2016/10/18B

Tuesday, October 18, 2016
11:39 AM

Generalization of J integral

- Dynamic loading
- Surface tractions on crack surfaces
- Body force
- Initial strains (e.g. thermal loading)
- Initial stress from pore pressures

Assumptions

1. static loading
2. No traction on crack surface
3. No body force
4. nonlinear elastic

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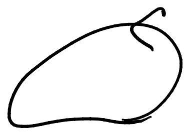
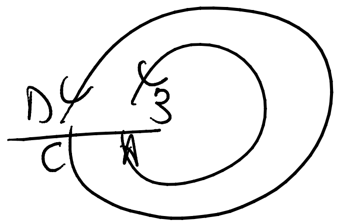
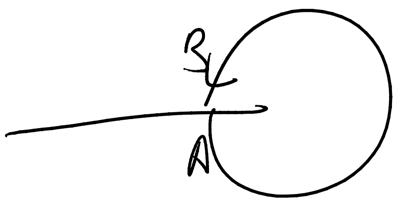
A. defining J integral

B. $J = \int_{\partial\Omega} \cdots = 0$  4, 1, 3

C. $J_{AB} = J_{CD}$ 

4, 2

D. 

$J_{AB} = G$ 4, 1, 2, 3

For more general setting (dynamic loading, complex non-elastic material response, non-traction free crack surfaces, etc.) refer to

cf. Saouma 13.11 & 13.12 for details

We'll also talk about generalization of J integral when we are in the computational section.

$$G = \frac{1}{E'}\left(K_I^2 + K_{II}^2\right) + \frac{1}{\mu} K_{III}^2$$

pure mode I: $\boxed{G = \frac{K^2}{E'}}$

K

$\boxed{J = \frac{K^2}{E'}}$

$\left(\sigma \sim \frac{K}{\sqrt{2\pi r}}\right)$ Only for LEFM

G $\longrightarrow$ J

works for nonlinear case

- K only works for LINEAR EFM (LEFM)
- However $K = \sqrt{JE'}$ works for LEFM & the integral expression for J provides a ROBUST way to calculate J

Basically even for the linear case, computing J is the preferred method for calculating K !

	LEFM	NLFM
Global response	$G = -\frac{d\pi}{da}$	$J_1 = \int_\Gamma \left(W n_1 - t_i \frac{du_i}{dx_1}\right) ds$
Local response	$\sigma \propto \frac{K}{\sqrt{2\pi r}}$	$\sigma \propto J^{\alpha} r^{\beta}$

generalization of local stress response

$G = \frac{K^2}{E'}$ $\quad \sigma \propto (G)^{\frac{1}{2}} r^{-\frac{1}{2}}$

Now that we know how G (energy release rate) extends to NLFM all left is to also extend the local stress solution to NLFM

5.3. 5. Plastic crack tip fields; Hutchinson, Rice and Rosengren (HRR) solution

Ramberg–Osgood model

linear part

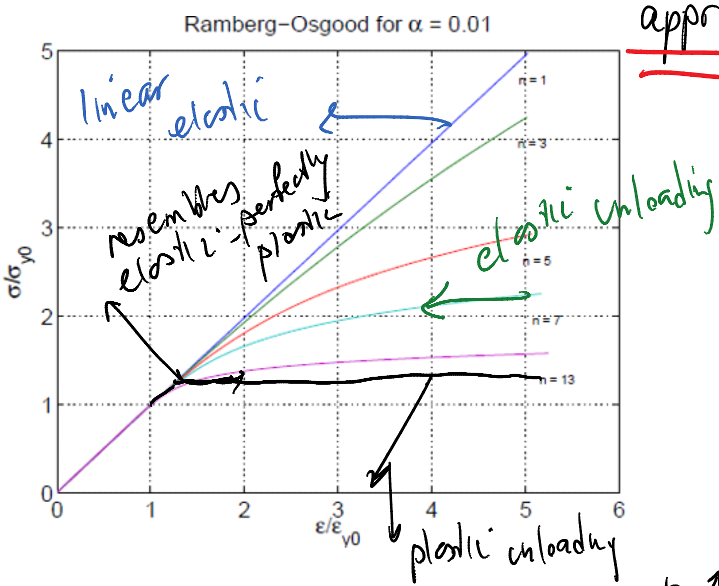
$$\frac{\varepsilon}{\varepsilon_{y0}} = \frac{\sigma}{\sigma_{y0}} + \alpha\left(\frac{\sigma}{\sigma_{y0}}\right)^n$$

$\sigma(\varepsilon)$ or $\varepsilon(\sigma)$ for elastic material

$\varepsilon_e + \varepsilon_p \rightarrow$ plastic

$$\underbrace{\varepsilon_e + \varepsilon_p}_{\text{elastic}} \rightarrow \text{plastic}$$

approximate



$$\left(\frac{\varepsilon}{\varepsilon_{y_0}}\right) = \left(\frac{\sigma}{\sigma_{y_0}}\right) + \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n$$

$n = 1$ $\quad \frac{\varepsilon}{\varepsilon_{y_0}} = \left(\frac{1+\alpha}{\sigma_{y_0}}\right)\sigma$

$n \uparrow \infty$

$n = 13$
large

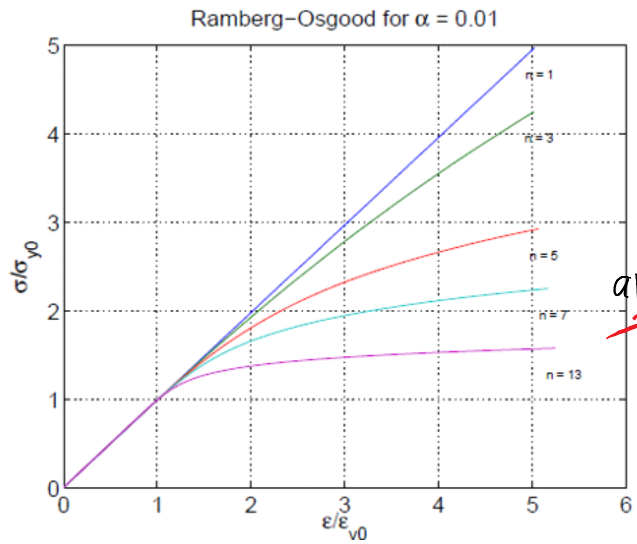
$$\left(\frac{\varepsilon}{\varepsilon_{y_0}}\right) = \left(\frac{\sigma}{\sigma_{y_0}}\right) + \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n$$

slightly $\frac{\sigma}{\sigma_{y_0}} > 1$

$$(\underbrace{1+\varepsilon}_{>1})^n \quad n \to \infty$$

So Ramberg-Osgood model provides a very realistic response of materials even if plastic deformation happens as long as there is no plastic unloading.

$(1.5)^{13}$ vs $1/(1.5)^{13}$



approximate

$$\frac{\varepsilon}{\varepsilon_{y_0}} = \underbrace{\left(\frac{\sigma}{\sigma_{y_0}}\right)}_{\varepsilon_e} + \underbrace{\alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n}_{\varepsilon_p}$$

dominant term

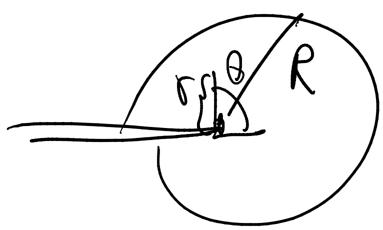
close to crack tip

Near the crack tip we can write

$$\boxed{\left(\frac{\varepsilon}{\varepsilon_{y_0}}\right) \approx \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n}$$

$$\underbrace{\text{Energy/let}}_{} \int W(\varepsilon)\,\underbrace{dA}_{r\,dr\,d\theta}$$

↓ internal energy in radius R



$$\sim \int^{R} \int^{2\pi} (\alpha\,\varepsilon)\,\sigma \ldots$$

in radius R $\approx \int_0 \int_0^{2\pi} (\sigma \varepsilon) r) \, dr \, d\theta$

for finite energy

$\sigma(r)\,\varepsilon(r)\,r$ should be bounded

$\sigma \propto \frac{A}{r^x}$ $\qquad \sigma . \varepsilon r = r^{1-x-y}$

$\varepsilon \propto \frac{B}{r^y}$ $\qquad 1-x-y \geq 0$

for finite energy

in fact for the elastic model considered

$$\boxed{x + y = 1}$$

Other equation

$$\frac{\varepsilon}{\varepsilon_{y_0}} \approx \alpha \left(\frac{\sigma}{\sigma_{y_0}} \right)^n$$

$$\varepsilon \propto \frac{1}{r^y}$$

$$\sigma \propto \frac{1}{r^x}$$

$\rightarrow$ $\frac{1}{r^y} = \frac{1}{r^{xn}}$

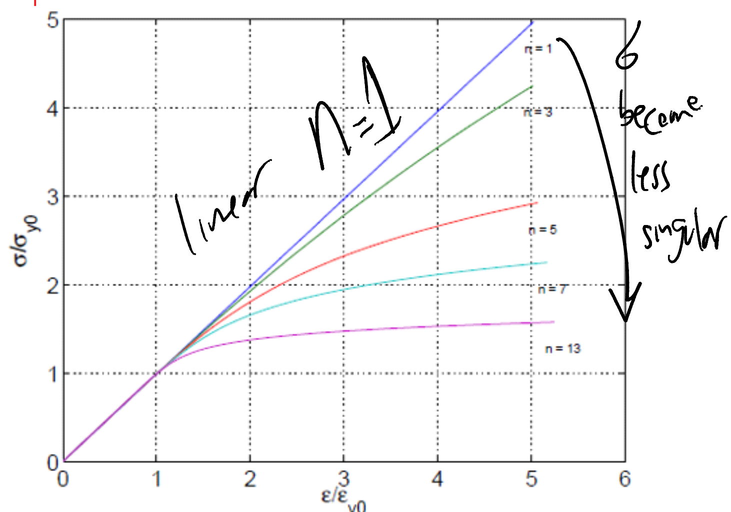
$\sigma \propto \frac{1}{r^x}$ $\quad\int$ ... $\Rightarrow \boxed{y = xn}$

$x + y = 1$

$\Rightarrow \quad y = \frac{n}{n+1}$

$x = \frac{1}{n+1}$

$$\boxed{\begin{aligned} \sigma &\propto \frac{1}{r^{1/(n+1)}} \\ \varepsilon &\propto \frac{1}{r^{n/(n+1)}} \end{aligned}}$$



linear $\sigma, \varepsilon \propto \frac{1}{\sqrt{r}}$ already knew

$n \uparrow$ $\quad \varepsilon \rightarrow$ more singular $\qquad \sigma \rightarrow$ less singular

$n \rightarrow \infty$ / $\quad \sigma \rightarrow$ bounded $\qquad \varepsilon \propto \frac{1}{r^1}$

power singularity)

Same argument

Hutchinson, Rice and Rosengren(HRR) solution

- Near crack tip "plastic" strains dominate:

$$\frac{\epsilon}{\epsilon_0} = \alpha\left(\frac{\sigma}{\sigma_0}\right)^n \qquad *$$

- Assume the following r dependence for σ and ε

$$\sigma = \frac{c_1}{r^x}$$
$$\varepsilon = \frac{c_2}{r^y}$$

1. Bounded energy:

$$\sigma\varepsilon \propto \frac{1}{r} \Rightarrow x + y = 1$$

2. ε - σ relation *

$$y = nx$$

$$\Rightarrow \quad x = \frac{1}{1+n}, \qquad y = \frac{n}{1+n}$$

If one actually solves the asymptotic stress solution around a crack tip with Osgood strain response:

HRR solution: Local stress field based on J

J appears as a factor the same way K

- Final form of HRR solution:

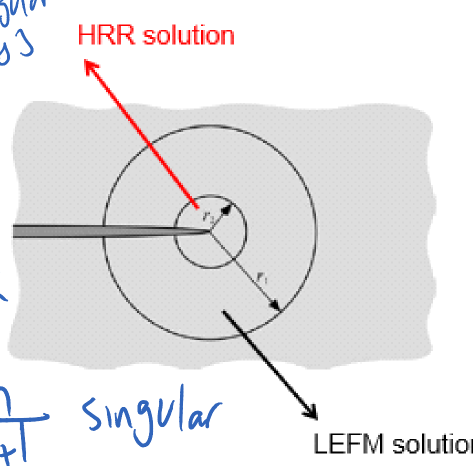
n+1 singular stress

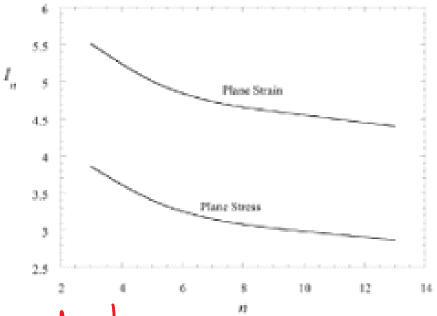
$$\sigma_{ij} = \sigma_0\left(\frac{EJ}{\alpha\sigma_0^2 I_n r}\right)^{\frac{1}{n+1}}\tilde{\sigma}_{ij}(n,\theta)$$

$$\epsilon_{ij} = \frac{\alpha\sigma_0}{E}\left(\frac{EJ}{\alpha\sigma_0^2 I_n r}\right)^{\frac{n}{n+1}}\tilde{\epsilon}_{ij}(n,\theta)$$

strain is $\frac{n}{n+1}$ singular

J plays the role of K for local σ, ε, u fields





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$\frac{n}{n+1}$ singular

LEFM solution

scaled σ, ε response for LEFM

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n, \theta)$$

generalization of $\sigma_{ij}(\theta)$

$$\epsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(n, \theta)$$

$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta)$

Nondimensional

$[E] = \sigma$

$[J] = \sigma \cdot L$

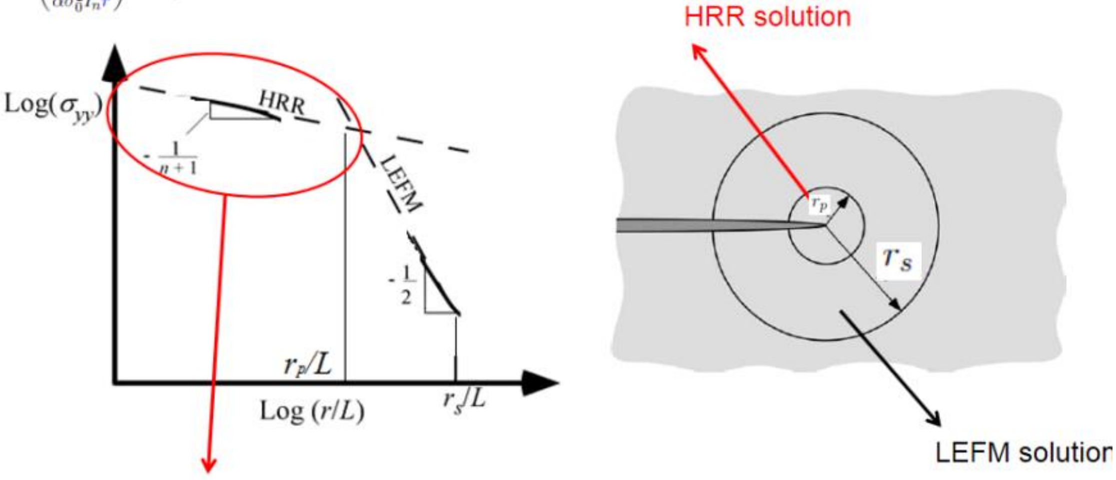
$[\sigma_0] = \sigma$

$[r] = L$

Have we been able to resolve the dilemma of stress going to infinity around the crack tip?

NO (unfortunately!!)

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n,\theta)$$



Stress is still singular but with a weaker power of singularity!

Going back to the relation between K and J

mode I

$G = \frac{K^2}{E'}$

LEFM

K

G $=$ J

NL elastic FM

$J = \frac{K^2}{E'}$ for LEFM only as we saw

K has No meaning in NLFM

Do the J integrals provide a way to calculate KI and KII for a mixed in-plane fracture problem? (Obviously we are focusing on LFEM)

$$J_1 = G = \frac{K_I^2 + K_{II}^2}{E'}$$

#2 K_I & K_{II} unknown

#1 G is known

cannot find K_I & K_{II}

Remember

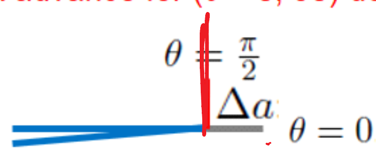
$$J_k = \int_{\Gamma} \left(w n_k - \vec{t} \cdot \frac{d\vec{u}}{dx_k} \right) ds$$

replacing k by 1 & 2 we get:

$$J_1 = \int_{\Gamma} \left(w dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$

$$J_2 = \int_{\Gamma} \left(-w dx - \mathbf{t} \frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

J1 & J2: crack advance for (θ = 0, 90) degrees



J_1: energy release rate when the crack propagates at $\theta = 0$ angle

J_2: energy release rate if the crack propagates at $\theta = \frac{\pi}{2}$ angle

$$J_1 = G = \frac{1}{E'}(K_I^2 + K_{II}^2)$$

$J_2 =$

$$\boxed{\begin{aligned} J_1 &= \frac{K_I^2 + K_{II}^2}{E'} \\ J_2 &= \frac{-2K_I K_{II}}{E'} \end{aligned}}$$

new equation

J_1, J_2 known $\Longrightarrow$ K_I & K_{II}

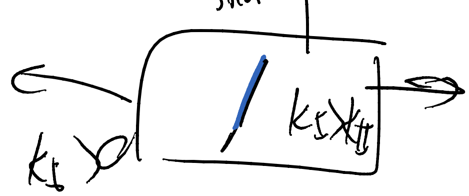
multiple solutions for that

a, b

$a > b$

K_I	a	b	$-a$	$-b$
K_{II}	b	a	$-b$	$-a$

solution



$$E' = \begin{cases} E & \text{plane strain} \\ \frac{E}{1-\nu^2} & \text{plane stress} \end{cases}$$

Note that if $K_I = a, K_{II} = b$ is a solution the general solution is:
$K_I = \pm a, K_{II} = \pm b$ and $K_I = \pm b, K_{II} = \pm a$

5.3. 4. Energy Release Rate, crack growth and R curves

Already know how to evaluate J1 = G from contour integral

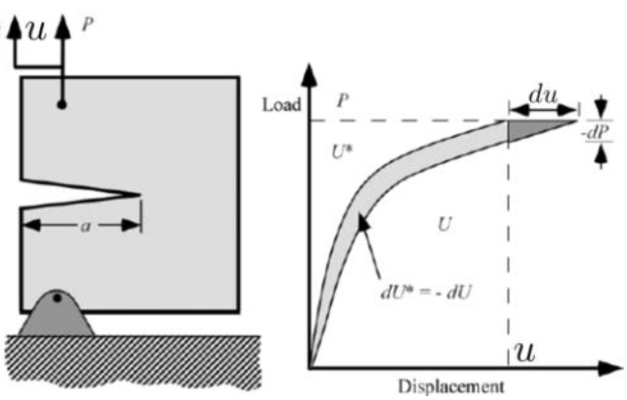
Can we evaluate J from experiments, say load displacement history, but now for nonlinear elastic material (or elastic/plastic with no unloading)

Goal: Obtain J from P-Δ Curve

$G = J = -\frac{\partial \Pi}{\partial A} = -\frac{1}{B}\frac{\partial \Pi}{\partial a}$

- $\Pi = U_e - W$: Potential energy
- W: External work
- $U_e = \int_V e\,\mathrm{dv}$: Internal energy
- $e(\epsilon_0)$ (or $w(\epsilon_0)$) $= \int_0^{\epsilon_0} \sigma(\epsilon)\mathrm{d}\epsilon$

1. Load Control (P fixed, u increases):

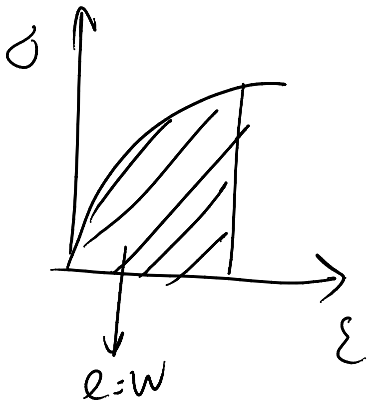


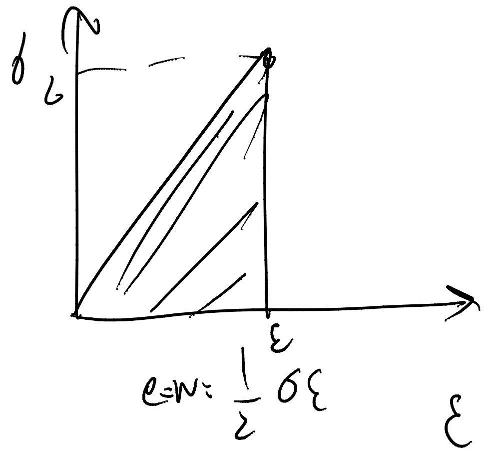
the only chang $U_e = \int_V e\,dV$

W = internal energy density

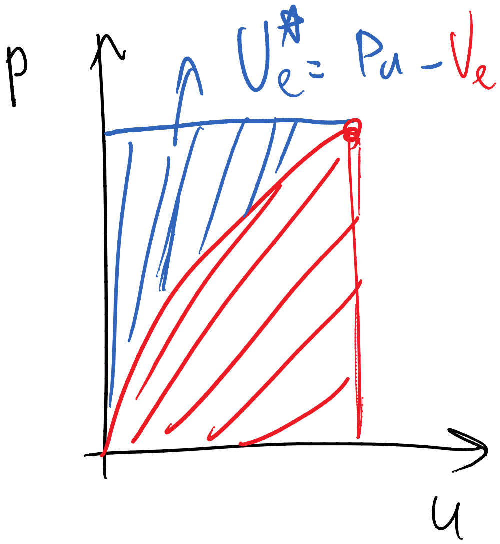
no longer

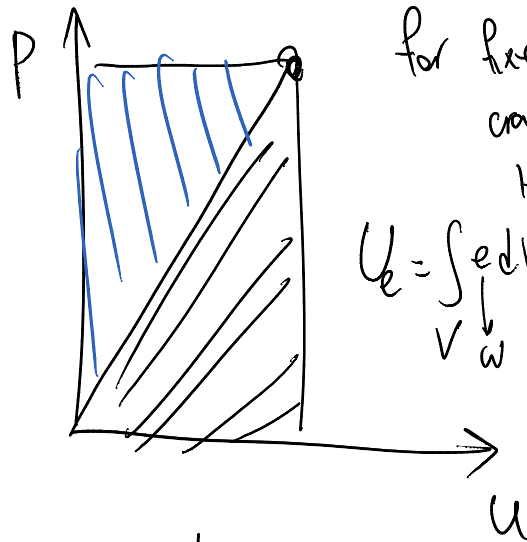
$e = W = \frac{1}{2}\sigma : \varepsilon$ is a more complex equation





linear case





for fixed crack length

$$U_e = \int_V e \, dV$$
($e \to w$)

linear case

$$U_e = \int_V w \, dV = \text{shaded area}$$

$$U_e^* = Pu - U_e = \text{Complementary potential energy}$$

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crack length a

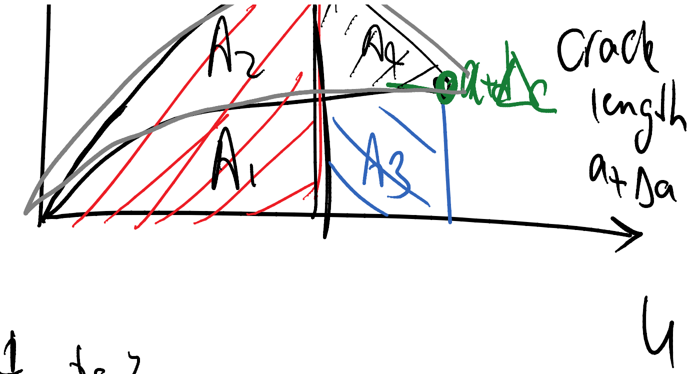
$$U_e(a) = A_1 + A_2$$

$$U_0(a+\Delta a) = A_1 + A_2$$

crack length a

Crack length

$U_e(a+\Delta a) = A_1 + A_3$



$W_{12} = A_3 + A_4$

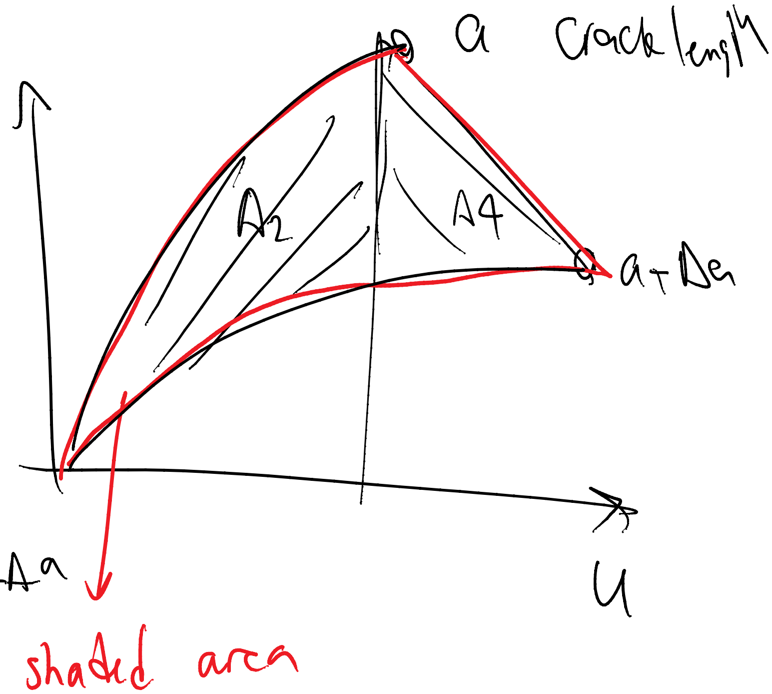
external work from state 1 to 2

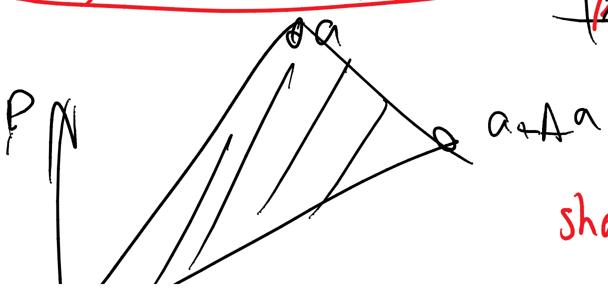
$$G = J = \frac{-1}{B}\frac{\Delta \Pi}{\Delta a} = \frac{-1}{B}\left(\frac{\Pi_2 - \Pi_1 - W_{12}}{\Delta a}\right)$$

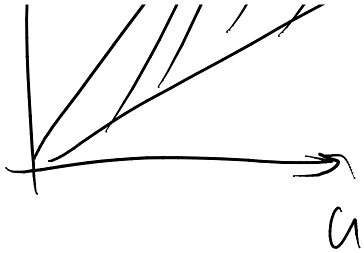
$$= \frac{-1}{B}\left(\frac{(\cancel{A_3} + \cancel{A_1}) - (\cancel{A_1} + A_2) - (\cancel{A_3} + A_4)}{\Delta a}\right)$$

$$\Rightarrow \boxed{G = \frac{1}{B\Delta a}\left(A_2 + A_4\right)}$$

$$\boxed{J = G = \frac{-1}{B}\frac{\text{shaded area}}{\Delta a} \quad \lim \Delta a \to 0}$$







linear case

shaded area