2016/10/11

Tuesday, October 11, 2016
8:39 AM

From last time

K - dominant radius $r_s \propto \left(\frac{K}{\sigma}\right)^2$

applied stress

plastic/nonlinear zone size $r_p \propto \left(\frac{K}{\sigma_y}\right)^2$

$\Rightarrow$

$$\boxed{\left(\frac{r_s}{r_p}\right) \propto \left(\frac{\sigma}{\sigma_y}\right)^2}$$

applied load

yield stress

$\frac{r_s}{r_p} \ll 1$

$\lesssim .1$ or $.5$

From last time

Von Mises Yield Criterion

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]^{1/2} = \sigma_y$$

Plane stress

$$\sigma_1 = \sigma_2 = \sigma_{yy}, \quad \sigma_3 = 0$$

$$\sigma_e = \sigma_1 = \sigma_y \quad \Rightarrow$$

$$\sigma_{ys} = \sigma_y \quad \Rightarrow$$

$$\boxed{r_p = \frac{K_I^2}{\pi\sigma_y^2}}$$

$$\boxed{r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2}$$

Plane strain

$$\sigma_1 = \sigma_2 = \sigma_{yy}$$

$$\sigma_3 = \nu(\sigma_{xx}+\sigma_{yy}) = 2\nu\sigma_{yy}$$

$$\sigma_e = (1-2\nu)\sigma_1 = \sigma_y \quad \Rightarrow$$

$$\sigma_{ys} = \frac{\sigma_y}{1-2\nu} \quad \Rightarrow$$

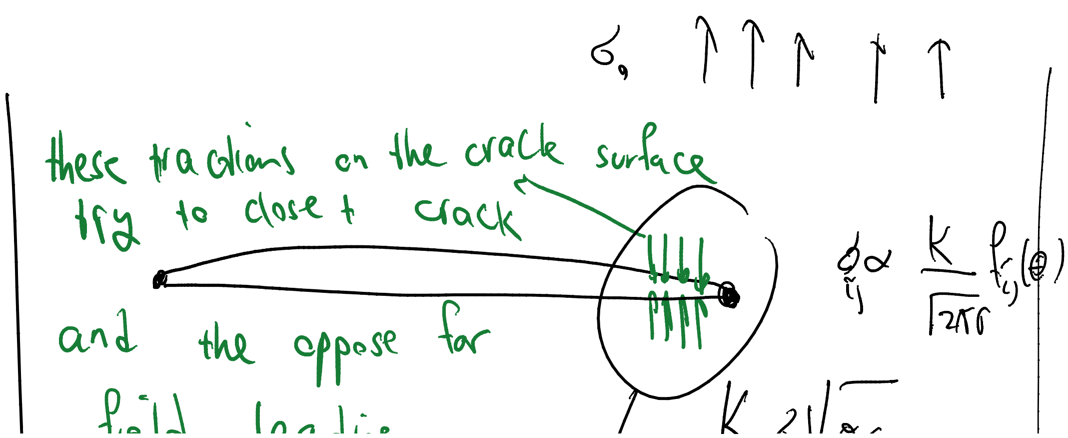
$$r_p = \frac{K_I^2}{\pi\sigma_{ys}^2} = (1-2\nu)^2\frac{K_I^2}{\pi\sigma_y^2} \quad \Rightarrow$$

$$\boxed{r_p \approx \frac{K_I^2}{3\pi\sigma_y^2}} \quad (1 - 2 \times 0.2)^2 = 0.36$$

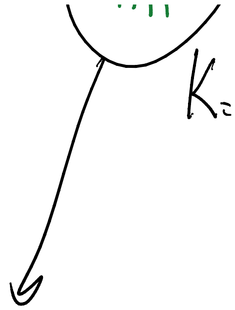
r_p is about 3 times smaller for plane strain.

More accurate estimate of r_p:

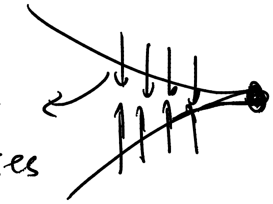
3. Strip Yield Model



and the opposit for field loading



$K = \sigma\sqrt{\pi a}$



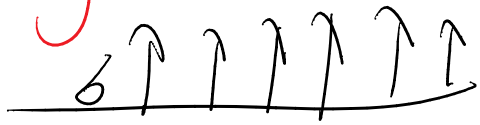
stress is finite

1st LEFM asymptotic term $\sigma_{ij} = \frac{K^{eff}}{\sqrt{2\pi r}} f_{ij}(\theta)$

from field loading & crack surface loading

so if K^{eff} (which is $K_{\text{for field load}}$ + $K_{\text{crack surface stress}}$ = 0 $\longrightarrow$

stress singularity is removed



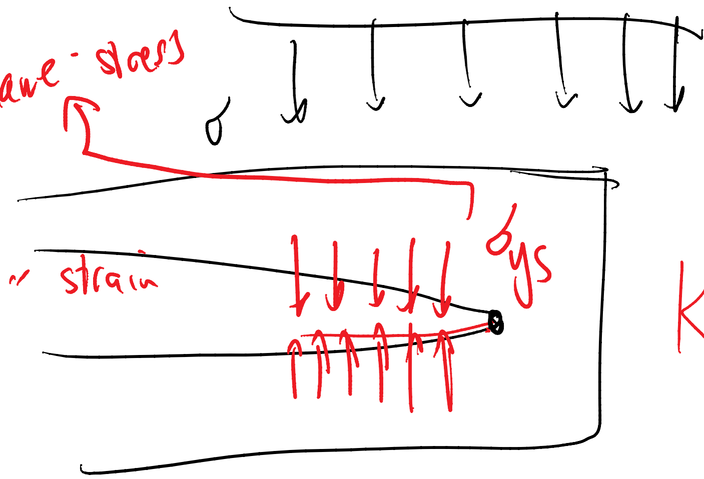
2a

$K_\sigma = +\sqrt{\pi a}\,\sigma$

$K_\sigma = +\sqrt{\pi a}\,\sigma$

yield stress

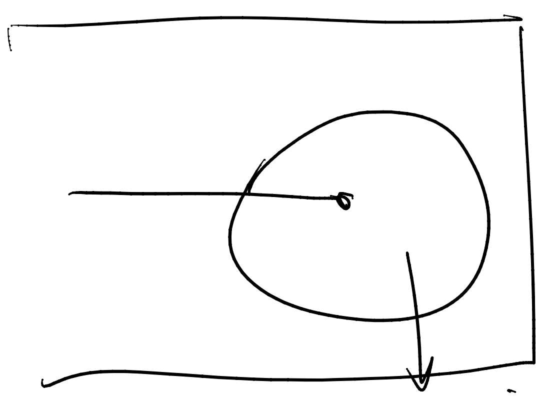
$\sigma_{ys} = \begin{cases} \sigma_y & \text{plane-stress} \\ \dfrac{\sigma_y}{1-2\nu} & \text{" strain} \end{cases}$



$K_\sigma = -\sqrt{\pi a}\,\sigma$

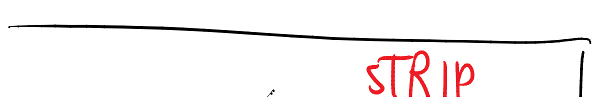
$K = K_\sigma + K_\sigma = 0$

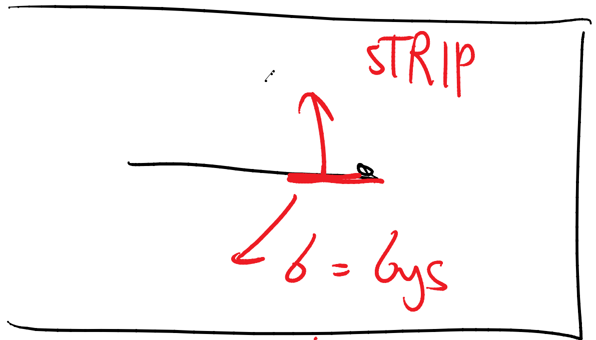
Real material



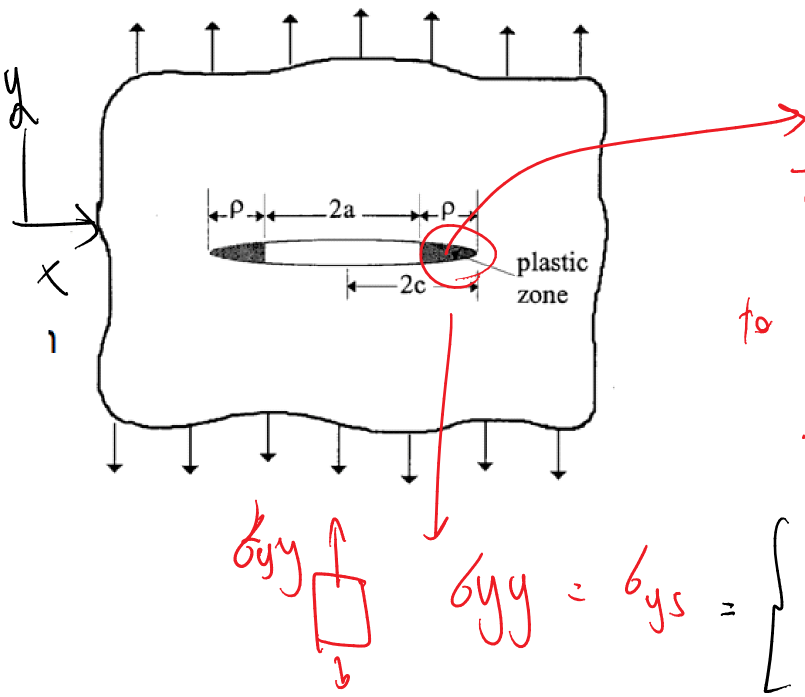
region that goes through plastic deformation

in STRIP YIELD Model we restrict plastic deformation to a STRIP behind crack tip





behind the crack tip

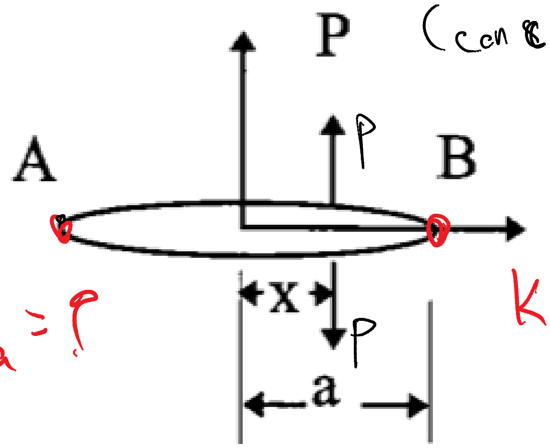


the yielding zone is restricted to strips behind the crack tip

$$\sigma_{yy} = \sigma_{ys} = \begin{cases} \sigma_y & \text{p. stress} \\ \dfrac{\sigma_y}{1-2\nu} & \text{p. strain} \end{cases}$$

Goal: Find ρ (plastic zone size) such that effective K around the crack tip is zero $\Longrightarrow$

We remove singularity around the crack tip



(concentrated force p with distance x from center of the crack

$K_a = ?$

$K_B = ?$ from load p

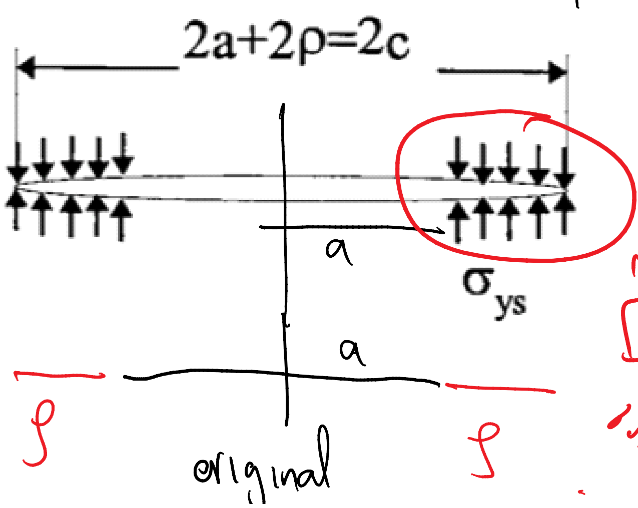
half crack length

$$K_A = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}}$$

$$K_B = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}}$$

Anderson p.64

$c = a + \rho$



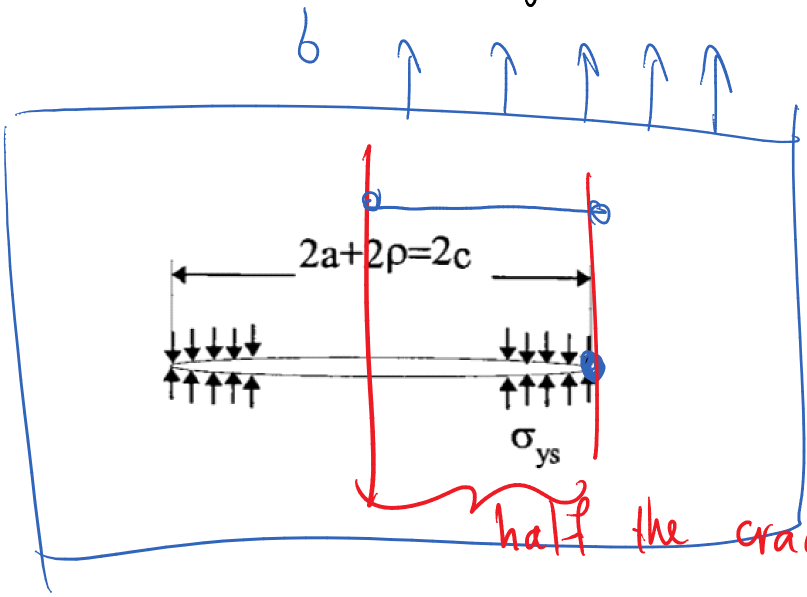
material has yielded

$\sigma_{yy} = \sigma_{ys} = \begin{cases} \sigma_y & \text{p-stress} \\ \frac{\sigma_y}{1-2\nu} & \text{strain} \end{cases}$

σ_{yy}

ρ original crack length $= 2a$

ρ_{ys} region that material has yielded (only on the line)



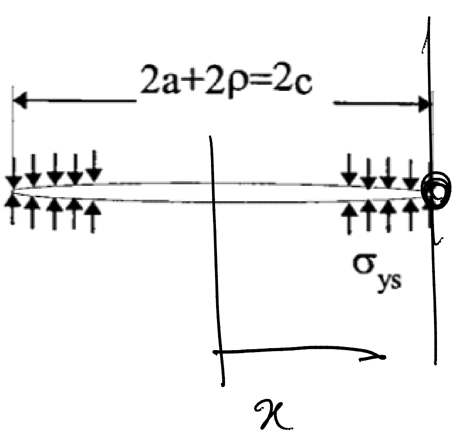
half the crack length

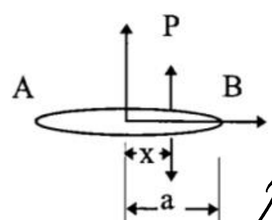
$$K = K_{\sigma} + K_{\sigma_{ys}}$$

K_{σ}: far field

$K_{\sigma_{ys}}$: on the crack surface

$$K_{\sigma} = \sigma\sqrt{\pi c}$$





crack length

$$K_A = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}}$$

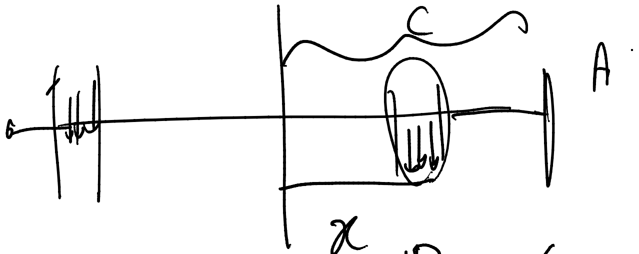
$$K_B = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}}$$

distance from ...

crack length $= c = a + \rho$

$K_B \quad \sqrt{\pi a}\sqrt{a+x}$ from center

crack length = $C = a + \rho$



$dP = -(\sigma_{ys}\, dx)$

$\underbrace{dK_{\sigma y}^{A}}_{\text{change in } K_{\sigma y}}$

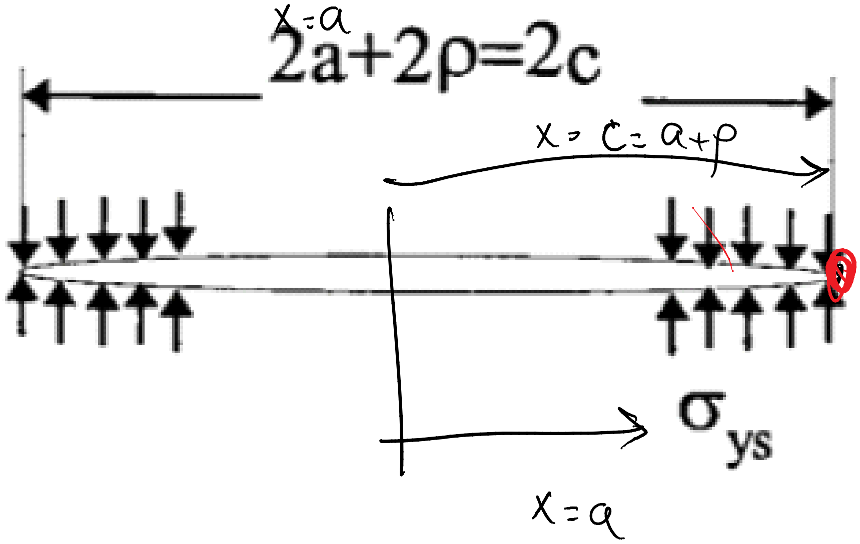
$$= \frac{dP}{\sqrt{\pi c}} \underbrace{\sqrt{\frac{c+x}{c-x}}}_{\text{from loading on the same side}}$$

crack length

$dP = -\sigma_{ys}\, dx$

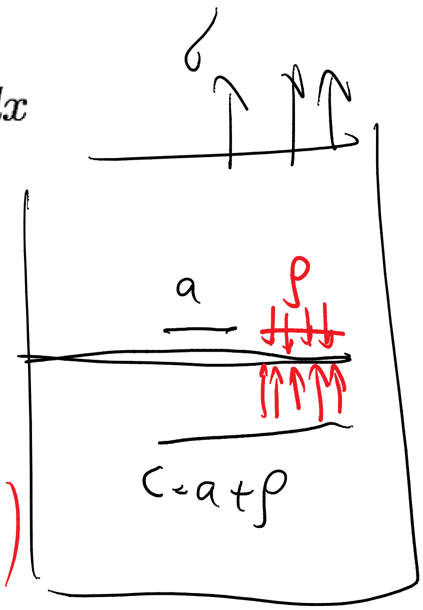
$$+ \frac{dP}{\sqrt{\pi c}} \sqrt{\frac{c-x}{c+x}}$$

$$K_{\sigma y} = \int_{x=a}^{x=c} dK_{\sigma y}$$



$K^{eff} = 0$

$$K_I^{\sigma_{ys}} = -\frac{\sigma_{ys}}{\sqrt{\pi c}}\int_a^c \left(\sqrt{\frac{c-x}{c+x}} + \sqrt{\frac{c+x}{c-x}}\right) dx$$



$$K_I^{\sigma_{ys}} = -2\sigma_{ys}\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right)$$

$$K^{\sigma_{ys}} = -2\sigma_{ys}\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right)$$

far field loading $\quad K^{\sigma} = \sigma\sqrt{\pi c}$

$$K^{eff} = K^{\sigma_{ys}} + K^{\sigma} = -2\sigma_{ys}\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right) + \sigma\sqrt{\pi c} = 0$$

$c \qquad c$

to remove stress singularity

$$\sigma\sqrt{\cancel{\pi(a+\rho)}} = 2\sigma_{ys}\sqrt{\cancel{\frac{a+\rho}{\pi}}}\cos^{-1}\left(\frac{a}{a+\rho}\right)$$

$$\left(\frac{\sigma}{2\sigma_{ys}}\right)\pi = \cos^{-1}\left(\frac{a}{a+\rho}\right) \Rightarrow$$

$$\frac{a}{a+\rho} = \cos\left(\frac{\sigma\pi}{2\sigma_{ys}}\right)$$

$$\frac{a}{a+\rho} = \frac{1}{1+\frac{\rho}{a}} \cong 1 - \frac{\rho}{a}$$

plastic zone much smaller than the crack length $\frac{\rho}{a} \ll 1$

$$\cos\left(\frac{\sigma\pi}{2\sigma_{ys}}\right) \cong 1 - \frac{1}{2}\left(\frac{\sigma\pi}{2\sigma_{ys}}\right)^2$$

$\cos x \cong 1 - \frac{x^2}{2}$

$|x| \ll 1$

we assume $\rightarrow$ $\left|\frac{\sigma}{\sigma_{ys}}\right| \ll 1$ holds which

we have a precedence of showing that it corresponds to SSY

$\Rightarrow$

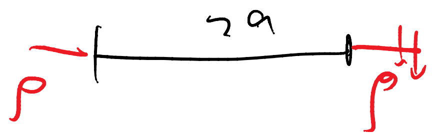
$$\Rightarrow 1 - \frac{\rho}{a} = 1 - \frac{1}{2}\left(\frac{\sigma\pi}{2\sigma_{ys}}\right)^2 \Rightarrow)$$

$$\frac{\rho}{a} = \frac{1}{2}\left(\frac{\sigma\pi}{2\sigma_{ys}}\right)^2 \Rightarrow$$

$$\frac{1}{a} = \frac{1}{2}\left(2\sigma_{ys}\right) \Longrightarrow$$

K for original crack

$$\rho = \frac{\left(\sigma\sqrt{\pi a}\right)^2}{\sigma_{ys}^2}\,\frac{\pi}{8}$$



$$\boxed{\rho = \frac{\pi}{8}\left(\frac{K_I}{\sigma_{ys}}\right)^2}$$

plastic zone size with strip yield model

$a + \rho$ $\left(2\sigma_{ys}\right)$

$$\rho = \frac{\pi^2\sigma^2 a}{8\sigma_{ys}^2} = \frac{\pi}{8}\left(\frac{K_I}{\sigma_{ys}}\right)^2$$

close to 0.392

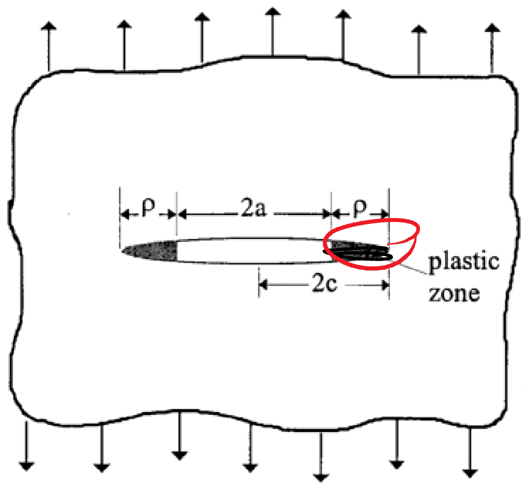
182

Irwin's result 0.318

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2$$

So we see that Irwin's model (even though it does not solve the balance of linear momentum) still provides a good approximation

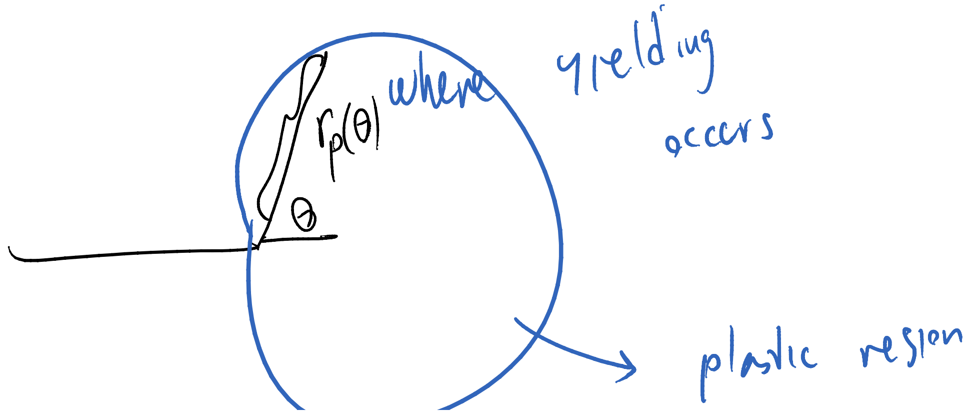
Strip yield model is exact:

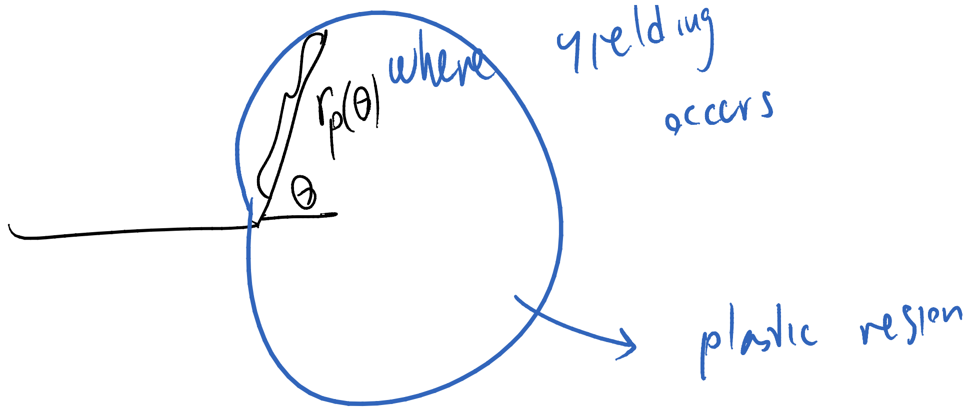


BUT the limitation is that the yield region is restricted to a strip ahead of the crack tip

What is the next level of accuracy? Some model even more realistic than strip yield model.

Next level is allowing the yielding not only ahead of the crack in a strip but in the entire 2D region





5.2.2 Plastic zone shape: 2D models

- 2D models
- plane stress versus plane strain plastic zones

von-Mises criterion

$$\sigma_v = \sqrt{3J_2}$$
$$= \sqrt{\frac{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)}{2}}$$
$$= \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2}{2}}$$
$$= \sqrt{\tfrac{3}{2}\, s_{ij}s_{ij}}$$

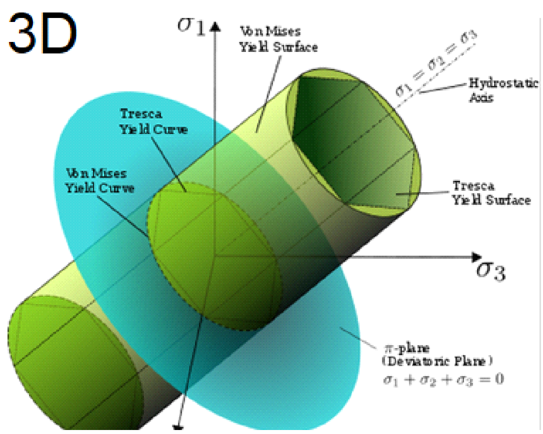
s is stress deviator tensor

$$\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\sigma})\,\mathbf{I}$$

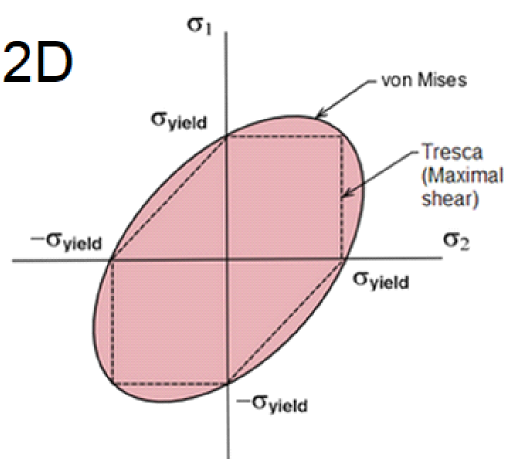
Tresca criterion
Maximum shear stress

$$\sigma_{tresca} = \sigma_1 - \sigma_3 > \sigma_{max}$$

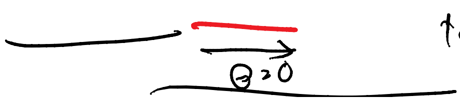
3D



2D



Before we used von Mises yield condition ONLY ahead of the crack



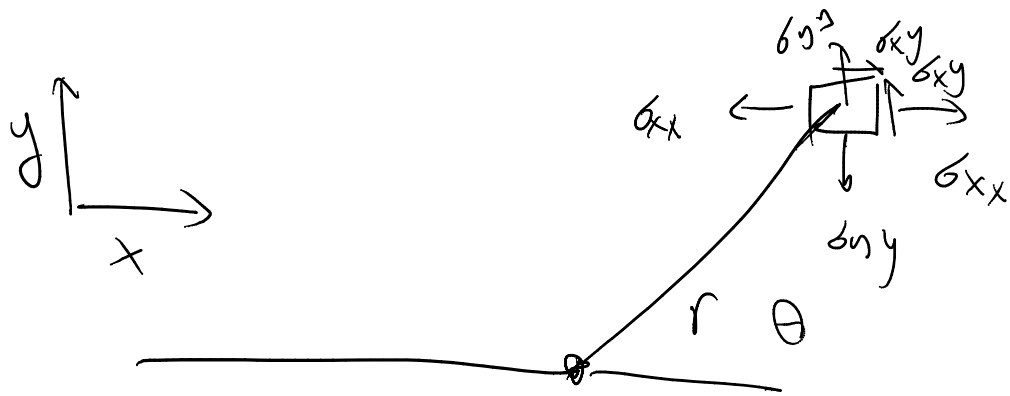
to obtain

$\xrightarrow{\theta=0}$ to obtain

$$\sigma_e = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}} = \begin{cases} \sigma_{yy} & \text{p. stress} \\ (1-2\nu)\sigma_{yy} & \text{p. strain} \end{cases}$$

for $\theta = 0$

How can we estimate plastic zone size in 2D region ahead of the crack?



$$\sigma_e = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}$$

$$\begin{cases} \sigma_{xx} = \dfrac{K_I}{\sqrt{2\pi r}} f_{xx}(\theta) \\ \sigma_{yy} = \text{〃} \ f_{yy}(\theta) \\ \sigma_{xy} = \text{〃} \ f_{xy}(\theta) \end{cases}$$

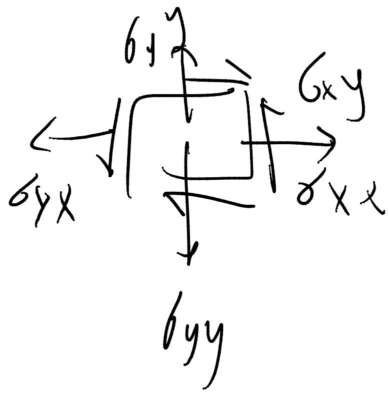
$\sigma_1, \sigma_2, \sigma_3$ = principal stresses

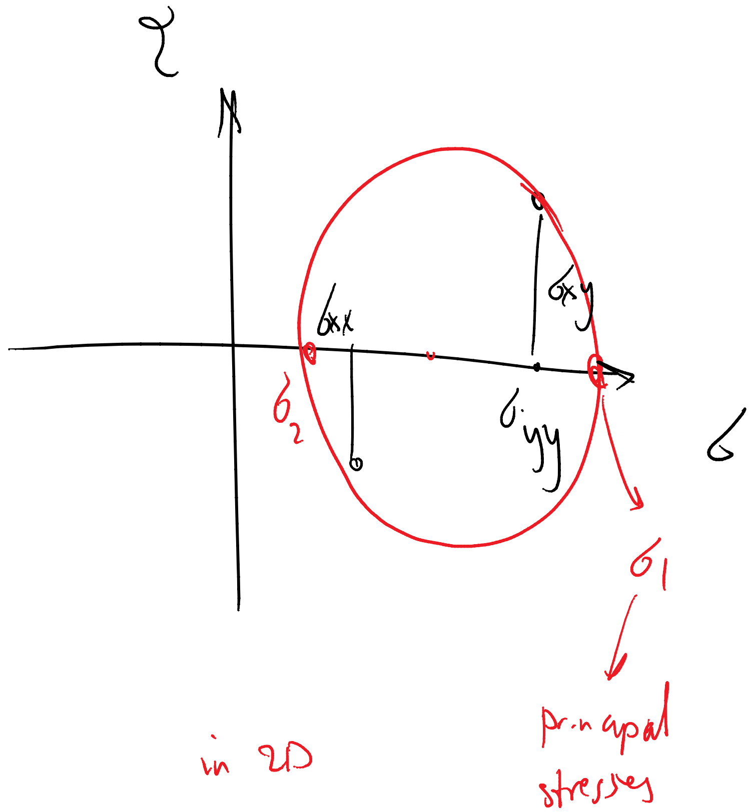
Slide 124

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$





$$\sigma_1, \sigma_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \left[\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}{}^2\right]^{1/2}$$

$$\sigma_3 = \begin{cases} 0 & \text{plane stress} \\ \nu(\sigma_1 + \sigma_2) & \text{plane strain} \end{cases}$$

$$\epsilon_3 = \frac{1}{E}\left(\sigma_3 - \nu(\sigma_1 + \sigma_2)\right) = 0$$

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)$$
$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)$$
$$\sigma_3 = \begin{cases} 0 & \text{plane stress} \\ \dfrac{2\nu K_I}{\sqrt{2\pi r}} \cos\dfrac{\theta}{2} & \text{plane strain} \end{cases}$$

principal stresses as functions of r, θ

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy}\ (r, \theta)$ known

Mohr's circle & $\sigma_3 = \begin{cases} 0 & \text{p. stress} \\ \nu(\sigma_1 + \sigma_2) & \text{p. strain} \end{cases}$ $\Longrightarrow$

$$\sigma_e = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}$$

$$\boxed{\sigma_e(r,\theta) \text{ is known}}$$

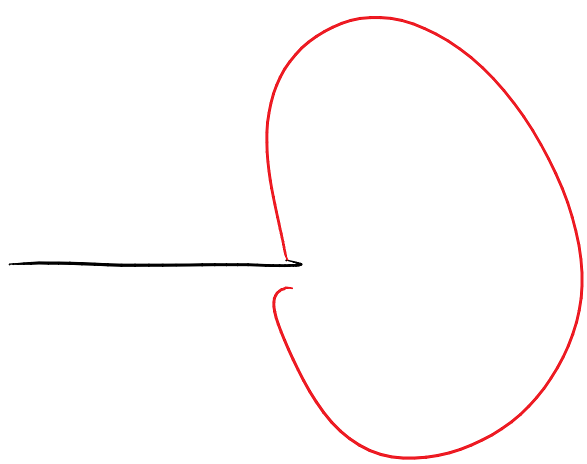
$\sigma_e \rightarrow$ known

$$\boxed{\sigma_e \geq \sigma_y}$$

label the point as yielded

as $r \to 0$

$\sigma_e \to \infty$



plastic zone

$$r_y(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2\left[1+\cos\theta+\frac{3}{2}\sin^2\theta\right]$$ plane stress

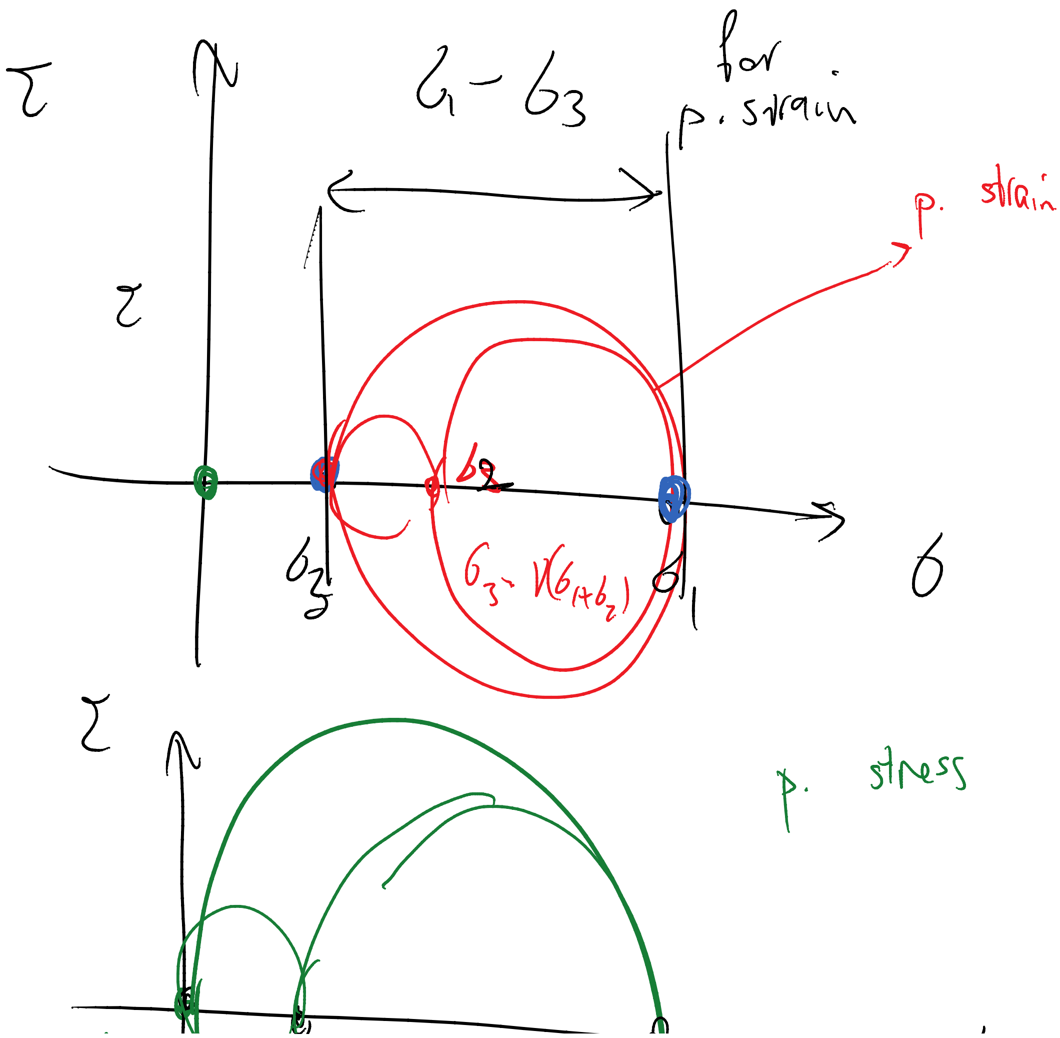
$$r_y(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2\left[(1-2\mu)^2(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right]$$ plane str

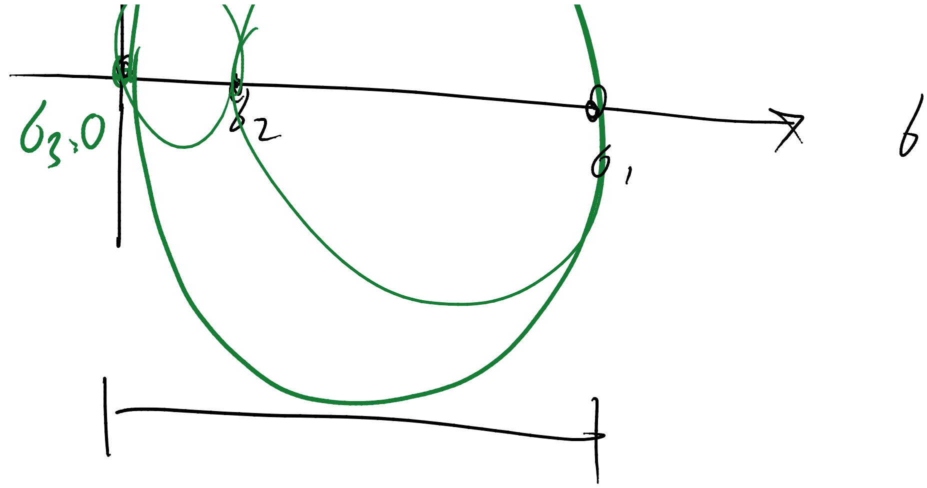
plastic zone radius as a function of r

(obtained from

$r_y(\theta) \longleftarrow \sigma_e(r,\theta) = \sigma_y$

plane stress gives LARGER plastic zone size





$\sigma_1 - \sigma_3 = \sigma_1$ p. stress

$> (\sigma_1 - \sigma_3)$ for p. strain

material yields easier

for plane stress $\Longrightarrow$

larger plastic zone

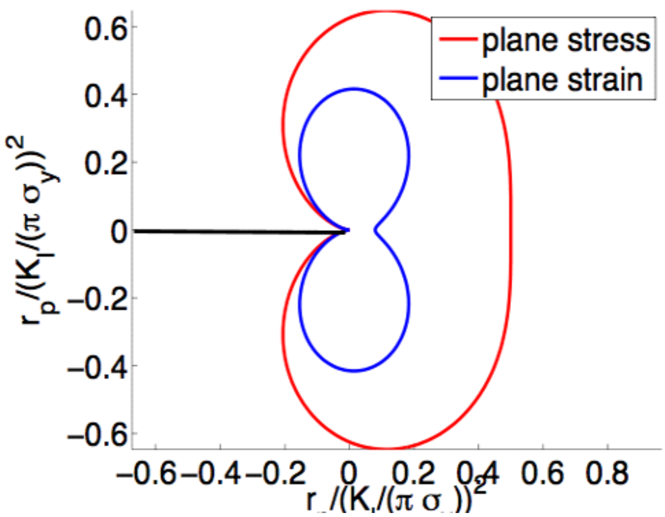
Shapes:

Plastic zone shape

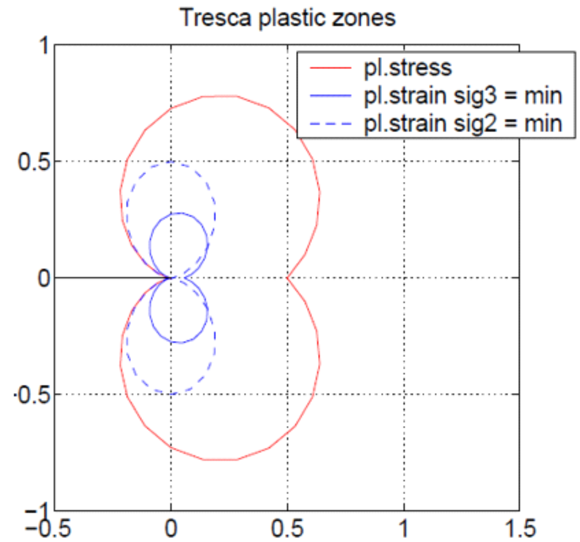
von-Mises criterion

$$r_y(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2\left[1 + \cos\theta + \frac{3}{2}\sin^2\theta\right]$$

plane stress



Tresca criterion



This is a crude approximation (better than 1D versions) but still the stress solution we have does not satisfy balance of linear momentum!

How do we get the correct solution?

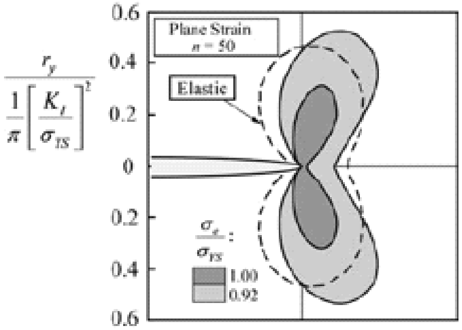
We need to solve the problem numerically because it's a difficult problem

Stress redistributed for 2D

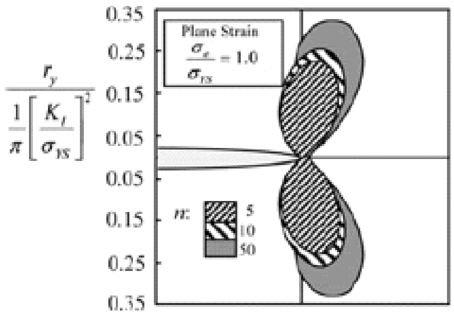
Dodds, 1991, FEM solutions
Ramberg-Osgood material model

$$\frac{\varepsilon}{\varepsilon_o} = \frac{\sigma}{\sigma_o} + \alpha\left(\frac{\sigma}{\sigma_o}\right)^n$$

- Low n: High strain-hardening.
- $n \to \infty$: Similar to elastic perfectly plastic.



Effect of definition of yield (some level of ambiguity)



Effect of strain-hardening:
Higher hardening (lower n) => smaller zone

198