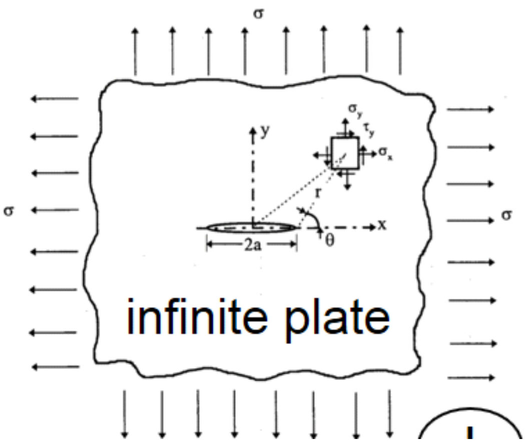
2019/09/20

Tuesday, September 20, 2016
8:41 AM

From last time we obtained

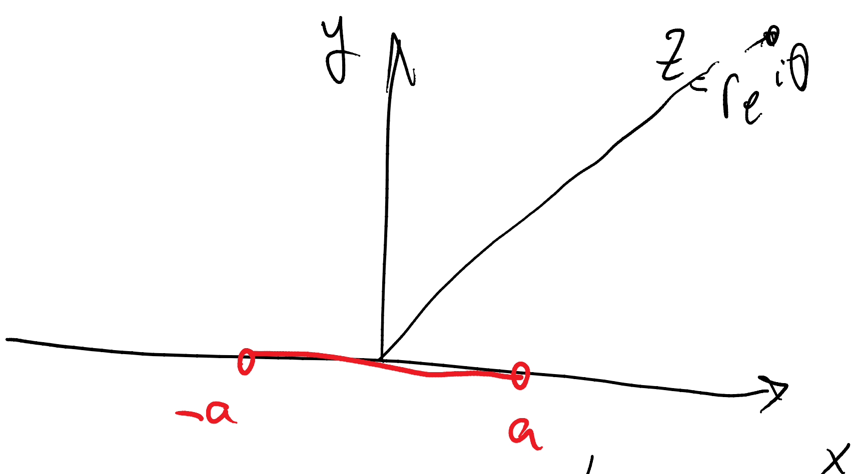


$$Z(z) = \frac{\sigma}{\sqrt{1-(a/z)^2}}$$

$$\boxed{\begin{aligned} \sigma_{xx} &= \mathrm{Re}Z - y\mathrm{Im}Z' \\ \sigma_{yy} &= \mathrm{Re}Z + y\mathrm{Im}Z' \\ \tau_{xy} &= -y\mathrm{Re}Z' \end{aligned}}$$

$$Z(z) = \frac{\sigma}{\sqrt{1-(a/z)^2}} \Rightarrow Z' = \sigma \frac{\frac{1}{2}\times 2\, a^2/z^2}{\left(\sqrt{1-a^2/z^2}\right)^3}$$

$$(1) \quad \boxed{\sigma_{yy} = \mathrm{Re}\frac{\sigma}{\sqrt{1-(a/z)^2}} + y\,\mathrm{Im}\left(\frac{\sigma a^2}{z^2\left(\sqrt{1-a^2/z^2}\right)^3}\right)}$$



The problem with equation (1) is that it only applies to this particular problem that we solved

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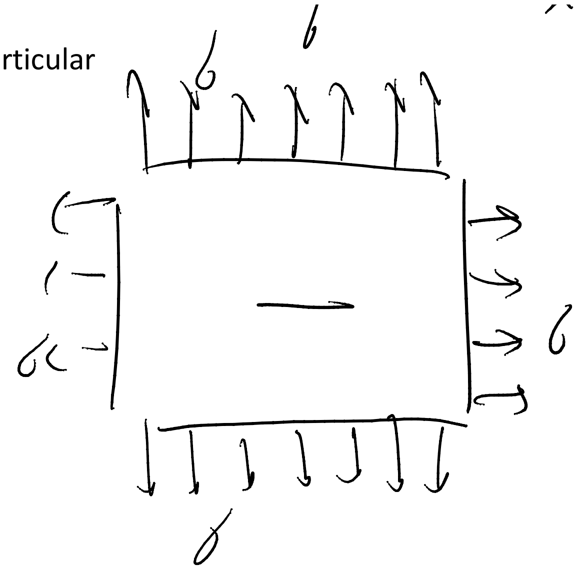
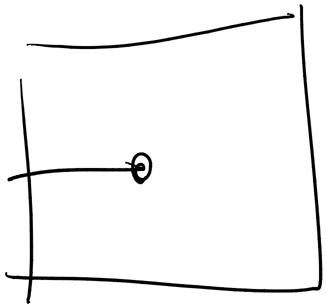
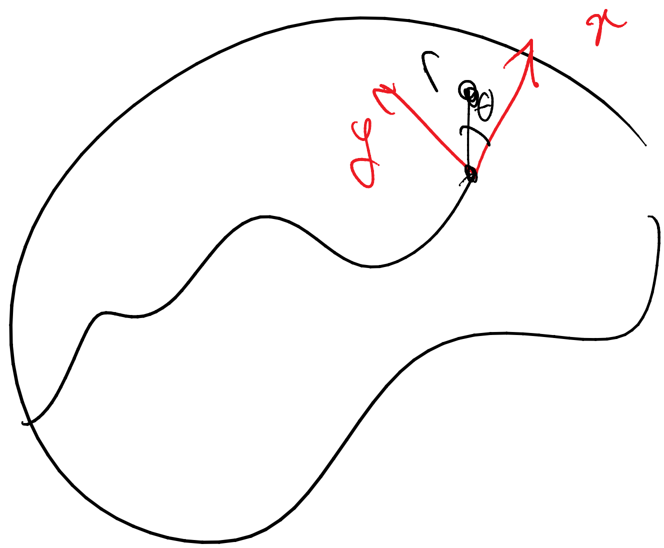


For example it does not apply to the problem below:

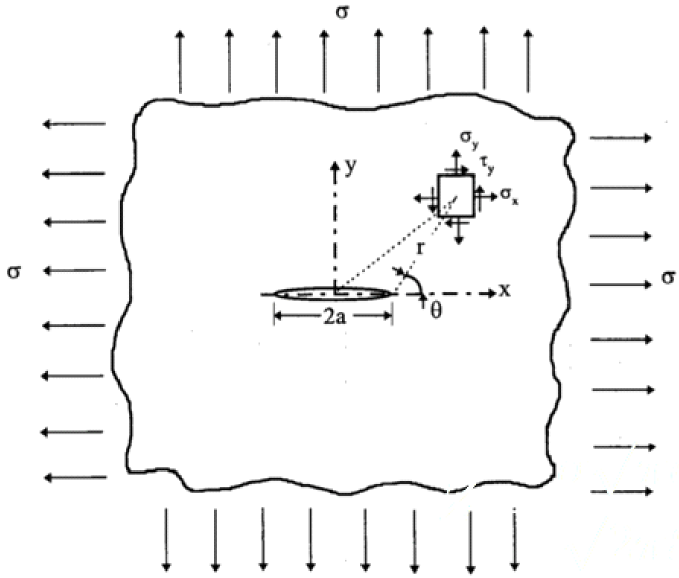


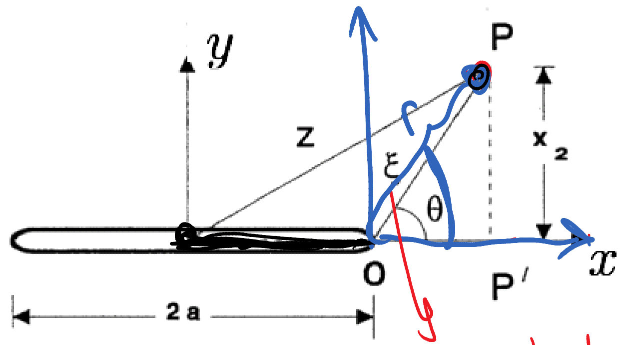
- Finite size domain
- side crack

However, we still can use equation (1) to express stress field but ONLY in the neighborhood of a crack tip for ANY DOMAIN and CRACK GEOMETRY



How do obtain asymptotic solution fields?





$Z = a + \xi$

all real

$\xi = r e^{i\theta}$

$$\sigma_{yy} = \mathrm{Re}\,\frac{\sigma}{\sqrt{1-\left(\frac{a}{z}\right)^2}} + y\,\mathrm{Im}\left(\frac{\sigma a^2}{z^2\left(\sqrt{1-\frac{a^2}{z^2}}\right)^3}\right)$$

1st term (2nd term)

we add this term later

Let's only focus on the first term

$z = a + \xi \qquad z = a + \xi$

$$\frac{\sigma}{\sqrt{1-\left(\frac{a}{z}\right)^2}} = \frac{\sigma z}{\sqrt{z^2-a^2}} = \frac{\sigma(a+\xi)}{\sqrt{(a+\xi)^2-a^2}}$$

$$= \frac{\sigma a \cdot \left(1+\frac{\xi}{a}\right)}{\sqrt{a^2+\xi^2+2\xi a-a^2}} = \frac{\sigma a}{\sqrt{\xi^2+2a\xi}}\cdot\left(1+\frac{\xi}{a}\right)$$

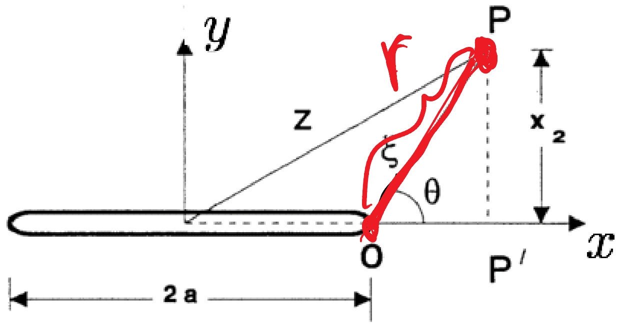
$$\frac{\sigma a}{} \cdot \frac{\left(1+\xi/a\right)}{}$$

$\sqrt{2a\xi}$ $\sqrt{1+\xi/2a}$

term from Z'

$$\sigma_{yy} = \mathrm{Re}\left\{\frac{\sigma\sqrt{a}}{\sqrt{2\xi}} \cdot \left(\frac{1+\xi/a}{\sqrt{1+\xi/2a}}\right)\right\} + \ldots$$

$$\sigma_{yy} = \mathrm{Re}\left\{\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}} \cdot \frac{1+\xi/a}{\sqrt{1+\xi/2a}}\right\} + \ldots$$



what happens to ξ/a around the crack tip

How can we discuss "closeness" of a point P to the crack tip?

The point P has distance r = $|\xi|$ to the crack tip. This has the physical scale of a length.

To say that this distance is "small" we need to compare it with another length scale provided by this problem.

- The other length scale in this problem is the crack length a $// |\xi|$

The point P is close to the crack tip IFF $\frac{r}{a} \ll 1$ $\frac{|\xi|}{a} \ll 1$

$$\sigma_{yy} \quad \mathrm{Re}\left\{\sigma\sqrt{\pi a} \cdot \frac{1+\xi/a}{\quad}\right\} + \ldots$$

$$\sigma_{yy} = Re\left\{\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}} \cdot \frac{1+\xi/a}{\sqrt{1+\xi/2a}}\right\} \quad + \text{t.n}$$

$$\frac{r}{a} = \frac{|\xi|}{a} \ll 1$$

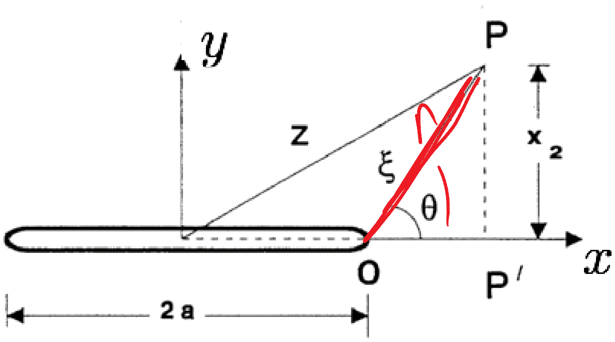
what is the limit of σ_{yy}

$$\lim_{\frac{r}{a} = \frac{|\xi|}{a} \to 0} \sigma_{yy} = \lim_{\xi/a \to 0} Re\left\{\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}} \cdot \frac{1+\xi/a}{\sqrt{1+\xi/2a}}\right\} + \text{2nd term}$$

$\to 1$

as $\xi/a \to 0$

$$\sigma_{yy} \approx Re\,\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}} \quad + \quad \text{second term}$$



$$\xi = re^{i\theta}$$

$$\sigma_{yy} = Re\,\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r e^{i\theta}}} \quad + \quad \text{second term}$$

$$= \mathrm{Re}\left(\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}\, e^{-i\frac{\theta}{2}}\right) + \quad " \quad "$$

$$= \mathrm{Re}\left(\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)\right) + \cdots$$

K_I

stress intensity factor

$$\sigma_{yy} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}\cos\frac{\theta}{2} + \lim y \,\mathrm{Im}\left(\frac{\sigma a^2}{z^2\left(\sqrt{1-\frac{a^2}{z^2}}\right)^3}\right)$$

Z'

$|\xi| \to 0$

$z = a + \xi$

$$Z(z) = \frac{K_I}{\sqrt{2\pi\xi}}, \boxed{K_I = \sigma\sqrt{\pi a}}$$

$$Z(z) = \frac{K_I}{\sqrt{2\pi r}} e^{-i\theta/2} \qquad \xi = re^{i\theta}$$

$$Z'(z) = -\frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\xi^{-3/2} = -\frac{K_I}{2r\sqrt{2\pi r}} e^{-i3\theta/2}$$

contribution from the second term

from the second term

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

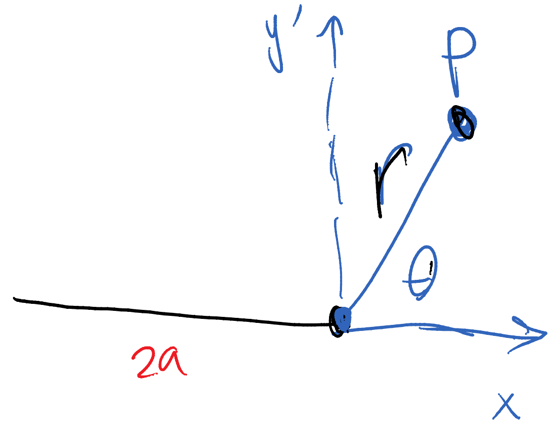
from the first term

$$\sigma_{yy} = \underbrace{\mathrm{Re}Z}_{\text{first term}} + \underbrace{y\mathrm{Im}Z'}_{\text{second term}}$$

first term $\quad \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}, \; K_I = \sigma\sqrt{\pi a}$

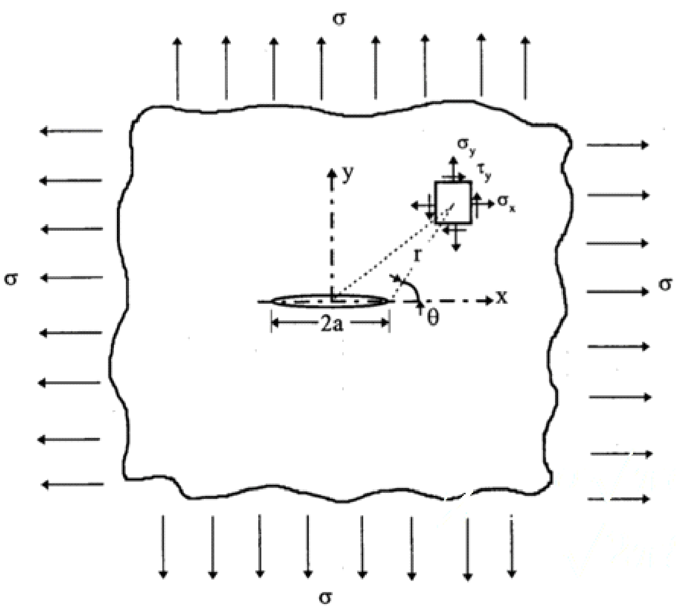
the only factor that affect stress solution only factor that affect stress solution

$$
\begin{cases}
\sigma_{xx} = \dfrac{K_I}{\sqrt{2\pi r}}\cos\left(\dfrac{\theta}{2}\right)\left[1-\sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{3\theta}{2}\right)\right] \\
\sigma_{yy} = \dfrac{K_I}{\sqrt{2\pi r}}\cos\left(\dfrac{\theta}{2}\right)\left[1+\sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{3\theta}{2}\right)\right] \\
\tau_{xy} = \dfrac{K_I}{\sqrt{2\pi r}}\cos\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{\theta}{2}\right)
\end{cases}
$$



σ's at point P with relative coordinate r, θ

$K_I = \sigma\sqrt{\pi a}$ for mid crack example



Observations:

1. All the stress components are SINGULAR with 1/2 as the power of singularity
2. What is the physical dimension of STRESS INTENSITY FACTOR K? For this particular problem

$\sigma_y \propto \dfrac{1}{\sqrt{r}} = r^{-\frac{1}{2}}$ as $r \to 0$

For this particular problem

$$K_I = \sigma\sqrt{\pi a}$$

physical dimension:

$$\boxed{[K_I] = [\sigma]\sqrt{L}}$$

as $r \to 0$

$\sigma_y \to \infty$

$$[\sigma] = \frac{[F]}{L^2}$$

force

area

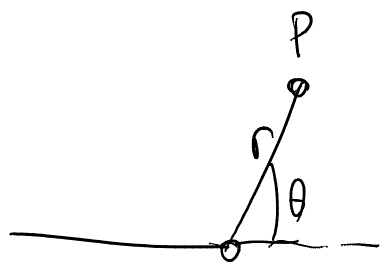
Physical dimension of K is always stress times square root of length

always applied load expressed in stress form

crack length

3. Why K is called stress intensity factor?

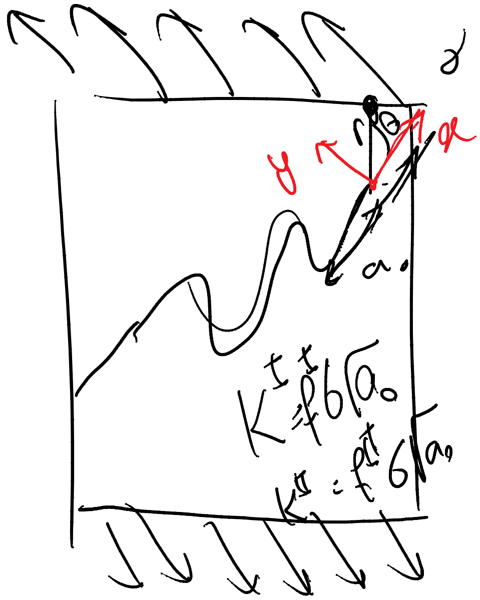
$$\begin{cases}
\sigma_{xx} = \dfrac{K_I}{\sqrt{2\pi r}}\cos\left(\dfrac{\theta}{2}\right)\left[1-\sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{3\theta}{2}\right)\right] \\
\sigma_{yy} = \dfrac{K_I}{\sqrt{2\pi r}}\cos\left(\dfrac{\theta}{2}\right)\left[1+\sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{3\theta}{2}\right)\right] \\
\tau_{xy} = \dfrac{K_I}{\sqrt{2\pi r}}\cos\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{\theta}{2}\right)
\end{cases}$$



in these expressions for stress solution the only "unknown" factor is K. Once we have

K all stress components are exactly characterized

4. How do obtain K for other problems and why these LOCAL (asymptotic) stress solutions apply to other geometries?



$$\boxed{\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f^I_{ij}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f^{II}_{ij}(\theta)}$$

K_I: stress intensity factor for mode I

K_{II}: " " " " mode II

ij $\quad i=1, j=1 \quad \sigma_{11} \ (\sigma_{xx})$

$\quad i=1, j=2 \quad \sigma_{12} \ (\sigma_{xy})$

$\quad i=2, j=2 \quad \sigma_{22} \ (\sigma_{yy})$

$f^I_{ij}(\theta)$: function depending only on angle θ for stress component σ_{ij} & mode I for stress component σ_{ij} & mode I

$f_{ij}^{II}(\theta)$: function depending only on angle θ for stress component σ_{ij} & mode II

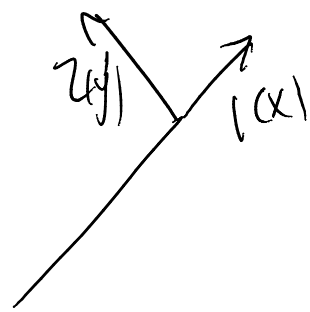
Example:

What are the expressions for f_{ij}^{I} ?

$$\sigma_{11} = \sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]}_{f_{11}^{I}(\theta)}$$

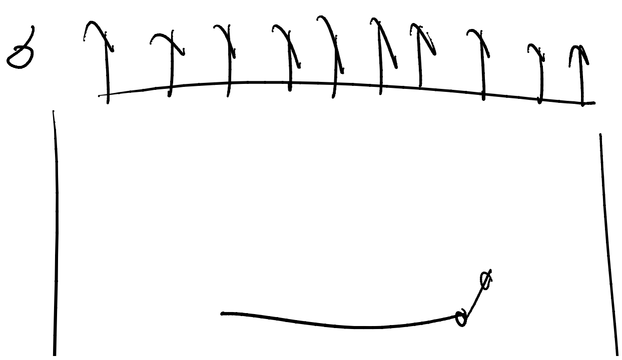
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]}_{f_{22}^{I}(\theta)}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}_{f_{12}^{I}(\theta)}$$



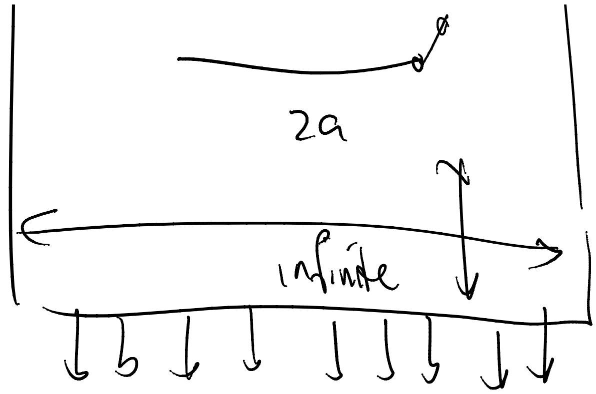
So basically for any complex domain and crack geometry we can always write local stress fields only as functions of two unknowns (in-plane problems) KI and KII

Example:

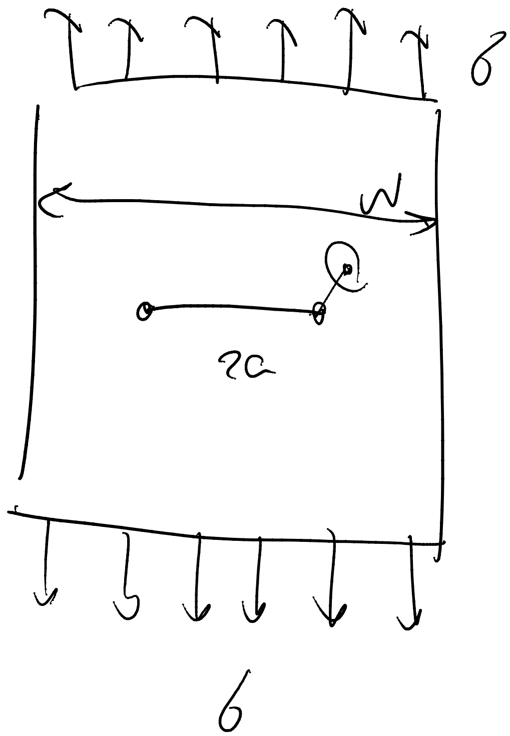


$K = \sigma\sqrt{\pi a}$

$$K_I = \sigma\sqrt{\pi a}$$



$$K_I = \underbrace{f(\text{geometry})}_{\text{correction factor}}\,\sigma\sqrt{\pi a}$$

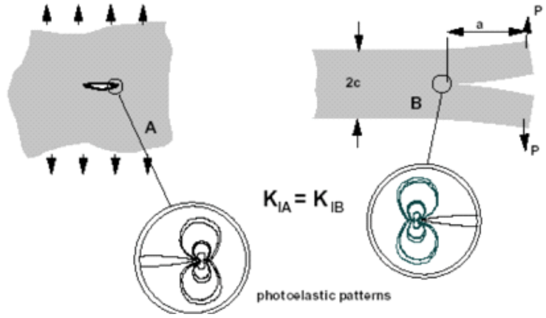


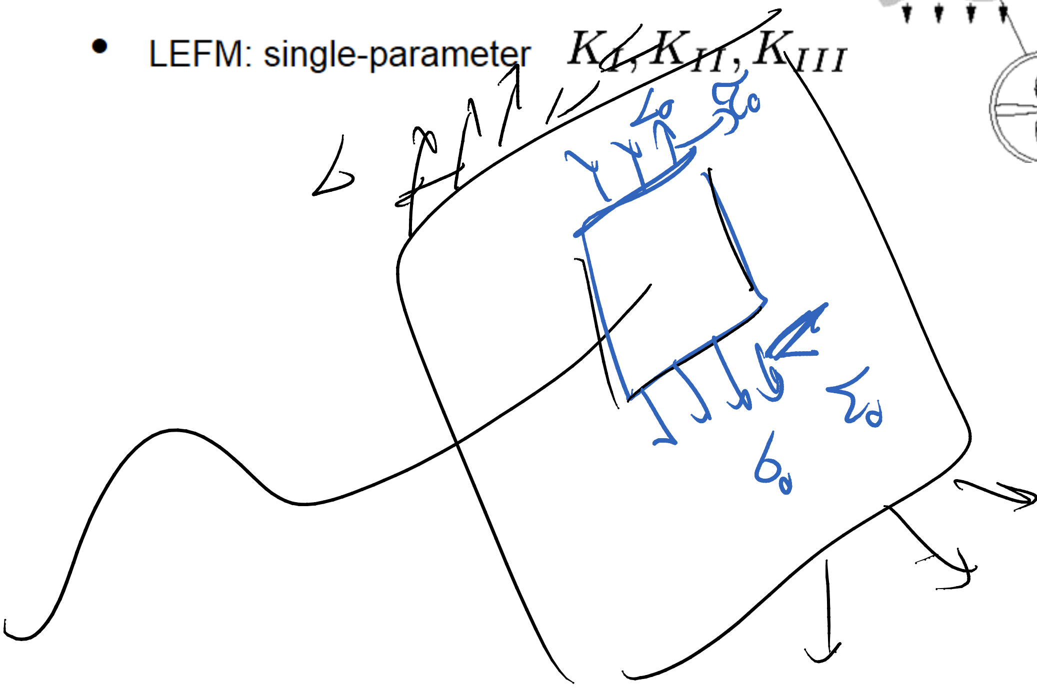
$$f = \frac{1}{\cos\left(\frac{\pi a}{W}\right)}$$ if $2a \to W$

$$f \to \frac{1}{\cos\frac{\pi a}{2a}} = \frac{1}{\cos\frac{\pi}{2}} \to \infty$$

- Stresses-K: linearly proportional
- K uniquely defines the crack tip stress field
- modes I, II and III:
- LEFM: single-parameter K_I, K_{II}, K_{III}

SIMILITUDE





$\sigma_0 = f_{11}\,\sigma + f_{12}\,\tau$

Σ_0

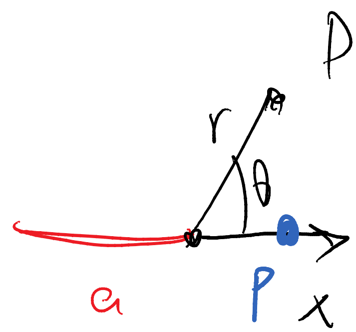
6. What is the stress solution along the crack direction?

$\theta = 0$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$



$\theta = 0$ $\quad \sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}$

$\sigma_{yy} = \frac{K_I}{}$

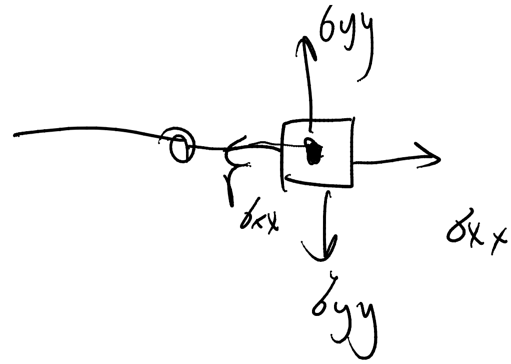
σ_{yy}

along the crack direction

$\theta = 0$ $\quad \sqrt{2\pi r}$

$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$

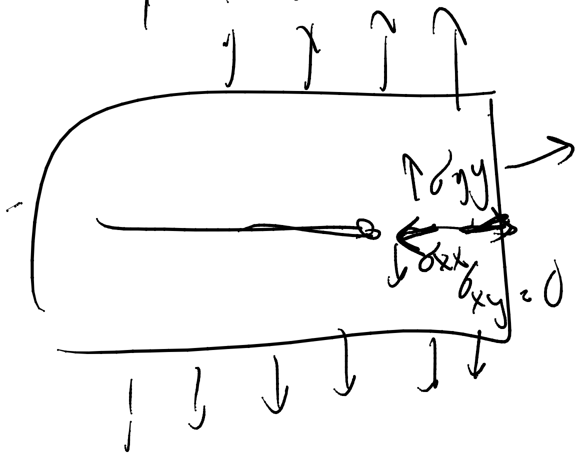
$\sigma_{xy} = 0$



direction

for mode I problem

$\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$

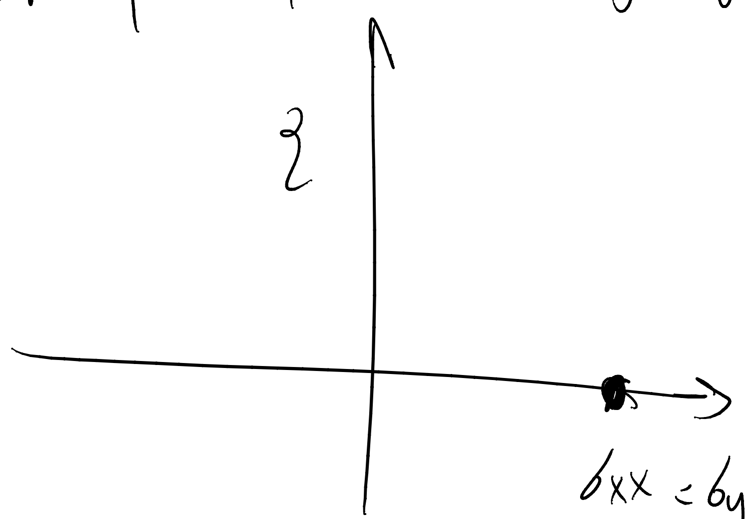


very high

$\sigma_{xx} = ?$

no idea whether σ_{xx} was going to be high or at least as high as σ_{yy}

Mohr circle for point P with ($\theta = 0$)



$\sigma_{xx} = \sigma_{yy} = \frac{K}{\sqrt{2\pi r}}$

for $\theta = 0$

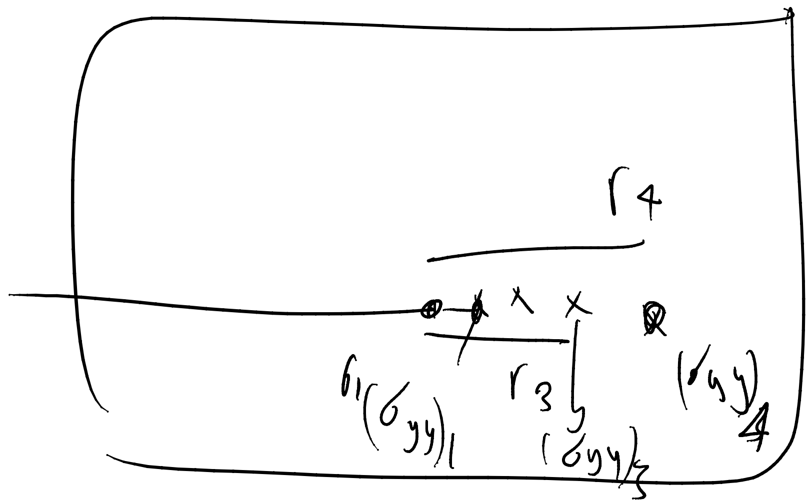
$12\pi r$

for $\theta = 0$

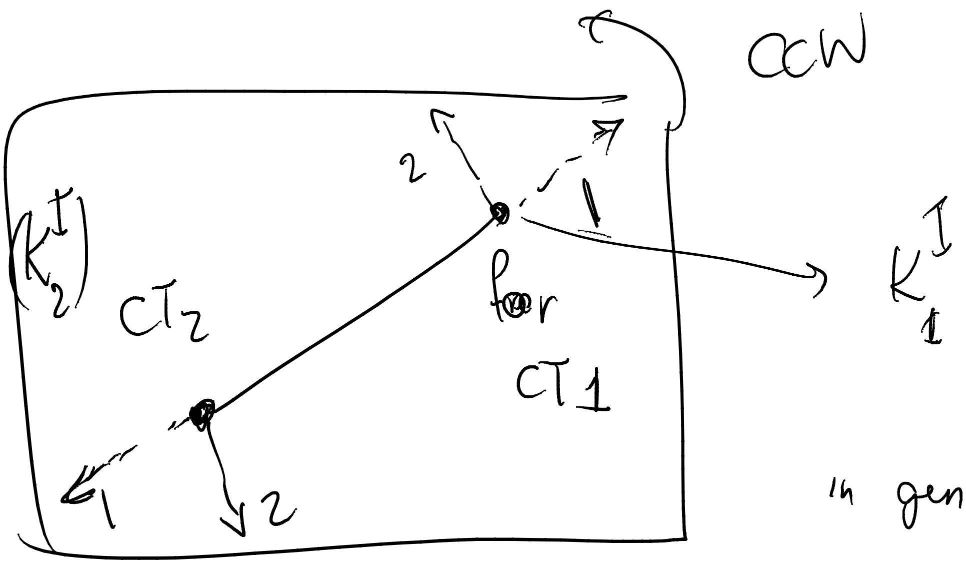
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \rightarrow$$

$$\boxed{\sigma_{yy}\sqrt{2\pi r} = K_I \quad \text{as } r \rightarrow 0}$$

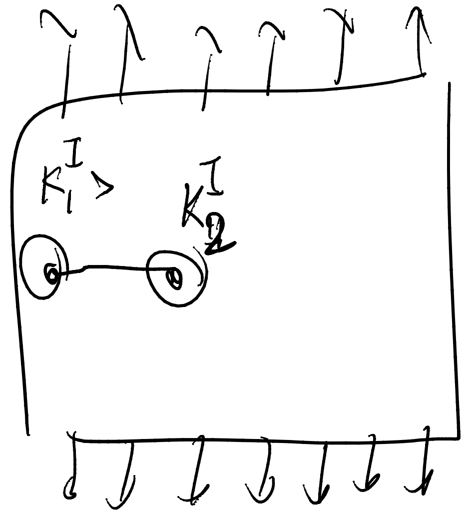
practical means to evaluate K_I!



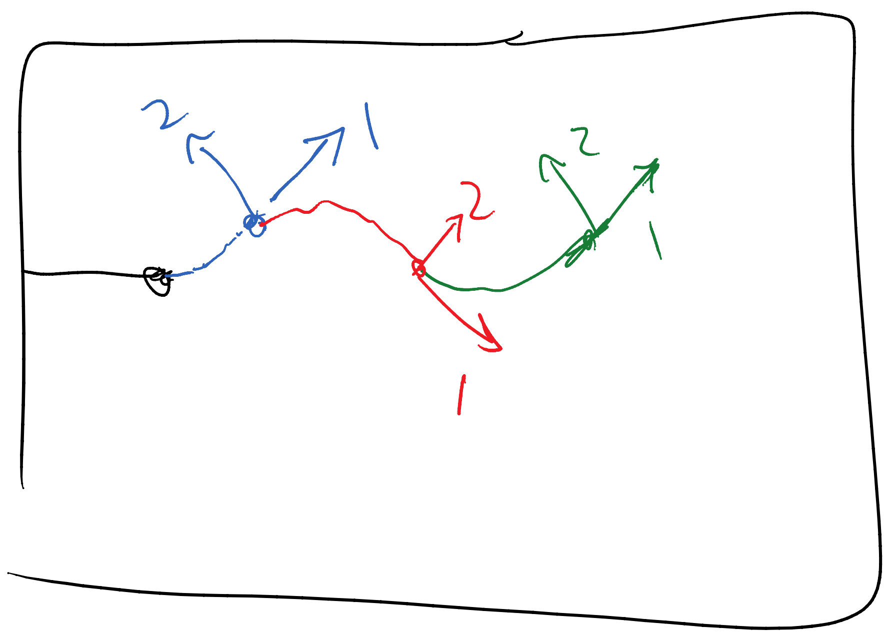
$\sqrt{2\pi r_i}\,(\sigma_{yy})_i \rightarrow K_I$ as $r_i \rightarrow 0$



in general $K_1^I \neq K_2^{II}$



Winding crack



How do we get solutions for mode II?

Mode II problem

Boundary conditions

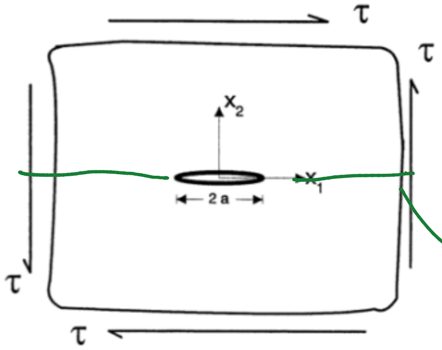
$$(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau$$
$$|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0$$

Stress function

only difference

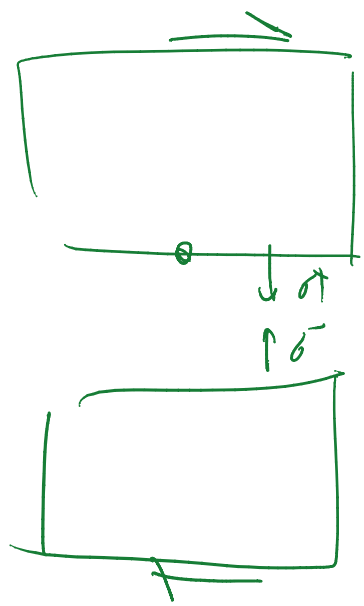
$$\boxed{Z = -\frac{i\tau z}{\sqrt{z^2 - a^2}}}$$

very similar to mode I problem



Check BCs

anti_symmetry boundary



$\sigma^+_{yy} = \sigma^-_{yy}$ from action reaction

$\sigma^+_{yy} = -\sigma^-_{yy}$ from anti-symmetry

$$\boxed{\Rightarrow \sigma_{yy} = 0}$$

$$\boxed{\begin{aligned}\sigma_{xx} &= \mathrm{Re}Z - y\mathrm{Im}Z' \\ \sigma_{yy} &= \mathrm{Re}Z + y\mathrm{Im}Z' \\ \tau_{xy} &= -y\mathrm{Re}Z'\end{aligned}}$$

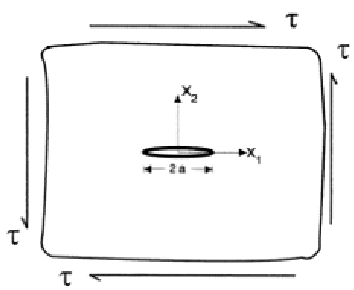
Mode II problem

Boundary conditions

$$\begin{aligned}&(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau \\ &|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0\end{aligned}$$

Stress function

$$Z = -\frac{i\tau z}{\sqrt{z^2 - a^2}}$$



$$\left.\begin{aligned}\sigma_{xx} &= -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \\ \sigma_{yy} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \\ \tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\end{aligned}\right\}$$

Solutions for mode II

mode II SIF

$$K_{II} = \tau\sqrt{\pi a}$$

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