2019/09/29

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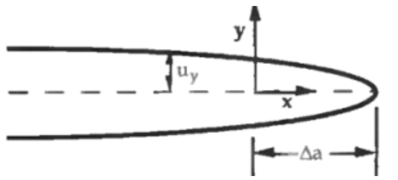
Relation between K and G

K: local behavior (tip stresses)
G: global behavior (energy)

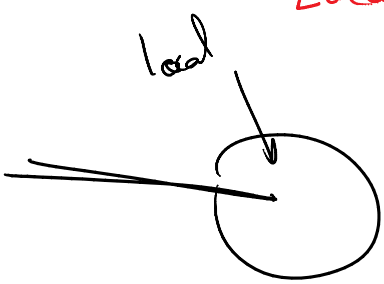


Physical dimension of K:

e.g $K = \sigma\sqrt{\pi a}$ $\qquad [K] = [\sigma]\sqrt{L}$

physical meaning of K :

Local quantity



determines the scaling of stress, displacement, & strain around a crack tip

$$\sigma^{I}_{ij} = \frac{K^{I}}{\sqrt{2\pi r}} f^{I}_{ij}(\theta)$$

G: Energy release rate:
How much energy released per unit area of the crack:

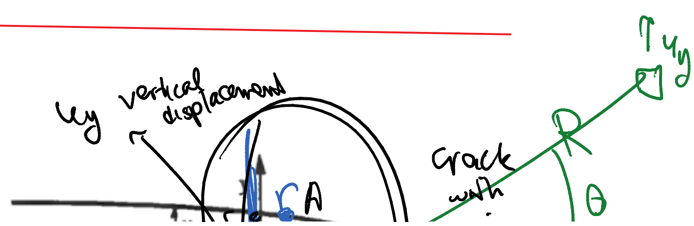
$$[G] = \frac{\text{Energy}}{L^2} = \frac{F.L}{L^2} = \left[\frac{F}{L^2}\right] L$$

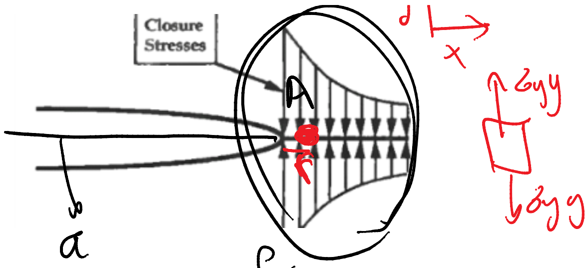
$[\sigma]\ L$

$$\boxed{\begin{aligned}[G] &= [\sigma]\ L \\ [K] &= [\sigma]\ L^{1/2}\end{aligned}}$$

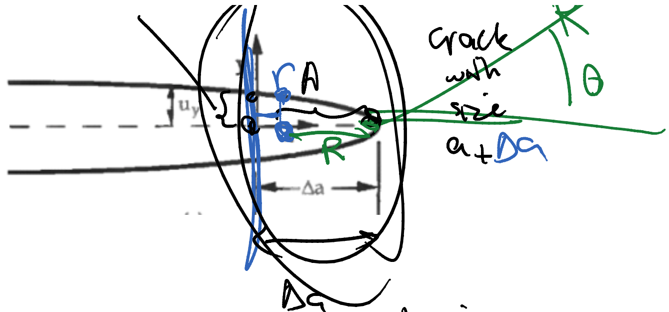
Irwin 1956 $\boxed{G \equiv -\frac{d\Pi}{dA}}$ $\longrightarrow$ G is a global quantity

We want to relate G and K





b/f for crack propagation



crack extension

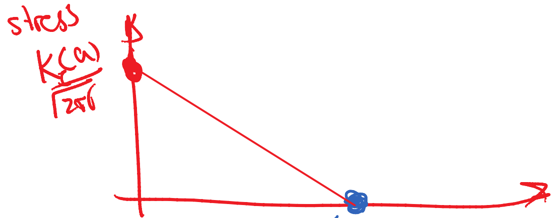
$$u_y = \frac{K^I(a+\Delta a)}{2\mu}\sqrt{\frac{R}{2\pi}}\left(1 + 2\cos^2\frac{\theta}{2}\right)$$

for point A

$\theta_A = \pi$

$R = \Delta a - r$

$$u_y = \frac{K^I(a+\Delta a)}{2\mu}\sqrt{\frac{\Delta a - r}{2\pi}}$$



$$\frac{K^I(a+\Delta a)}{2\mu}\sqrt{\frac{\Delta a - r}{2\pi}}$$

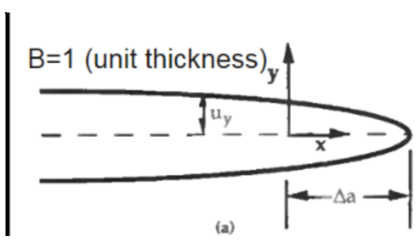
state 2

for point A (after crack has extended)

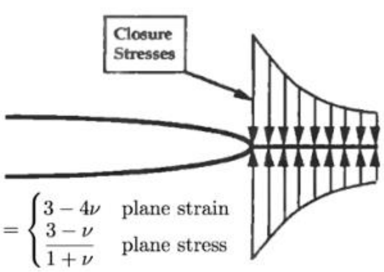
The work done at this point is the area under the triangle: 1/2 Height x base

$$\frac{1}{2} u_y(\text{state 2}) * \underbrace{Bdx\ \sigma_{yy}(\text{state 1})}_{dF_y} * 2$$

Half of displacement is going up half is going down



$$dU(x) = 2\frac{1}{2}\sigma_{yy}(x)u_y(x)dx$$



$$\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

work of crack closure $\Delta U = \int_0^{\Delta a} dU(x)$

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$$u_y = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 - 2\cos^2\frac{\theta}{2}\right),\ \theta = \pi\ \Rightarrow$$

$$\boxed{\begin{aligned} u_y &= \frac{(\kappa+1)K_I(a+\Delta a)}{2\mu}\sqrt{\frac{\Delta a - x}{2\pi}} \\ \sigma_{yy} &= \frac{K_I(a)}{\sqrt{2\pi x}} \end{aligned}}$$

$$G = \lim_{\Delta a \to 0}\frac{(\kappa+1)K_I^2}{4\pi\mu\Delta a}\int_0^{\Delta a}\sqrt{\frac{\Delta a - x}{x}}\,dx$$

In the denominator

After integration we get

$$\boxed{G = \frac{(\kappa+1)K_I^2}{8\mu}} \qquad \kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

Shear modulus $\mu = \frac{E}{2(1+\nu)}$

Scale of K: [stress] * sqrt(L) -> scale of K^2 = [stress]^2 L
Scale of mu: [stress]

What is the scale of the RHS: [stress] * L which is what we expected for G

K-G relationship (cont.)

Mode I

$$\boxed{G_I = \begin{cases} \frac{K_I^2}{E} & \text{plane stress} \\ (1-v^2)\frac{K_I^2}{E} & \text{plane strain} \end{cases}}$$

Mixed mode

$$\boxed{G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}} \qquad E' = \begin{cases} \frac{E}{1-\nu^2} & \text{for plane strain} \\ E & \text{for plane stress} \end{cases}$$

We know a crack can grow when G = Gc
At the same time for pure mode I we know

G = K^2 / E
Can we define a critical stress intensity factor?

G = Gc crack can grow
G= K^2/ E = Gc = Kc^2/ E -> Kc = sqrt(Gc E)

So if we have the value of K we can decide if the crack grows by

1) Directly comparing K with Kc (Kc = sqrt(Gc E))

2) Compute G = K^2 / E'   and then compare G with Gc ®

How does the evaluation of K come to use in practice?

K as a failure criterion

Failure criterion $\boxed{K = K_c}$ $\qquad f(a/W)\sigma\sqrt{\pi a} = K_c$

Basically we have stress, crack length a, and Kc
If we have 2 we can solve for the third one

In practice we have two available and want to find the third one

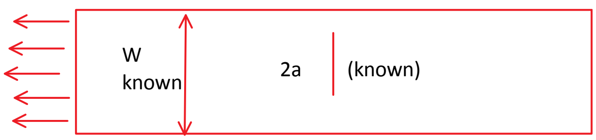
- Problem 1: given crack length a, compute the maximum allowable applied stress

$$\sigma_{\max} = \frac{K_c}{f(a/W)\sqrt{\pi a}}$$

Kc is known (material is given)
Crack length is known

Question: How much load can this specimen withstand before crack can propagate

How much load can
We apply before the
crack propagates



Material is known -> Kc

- Problem 2: for a specific applied stress, compute the maximum permissible crack length (critical crack length)

$$f(a_c/W)\sigma\sqrt{\pi a_c} = K_c \longrightarrow a_c$$

This is the case that the loading is known (sigma)
Material is known (Kc)

We are looking for a critical crack length ac

If we replace / fix a part when the crack length is smaller than ac then we are safe

Because ac appears in f and sqrt(pi ac) we may need to solve a nonlinear equation

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Problem 3: compute K_c provided crack length and stress at fracture

$$K_c = f(a_c/W)\sigma\sqrt{\pi a_c}$$

This case load and a crack length (for example the resolution of monitoring system) are known -> Looking for a material than can take such loading.

--- In the following example right now we are not going to talk about yield stress part (plasticity)
And will solve it only for steel (first row)

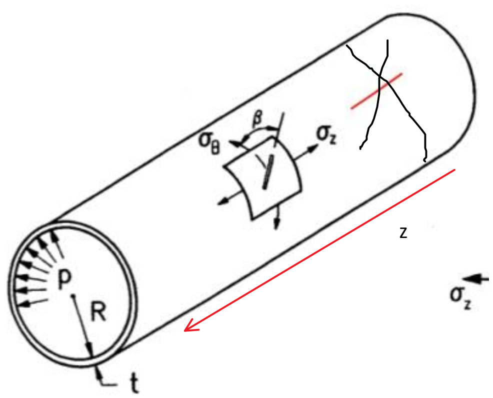
Example

A cylindrical pressure vessel with closed ends has a radius $R = 1$ m and thickness $t = 40$ mm and is subjected to internal pressure p. The vessel must be designed safely against failure by yielding (according to the von Mises yield criterion) and fracture. Three steels with the following values of yield stress σ_Y and fracture toughness K_{Ic} are available for constructing the vessel.

Steel	σ_Y(MPa)	K_{Ic}(MPa $\sqrt{m}$)
A: 4340	860	100
B: 4335	1300	70
C: 350 Maraging	1550	55

Fracture of the vessel is caused by a long axial surface crack of depth a. The vessel should be designed with a factor of safety $S = 2$ against yielding and fracture. For each steel:

R = 1 m
t = 40 mm
p = internal pressure
KIc = 100 Mpa sqrt(m)   critical K for mode I



$$K_I = \frac{pR}{2t}\sqrt{\pi a}(1+\sin^2\beta)$$
$$K_{II} = \frac{pR}{2t}\sqrt{\pi a}\sin\beta\cos\beta$$

$$\sigma_z = \frac{pR}{2t}$$
$$\sigma_\theta = \frac{pR}{t}$$

If we only restrict ourselves to mode I fracture (not worry about KII)
beta = 90 degrees (e.g. the red crack in the figure) has the most critical angle

K = pR/t   sqrt(pi a)   for a crack that this along the z direction

(a) Plot the maximum permissible pressure p_c versus crack depth a_c;

K = pR / t sqrt(pi a)

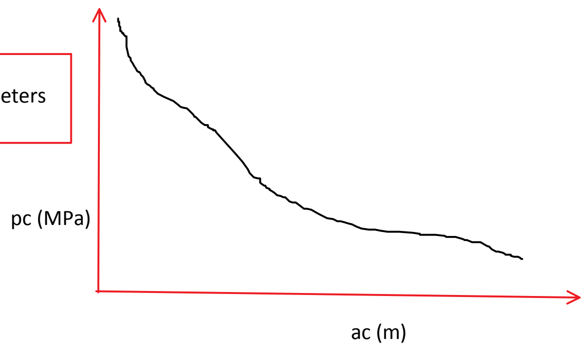
K = KIc

KIc = pc R / t sqrt(pi ac) KIc, R, t are known -> we want to obtain a relation between ac (critical crack length) and pc (critical pressure)

KIc = 100 Mpa sqrt(m)
R =1 m
t = 40 mm = 0.04 m
pi = 3.141592653589793

100 Mpa sqrt(m) = pc (1m) / (0.04 m) sqrt(3.14159 ac)

pc (Pa) = 4e6/sqrt(ac) - ac in meters



For very small crack lengths for this problem yielding will be the mechanism that causes the failure of this pressure vessel

In general we should design against all failure modes:

Fracture, ultimate stress, buckling, …

(b) Calculate the maximum permissible crack depth a_c for an operating pressure $p = 12$ MPa;

pc (Pa) = 4e6/sqrt(ac) -> pc . sqrt(ac) = 4e6 put p = 12e6 -> Sqrt(a) = 1 / 3 -> a = 1/9 = 0.1111 m = 11cm

When the crack length gets close to this value we must fix it

(c) Calculate the failure pressure p_c for a minimum detectable crack depth $a =$ 1 mm.

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P = $e6/sqrt(1e-3m) => p = 126 Mpa

This 1 mm can be the resolution of monitoring system so we use that to calculate how much load (pressure) the pressure vessel can take safely