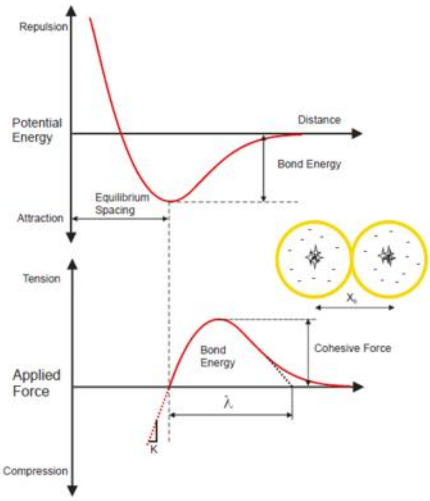
From last time:



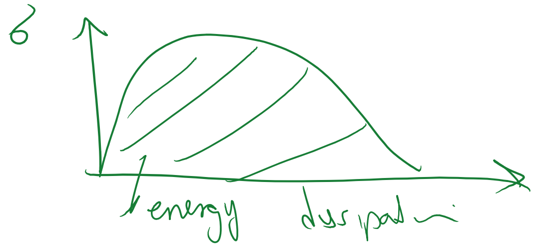
$$\sigma_c \approx \frac{E}{\pi}\,\frac{\lambda}{x_0}$$

λ: atomistic disp. for full separation

x_0: eq. atomistic distance

This is not realistic! For steel $\sigma_c \approx 250\text{MPa}$, $E = 200\text{GPa}$

Fracture energy



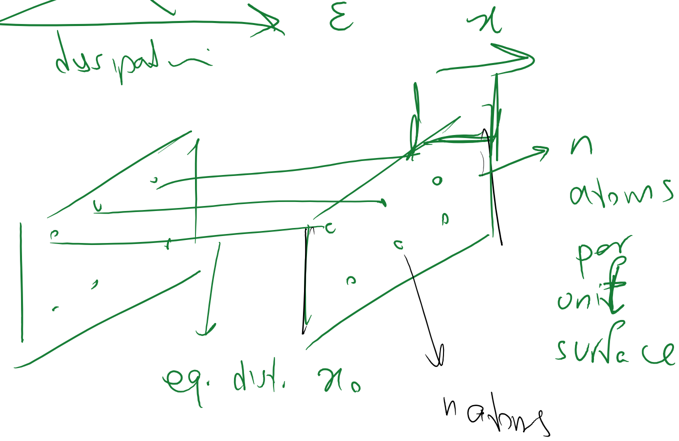
$$P = P_c \sin\left(\frac{\pi x}{\lambda}\right)$$

for one atom

$$\text{Work} = \int P\,dx$$

total work

$$W = n \int P_c \sin\left(\frac{\pi x}{\lambda}\right) = \frac{2n\lambda}{\pi} P_c$$

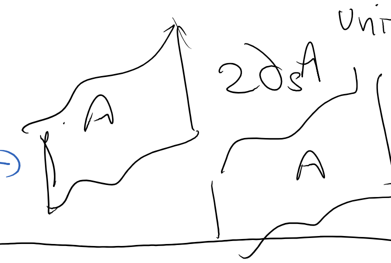


$$\frac{\text{Work of separation}}{\text{Area}} = \frac{W}{A} = \underbrace{\frac{nP_c}{A}}_{\sigma_c}\,\frac{2\lambda}{\pi} = 2\,\gamma_s$$

two sides

γ_s → fracture toughness

energy needed to create unit area of fractured surface on each side



$$\rightarrow \quad \gamma_s = \frac{\sigma_c \lambda}{\pi} \rightarrow \sigma_c = \frac{\pi \gamma_s}{\lambda}$$

$$\sigma_c = \frac{E}{\pi}\,\frac{\lambda}{x_0}$$

another $\sqrt{E\gamma_s}$ fracture toughness

$$\sigma_c^2 = \frac{E\gamma_s}{x_0} \rightarrow$$

another useful equation: $\sigma_c = \sqrt{\frac{E\gamma_s}{x_0}}$

toughness

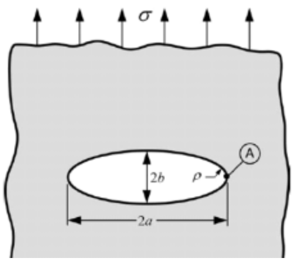
atomistic eq. distance

other eqn for σ_c: $\sigma_c = \frac{E}{\pi}\frac{\lambda}{x_0}$

Stress concentration (from defects or other features) is the main reason we do experience much higher stresses pointwise compared to average applied stress

Elliptic hole

Inglis, 1913, theory of elasticity



$$\sigma_A = \sigma\left(1 + \frac{2a}{b}\right)$$

radius of curvature

$$\rho = \frac{b^2}{a}$$

$$\sigma_A = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$

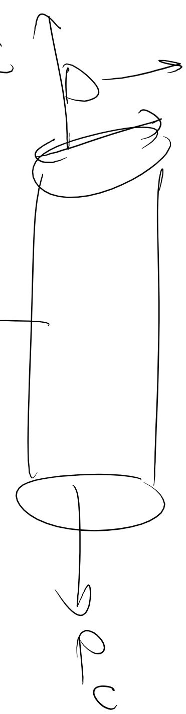
!!! ∞

Griffith's size effect experiment

TABLE 1.1. Strength of glass fibers according to Griffith's experiments.

Diameter (10^{-3} in)	Breaking stress (lb/in^2)	Diameter (10^{-3} in)	Breaking stress (lb/in^2)
40.00	24 900	0.95	117 000
4.20	42 300	0.75	134 000
2.78	50 800	0.70	164 000
2.25	64 100	0.60	185 000
2.00	79 600	0.56	154 000
1.85	88 500	0.50	195 000
1.75	82 600	0.38	232 000
1.40	85 200	0.26	332 000
1.32	99 500	0.165	498 000
1.15	88 700	0.130	491 000

$D \downarrow$ $\sigma_c \uparrow$



P_c

changing

glass fiber

$\sigma_c = \frac{P_c}{A}$

$\pi (d/2)^2$

P_c

Explanation

max defect size ... fiber diameter

max defect size $\propto$ fiber diameter $\pi(d/2)^2$ r_c

$\Longrightarrow$ smaller diameter fibers have less critical defect

Fracture stress: discrepancy between theory and experiment

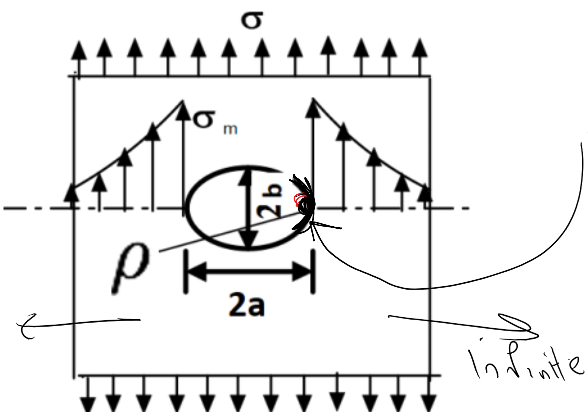
$$\sigma_{th} = \sqrt{\frac{E\gamma}{a_0}}$$

	a_0 [m]	E [GPa]	σ_{th} [GPa]	σ_b [MPa]	σ_{th}/σ_b
glass	$3 * 10^{-10}$	60	14	170	82
steel	10^{-10}	210	45	250	180
silica fibers	10^{-10}	100	31	25000	1.3
iron whiskers	10^{-10}	295	54	13000	4.2
silicon whiskers	10^{-10}	165	41	6500	6.3
alumina whiskers	10^{-10}	495	70	15000	4.7
ausformed steel	10^{-10}	200	45	3000	15
piano wire	10^{-10}	200	45	2750	16.4

$\dfrac{\text{theoretical strength} \left(\frac{E}{\pi}\frac{\lambda}{x_c}\right)}{\text{real strength}}$

bulk material: large discrepancy

very close



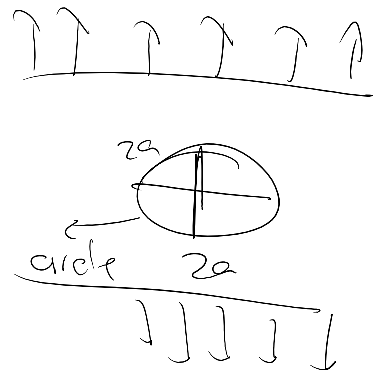
infinite

$$\sigma_{max} = \sigma\left(1 + 2\frac{a}{b}\right)$$

writing this in terms of radius of curvature

$$\sigma_{max} = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$

Ex.



$\sigma_{max} = 3\sigma$

$\rho = \frac{b^2}{a}$

$a/b \to \infty$ (more crack like) $\rho \to 0$

as $\rho \to 0$

$$\boxed{\sigma_{max} \approx 2\sigma\sqrt{\frac{a}{\rho}}}$$

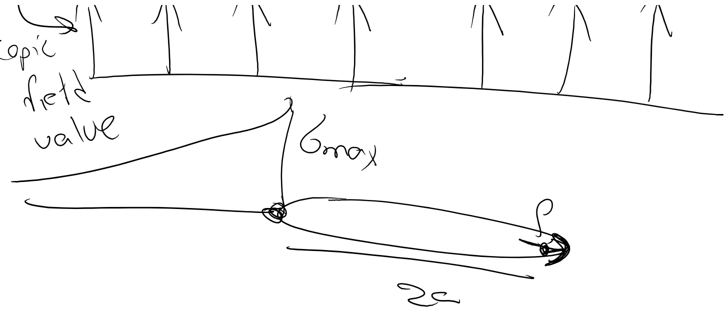
macroscopic far field

at failure $\sigma = \sigma_f$

$\sigma_{-} = 2(\sigma)\sqrt{a}$

$$\sigma_c = 2\sigma_f \sqrt{\frac{a}{\rho}}$$

$\sigma_c \to \sigma_{max}$, $\sigma_f \to \sigma$



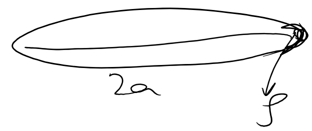
bulk
Strength for a material: σ_f
with characteristic defect size a & radius of curvature ρ

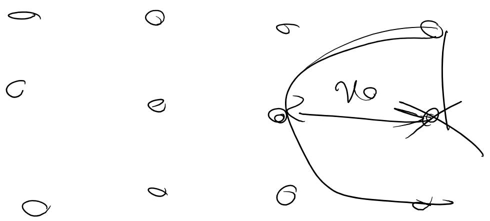
Locally where we have stress concentration, we can get to around the theoretical estimate σ_c ($\sigma_{max} \approx \sigma_c$)

$$\sigma_c = 2\sigma_f \sqrt{\frac{a}{\rho}} \longrightarrow \sigma_f = \sqrt{\frac{\rho}{4a}}\,\sigma_c$$

theoretical strength $\approx \frac{E}{\pi}\frac{\lambda}{x_c}$

real strength





$\rho_{min} \approx x_0$ worse case

worse case (very sharp crack)

$$\sigma_f = \sqrt{\frac{x_0}{4a}}\,\sigma_c = \sqrt{\frac{x_0}{4a}}\sqrt{\frac{E\gamma_s}{x_0}} = \sqrt{\frac{E\gamma_s}{4a}}$$

Atomistic: **Continuum with sharp crack $2a$**

very high $\sigma_c = \sqrt{\frac{E\gamma_s}{x_0}}$ very small $\Leftrightarrow$ $\sigma_f = \sqrt{\frac{E\gamma_s}{4a}} \Rightarrow$

$$\frac{\sigma_f}{\sigma_c} = \sqrt{\frac{x_0}{4a}}$$

$a \gg x_0 \Rightarrow \sigma_f \ll \sigma_c$

How can we test this hypothesis (the effect of existing crack lengths on fracture strength)?

Griffith's verification experiment

- Glass fibers with artificial cracks (much larger than natural crack-like flaws), tension tests

	Crack Length, $2a$ mm	Measured Strength, σ_f MPa	$\sigma_f\sqrt{a}$ MPa$\sqrt{\text{m}}$
sample 1	3.8	6.0	0.26
sample 2	6.9	4.3	0.25
sample 3	13.7	3.3	0.27
sample 4	22.6	2.5	0.27

(Data from the Griffith experiment)

$$\sigma_f = \sqrt{\frac{E\gamma_s}{4a}} \qquad \boxed{\sigma_f\sqrt{a} = \sqrt{\frac{E\gamma_s}{4}} = \text{const.}}$$

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Energy balance during crack growth

external work

kinetic energy

$$\dot{W} = \dot{U}_e + \dot{U}_p + \boxed{\dot{U}_k} + \dot{U}_\Gamma$$

surface energy

rate

$(\dot{x}) = \frac{dx}{dt}$

work rate ($\dot{}$) = rate of change of elastic internal energy

plastic internal energy

- "quasi-brittle" response in the bulk. we can ignore $\dot{U}_p$ from plastic deformation
- quasi-static problem Kinetic energy can be ignored

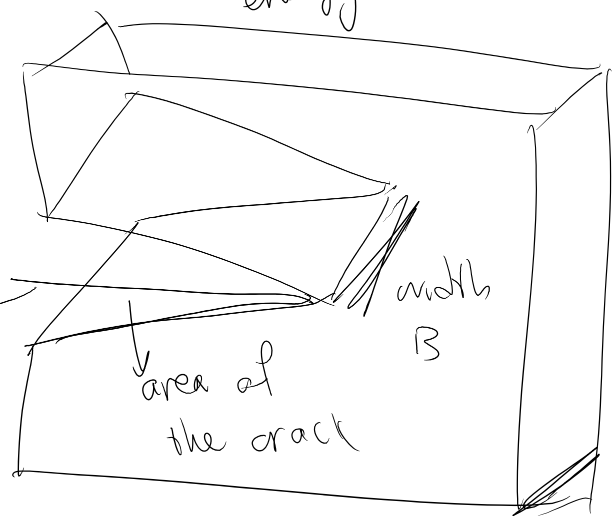
$$\dot{W} \approx \dot{U}_e + \dot{U}_\Gamma$$

$$\dot{W} \approx \dot{U}_e + \dot{U}_p$$

$$\Pi = U_e - W$$

potential energy — internal energy — external work

$$\boxed{\dot{\Pi} = \dot{U}_\Gamma}$$



$$\frac{\partial U_\Gamma}{\partial t} = \frac{\partial U_\Gamma}{\partial a}\left(\frac{\partial a}{\partial t}\right)$$

crack speed

crack length

(V_c)

$$\frac{\partial U_\Gamma}{\partial t} = -\frac{\partial \Pi}{\partial t} \rightarrow \cancel{V_c}\,\frac{\partial U_\Gamma}{\partial a} = -\cancel{V_c}\,\frac{\partial \Pi}{\partial a}$$

$$\rightarrow B\,\frac{\partial U_\Gamma}{\underbrace{\partial a(B)}_{A}} = -\frac{\partial \Pi}{\partial a}$$

$$B\,\frac{\partial U_\Gamma}{\partial A} = -\frac{\partial \Pi}{\partial a} \rightarrow \boxed{2\gamma_s = -\frac{1}{B}\frac{\partial \Pi}{\partial a} = -\frac{\partial \Pi}{\partial A}}$$

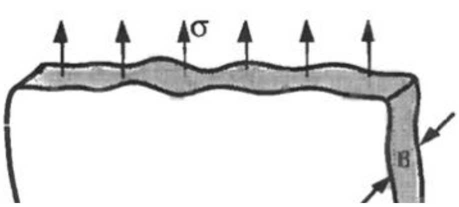
$A = Ba$

B thickness

area of crack $= Ba$

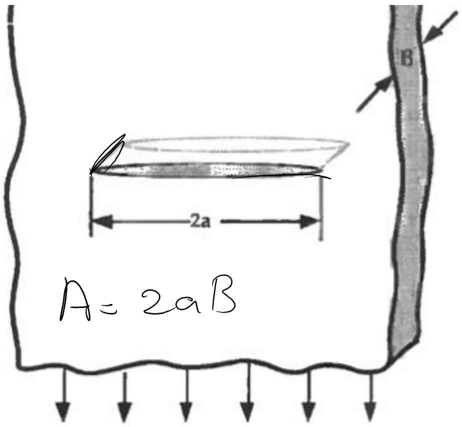
fracture toughness $= 2\gamma_s$

deriving γ_s from measurement of potential energy



$\Pi = U_e - W$ is obtained by elasticity solutions

2.14...



$\Pi = U_e - W$ is obtained by elasticity solutions

$$\boxed{\Pi = -\frac{\pi a^2 \sigma^2 B}{E}}$$

3.14...

$$-\frac{d\Pi}{dA} = -\frac{d(-\pi a^2 \sigma^2 B/E)}{d(2aB)} = \frac{\pi \sigma^2 a}{E} = 2\gamma_s$$

$$2\gamma_s = \frac{-d\Pi}{dA} \rightarrow$$

$$\frac{\pi \sigma^2 a}{E} = 2\gamma_s \rightarrow \boxed{\sigma = \sqrt{\frac{2}{\pi}\frac{\gamma_s E}{a}}}$$

$\sigma_{failure}$

stress need to initiate crack propagation for a crack of length 2a

Stress criteria	Energy approach
$\sigma_{failure} = \sqrt{\frac{E\gamma_s}{4a}}$	$\sigma_{failure} = \sqrt{\frac{E\gamma_s}{\frac{\pi}{2} a}}$

the only difference of the energy & stress approaches

Stress approach:
Stress Concentration

$$\sigma_f = 0.5\sqrt{\frac{E\gamma_s}{a}}$$

Energy approach:
Griffith

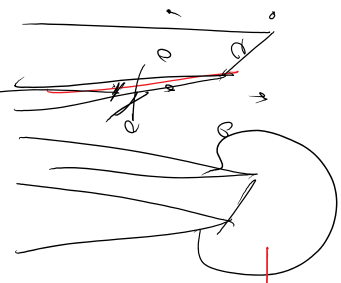
$$\sigma_f = \sqrt{\frac{2}{\pi}}\sqrt{\frac{E\gamma_s}{a}} \approx 0.8\sqrt{\frac{E\gamma_s}{a}}$$

Energy equation for ductile materials

Plane stress

$$\sigma_c = \sqrt{\frac{2E\gamma_s}{\pi a}}$$ Griffith (1921), ideally brittle solids

$$\boxed{\sigma_c = \sqrt{\frac{2E(\gamma_s + \gamma_p)}{\pi a}}}$$ Irwin, Orowan (1948), metals



need to add plastic energy dissipation per unit area of crack advance

plastic deformation ahead of crack contributes to energy dissipation

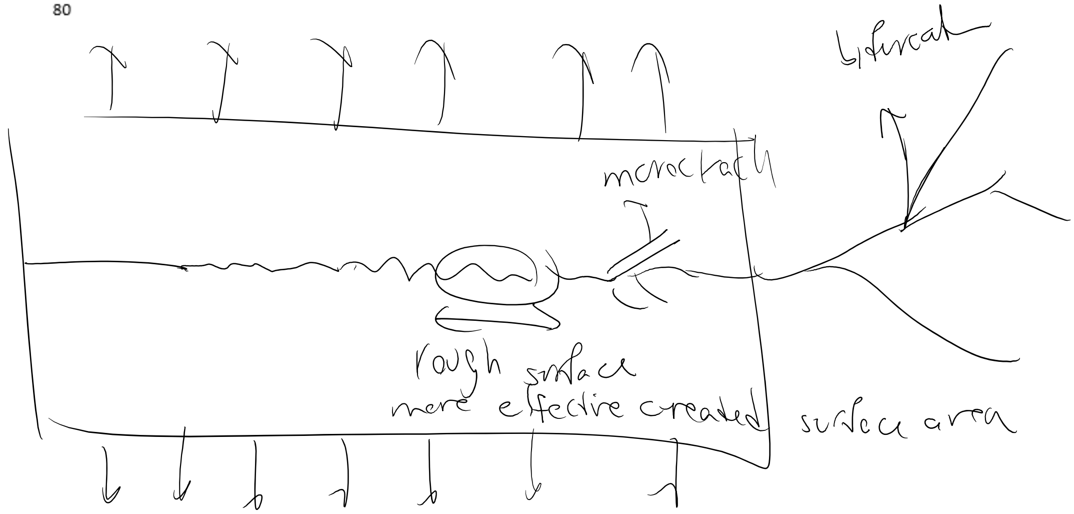
γ_p plastic work per unit area of surface created

$\gamma_p \gg \gamma_s$

$\gamma_p \approx 10^3 \gamma_s$ (metals)

80

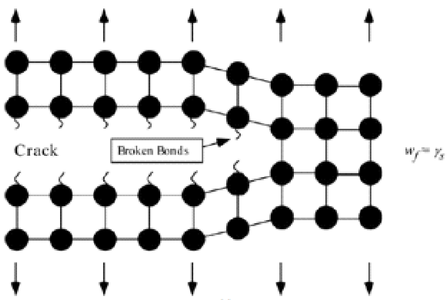
quasi-brittle

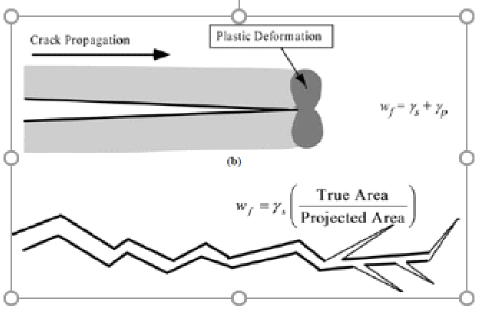


Generalization of Energy equation

$$\sigma_f = \sqrt{\frac{2E w_f}{\pi a}}$$

- w_f: Fracture energy from plastic, viscoelastic, or viscoplastic effects
- w_f can also be influenced by crack meandering and branching
- Caution: If nonlinear displacement regions are large enough this equation is not accurate as it is based on linear elastic solution ($\Pi = \Pi_0 - \frac{\pi\sigma^2 a^2 B}{E}$)





Energy Release rate versus fracture resistance

Energy released is

potential energy $= U_e - W$

$$G = -\frac{d\Pi}{dA}$$

$dA \rightarrow$ surface of crack

how much energy is released per unit area of crack

Energy release rate (Irwin 1956)

$R = 2\gamma_s$ Fracture resistance (toughness)

How much energy is needed to create unit surface of crack.

a few minutes ago we derived $\underbrace{-\frac{d\Pi}{dA}}_{G} = \underbrace{2\gamma_s}_{R}$

for quasi-static crack growth

For crack growth we need to have

$$G \geq R$$