{2\pi r}} f^{I}(\theta) + K_{II} f^{II}(\theta)$

$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}} f^{yy}(\theta)$



Part I: elasticity theory for the solution of this problem

Elastodynamics Boundary value problem

Kinematics

displacement u

velocity v $= \dot{u}$

acceleration a $= \ddot{u}$

strain ε $= \frac{1}{2}(\nabla u + \nabla u^T)$ $\quad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \Rightarrow \begin{cases} \epsilon_{11} = u_{1,1} \\ \epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \epsilon_{22} = u_{2,2} \end{cases}$

Kinetics

(force_like fields)

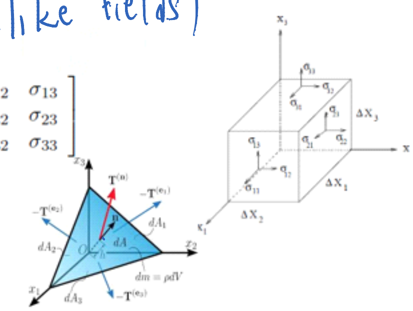
stress $\sigma = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

$\sigma = \sigma^T$

traction $\mathbf{T} = \sigma n$

linear momentum p

body force b



Elastostatics Boundary value problem

Constitutive equation (Relate force-like fields to kinematic fields)

Hook's law Isotropic 3D $\boxed{\sigma_{ij} = D_{ijkl}\varepsilon_{kl}}$ $\quad i,j,k,l = 1,2,3$

4th order Elasticity tensor

Isotropic $T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$ or $\mathbf{T} = \lambda\mathbf{I}_E + 2\mu\mathbf{E}$

2D (plane strain) Isotropic
$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & 0\\ \nu & (1-\nu) & 0\\ \nu & \nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}$$

2D (plane stress) isotropic
$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix} = \frac{1}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}$$

mass density

$\boxed{p = \rho\mathbf{v}}$ velocity

1D : $\sigma = E\varepsilon$ Hook's law

linear momentum density

$p = \rho\mathbf{v}$

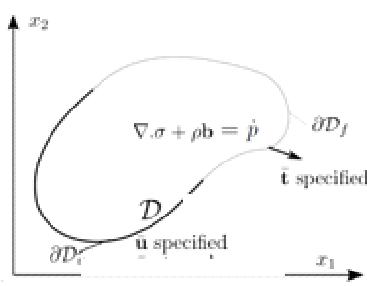
Balance of linear momentum

$\boxed{\nabla.\sigma + \rho\mathbf{b} = \dot{p}}$ or $\boxed{\sigma_{ij,j} + \rho b_j = \rho\ddot{u}_i}$

indicial notation expansion

summation on j

- Static: $\dot{\mathbf{p}} = 0$
- No body force $\mathbf{b} = 0$



108

Finally we close the system by using the balance law

For the moment we only consider static case where the crack propagation can be considered to be in quasi-static mode

$$\nabla.\sigma + \rho b = \dot{p} = \frac{\partial p}{\partial t}$$

zero under static loading

$$\frac{\partial(\cdot)}{\partial t} = 0$$

2nd simplification

- 2nd simplification:

set $b=0$ (body force, e.g. the effect from the weight of a material)

Under these two simplifications we are going to solve:

$$\nabla \cdot \sigma = \vec{0} \iff \sigma_{ij,j} = 0$$

i: free index

j: repetition on j

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \end{cases}$$

Displacement approach

$\sigma_{ij,j} = 0$ PDE is for σ

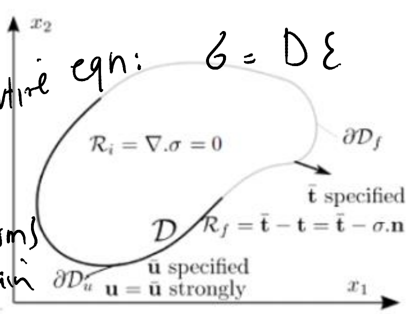
$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right)$$

constitutive eqn: $\sigma = D\varepsilon$

$$\boxed{\frac{E}{1+\nu}\left(\varepsilon_{ij,j} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk,j}\right) = 0}$$

PDE in terms of strain

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ strain in terms of displacement



$(\lambda+\mu)\nabla\nabla.\mathbf{u} + \mu\nabla.\nabla\mathbf{u} = 0$ or $(\lambda+\mu)u_{j,ji} + \mu u_{i,jj} = 0$

2 derivatives

BC's

PDE + BC for $\mathbf{u} \Rightarrow \epsilon \Rightarrow \sigma$

BCs

PDE + BC for $\mathbf{u} \Rightarrow \epsilon \Rightarrow \sigma$

2 derivatives

step 1

solving for $\vec{u}$ from PDE+BC

$$(\lambda+\mu)\nabla\nabla.\mathbf{u}+\mu\nabla.\nabla\mathbf{u}=0 \quad \text{or} \quad (\lambda+\mu)u_{j,ji}+\mu u_{i,jj}=0$$

2nd order der PDE for

Vector field displacement $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

+ BCs $\Longrightarrow$ Solve u

step 2

solving for strain

$$\epsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$$

step 3

solving for stress field

$$\sigma_{ij}=D_{ijkl}\,\epsilon_{kl}$$

from constitutive equation we obtain stress

Summary

2nd order PDE for displacement $u \Rightarrow \vec{u}$

+ BC

$\Downarrow$ $\quad\quad\downarrow$

$$\Downarrow \qquad\qquad \downarrow$$

use $\epsilon = \frac{1}{2}(\nabla u + \nabla u^T) \Rightarrow \quad \varepsilon$

Use constitutive equation

$\sigma = D\varepsilon \qquad \sigma$

Second approach: we use it to solve the stress field for modes I and II crack (tip) stress fields

The idea is having a potential function

Stress function approach

What are Airy stress function approach?

Use of stress function ⇨

Balance of linear momentum is automatically satisfied (no body force, static)

$$\boxed{\underbrace{\psi(x_1, x_2)}_{\text{scalar function of } \vec{x}} \quad \to \quad \sigma_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk} \quad \to \quad \sigma_{ij,j} = 0}$$

We obtain stress by differentiation of Airy stress function

$$\sigma_{ij} = -\psi_{,ij} + \delta_{ij}\,\psi_{,kk}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

2D case $\quad i = 1, 2$

$\qquad\quad j = 1, 2$

repeated index means the expression

$j = 1, 2$

$i=1, j=1 \qquad \sigma_{11} = -\psi_{,11} + \overset{1}{\delta_{11}} \underbrace{(\psi_{,kk})}_{\{\psi_{,11} + \psi_{,22}\}}$

repeated index means the expression is summed over k

$\sigma_{11} = -\psi_{,11} + \psi_{,11} + \psi_{,22} \Rightarrow \boxed{\sigma_{11} = +\psi_{,22}}$

$i=2, j=2 \qquad \sigma_{22} = -\psi_{,22} + \overset{1}{\delta_{22}}(\psi_{,11} + \psi_{,22}) \Rightarrow \boxed{\sigma_{22} = \psi_{,11}}$

$i=1, j=2 \qquad \sigma_{12} = -\psi_{,12} + \underset{0}{\delta_{12}}(\psi_{,11} + \psi_{,22}) \Rightarrow \boxed{\sigma_{12} = -\psi_{,12}}$

$i=2, j=1 \qquad \sigma_{21} = -\psi_{,21} = -\psi_{,12} = \sigma_{12}$ expected this

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \quad \text{is symmetric}$$

$$\left.\begin{aligned} \sigma_{11} &= \psi_{,22} \\ \sigma_{22} &= \psi_{,11} \\ \sigma_{21} = \sigma_{12} &= -\psi_{,12} \end{aligned}\right\}$$

let's check if the PDE on σ is satisfied?

$$\nabla \cdot \sigma = 0 \Leftrightarrow \sigma_{ij,j} = 0 \qquad \begin{cases} \sigma_{11,1} + \sigma_{12,2} = 0 \Rightarrow (\psi_{,22})_{,1} + (-\psi_{,12})_{,2} = \psi_{,221} - \psi_{,122} = 0 \\ \sigma_{21,1} + \sigma_{22,2} = 0 \quad (-\psi_{,12})_{,1} + (\psi_{,11})_{,2} = -\psi_{,121} + \psi_{,112} = 0 \end{cases}$$

This is too good to be true!
We will end up having to satisfy another PDE

Why?

Example:

$$\psi = x_1^4$$

$$\boxed{\begin{aligned} \sigma_{11} &= \psi_{,22} = 0 \\ \sigma_{12} &= -\psi_{,12} = 0 \\ \sigma_{22} &= \psi_{,11} = 12x_1^2 \end{aligned}}$$

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} = 0 \\ \sigma_{21,1} + \sigma_{22,2} = 0 \end{cases} \checkmark$$

Assume $\nu = 0$ Poisson ratio

$$\begin{cases} \epsilon_{11} = \dfrac{\sigma_{11}}{E} = 0 \\ \epsilon_{12} = \dfrac{\sigma_{12}}{G} = 0 \\ \epsilon_{22} = \dfrac{\sigma_{22}}{E} = \dfrac{12}{E} x_1^2 \end{cases}$$

$$\epsilon_{11} = u_{1,1} = 0 \quad (1.1)$$

$$\epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = 0 \quad (1.2)$$

$$\epsilon_{22} = u_{2,2} = \frac{12x_1^2}{E} \quad (1.3)$$

(1.1) $\epsilon_{11} = u_{1,1} = 0 \Rightarrow \boxed{u_1(x_1, x_2) = f(x_2)}$ (2.1)

$$\left\{ f'(x) = 0 \rightarrow f(x) = \underset{\text{constant}}{C} \right)$$

$$\epsilon_{22} = u_{2,2} = \frac{12x_1^2}{E} \Rightarrow u_{2,2}(x_1, x_2) = \frac{12x_1^2}{E} \Rightarrow$$

$$\boxed{u_2(x_1, x_2) = \frac{12 x_1^2 x_2}{E} + g(x_1)}$$

(2.2)

plug (2.1) & ($u_{2,2}$) in (1.2)

$$\epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = 0$$

$$\Rightarrow \quad u_{1,2} + u_{2,1} = \{f(x_2)\}_{,2} + \left\{\frac{12 x_1^2 x_2}{E} + g(x_1)\right\}_{,1} = 0$$

$$\Rightarrow \quad \underbrace{f'(x_2)}_{\text{only a function of } x_2} + \underbrace{g'(x_1)}_{\text{only a function of } x_1} + \underbrace{\frac{24 x_1 x_2}{E}}_{\text{function of both } x_1 \,\&\, x_2} = 0$$

this equation cannot be satisfied !

The problem is that we cannot integrate arbitrary strain fields to get displacement field

in 2D

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \underbrace{\epsilon_{21}}_{=\epsilon} & \epsilon_{22} \end{bmatrix}$$

how many independent strain fields we have?

$$\begin{bmatrix} \epsilon_{21} & \epsilon_{22} \end{bmatrix}$$
$\epsilon_{21} = \epsilon_{12}$

we have?

③ $\Rightarrow$ 3 equation

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

same question on u

② 2 unknowns

$$\underline{\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)}$$

3 eqns
$$\begin{cases} \epsilon_{11} = u_{1,1} \\ \epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \epsilon_{22} = u_{2,2} \end{cases}$$

2 unknowns u_1 & u_2

$3 - 2 = 1$

we need one constraint put on ϵ so that we can integrate ϵ to get $\underline{u}$

$$\left.\begin{aligned} \epsilon_{11} &= u_{1,1} \\ \epsilon_{12} &= \tfrac{1}{2}(u_{1,2} + u_{2,1}) \\ \epsilon_{22} &= u_{2,2} \end{aligned}\right\} \begin{aligned} &,2 \\ &,1 \\ &,1 \end{aligned} \quad \begin{aligned} \epsilon_{11,2} &= u_{1,12} \\ \epsilon_{12,1} &= \tfrac{1}{2}(u_{1,21} + u_{2,11}) \\ \epsilon_{22,1} &= u_{2,21} \end{aligned} \quad \begin{aligned} &,2 \\ &,2 \\ &,1 \end{aligned}$$

$$\begin{aligned} \epsilon_{11,22} &= u_{1,122} \\ \epsilon_{12,12} &= \tfrac{1}{2}(u_{1,212} + u_{2,112}) \\ \epsilon_{22,11} &= u_{2,211} \end{aligned} \Bigg\} \Longrightarrow$$

same

same

$$
\left.
\begin{aligned}
\epsilon_{11,22} &= u_{1,122} \\
\epsilon_{12,12} &= \tfrac{1}{2}\left(u_{1,212} + u_{2,112}\right) \\
\epsilon_{22,11} &= u_{2,211}
\end{aligned}
\right\} \Longrightarrow
$$

same

same

$$
\boxed{\epsilon_{11,22} + \epsilon_{22,11} - 2\,\epsilon_{12,12} = 0}
$$

Strain compatibility equation

If strains satisfy strain compatibility equation, then we can in fact integrate strains to obtain displacement field

ψ :

$$
\begin{cases}
\sigma_{11} = \psi_{,22} \\
\sigma_{22} = \psi_{,11} \\
\sigma_{12} = -\psi_{,12}
\end{cases}
$$

$$\sigma \downarrow \epsilon$$

$$\sigma = D\epsilon \Longrightarrow \epsilon = D^{-1}\sigma$$

D: stiffness 4th order tensor

D^{-1}: compliance 4th order tensor

strain MUST

strain **MUST** satisfy strain compatibility equation

$$\boxed{\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0}$$

if so

we can integrate

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \begin{cases} \epsilon_{11} = u_{1,1} \\ \epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \epsilon_{22} = u_{2,2} \end{cases}$$

to obtain $\vec{u} = \begin{cases} u_1 \\ u_2 \end{cases}$ $\qquad \vec{u}$

$$\left.\begin{aligned} \sigma_{ij} &= -\psi_{,ij} + \delta_{ij}\psi_{,kk} \\ \epsilon_{ij} &= (D^{-1})_{ijkl}\,\sigma_{kl} \\ \text{in 2D} \quad \epsilon_{11,22} &+ \epsilon_{22,11} - 2\epsilon_{12,12} = 0 \end{aligned}\right\} \Rightarrow$$

Stress function approach

$$\left.\begin{aligned} \varepsilon_{ij} &= \frac{1+\nu}{E}\{-\psi_{,ij} + (1-\nu)\delta_{ij}\psi_{,kk}\} \\ 2\varepsilon_{12,12} &- \varepsilon_{11,22} - \varepsilon_{22,11} = 0 \end{aligned}\right\}$$

$$2\psi_{,1122} + \psi_{,2222} + \psi_{,1111} = 0 \quad \rightarrow$$

4th order eqn on ψ

$$(\psi_{,11} + \psi_{,22})_{,11} + (\psi_{,11} + \psi_{,22})_{,22} = 0$$

$$\underbrace{(\psi_{,11}+\psi_{,22})}_{}{}_{,11}+(\psi_{,11}+\psi_{,22})_{,22}=0$$

Laplacian $\nabla^2\psi = \Delta\psi = \psi_{,11} + \psi_{,22}$

$$(\Delta\psi)_{,11} + (\Delta\psi)_{,22} = 0$$

$$\boxed{\Delta\Delta\psi = 0} \qquad \boxed{\nabla^2\nabla^2\psi = 0}$$ Bi-harmonic equation

* A function that satisfies Laplacian is called Harmonic function
* A function that satisfies bi-harmonic equation is called bi-harmonic function

---

Any bi-harmonic function ψ (that is, it satisfies bi-harmonic equation)

$$\nabla^2(\nabla^2\psi) = \psi_{,1111} + 2\psi_{,1122} + \psi_{,2222} = 0$$

Can generate analytical σ, ϵ, u fields

$\epsilon = D^{-1}\sigma$

$$\boxed{\begin{aligned}\sigma_{11} &= \psi_{,22}\\ \sigma_{22} &= \psi_{,11}\\ \sigma_{12} &= -\psi_{,12}\end{aligned}} \Rightarrow \varepsilon_{ij} = \frac{1+\nu}{E}\{-\psi_{,ij} + (1-\nu)\delta_{ij}\psi_{,kk}\} \xrightarrow{\text{integrate}} \text{to}$$

get displacement

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Comparison

Dispacement approach	Stress approach
PDE	PDE
$(\lambda+\mu)\nabla\nabla.\mathbf{u} + \mu\nabla.\nabla\mathbf{u} = 0$ or $(\lambda+\mu)u_{j,ji} + \mu u_{i,jj} = 0$	$\nabla^2\nabla^2\psi = \psi_{,1111} + 2\psi_{,1212} + \psi_{,2222} = 0$
2nd order for VECTOR $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$	4th order for SCALAR ψ

$u \rightarrow \epsilon \rightarrow$

$\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)$

$\sigma \rightarrow \epsilon \rightarrow$

$\epsilon = D^{-1}\sigma$ $\quad \varepsilon_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk}$

$\sigma \qquad \sigma = D\epsilon$

$u \quad$ integrate $\quad \epsilon = \frac{1}{2}(\nabla u + \nabla u^T)$