# G in terms of compliance

**Fixed grips**

$$G = \frac{1}{2B}\frac{u^2}{C^2}\frac{dC}{da} = \frac{1}{2B}P^2\frac{dC}{da}$$

**Fixed loads**

$$G = \frac{1}{2B}P^2\frac{dC}{da}$$

for general case $\quad G = \frac{1}{2B}P^2\frac{dC}{da} \quad C = \text{compliance} = \frac{u}{P}$

2D planar crack growth

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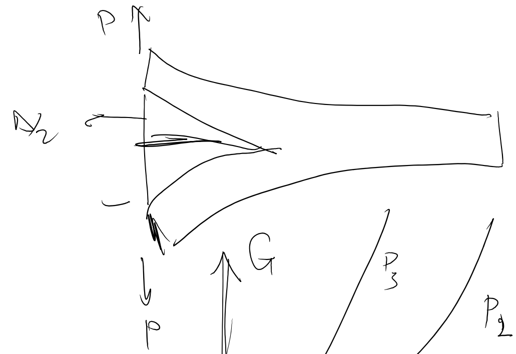
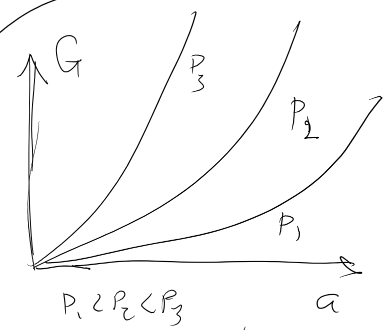
$$\frac{\Delta}{2} = \frac{Pa^3}{3EI} \rightarrow C = \frac{\Delta}{P} = \frac{2a^3}{3EI}$$

$$G = \frac{1}{2B}P^2\frac{dC}{da} \rightarrow$$

$$\boxed{G = \frac{P^2 a^2}{EI}}$$ for this problem




$P_1 < P_2 < P_3$

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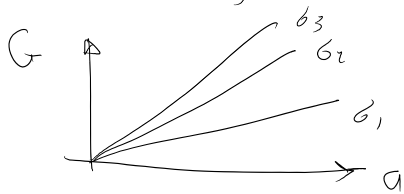
Another example


$$G = \frac{\pi \sigma^2 a}{E}$$




$\sigma_1 < \sigma_2 < \sigma_3$

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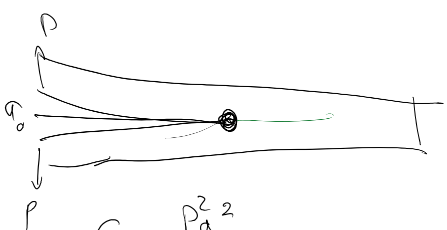
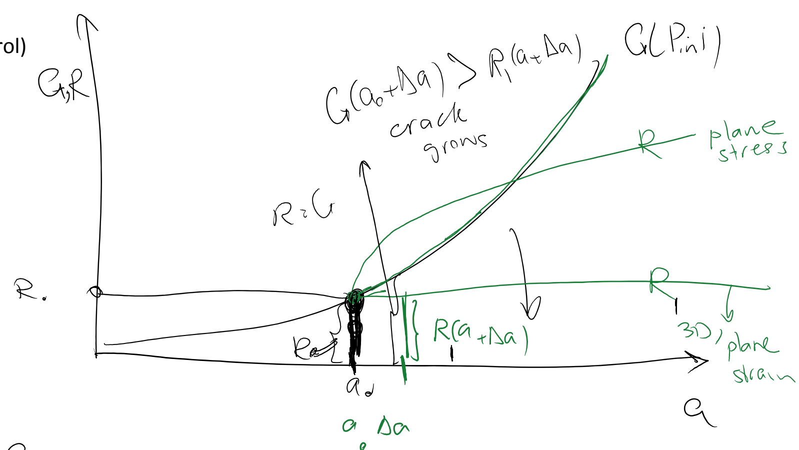
Crack growth for the cantilever geometry (load control)




$$G = \frac{P^2 a^2}{BEI}$$




$$\begin{cases} G(a_0+\Delta a) > R_1(a_0+\Delta a) \\ G(a_0) = R_1(a_0) \end{cases}$$

$$\lim_{\Delta a \to 0} \frac{G(a_0+\Delta a) - G(a_0)}{\Delta a} \geq \frac{R_1(a_0+\Delta a) - R_1(a_0)}{\Delta a}$$

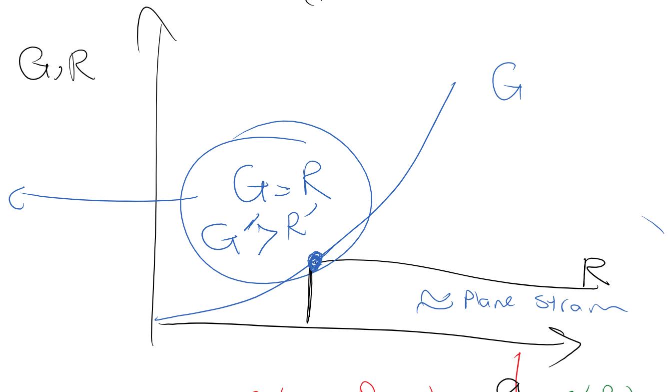
then the crack grows

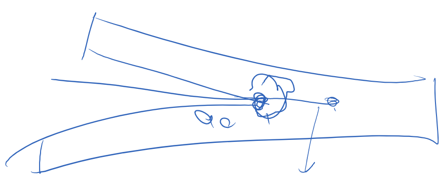
$$\frac{dG}{da}(a_0) \geq \frac{dR_1}{da}(a_0)$$

$G(a) = R(a)$

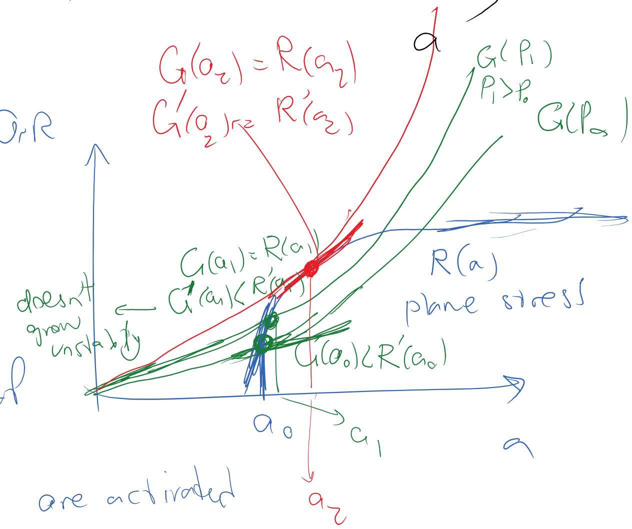
$G'(a) \geq R'(a)$ crack grows

crack unstably grows




process zone forms around crack tip and at early stages of crack growth resistance mechanisms are activated




For this case from $P = P_0$ to $P_2$ crack grows stably

→ we need to increase the load for ~~the~~ crack to grow

Post $P_2$ crack grows unstably → for constant $P_2$ crack grows unboundedly in a dynamic fashion

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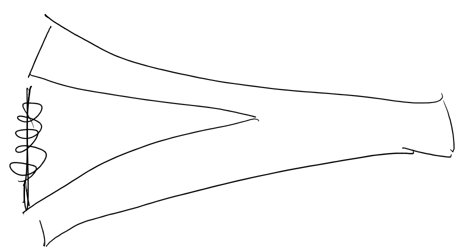
# Displacement control:

Instead of applying a known load at the end points, we pull it with specified displacement

open screws to increase Δ ← Δ



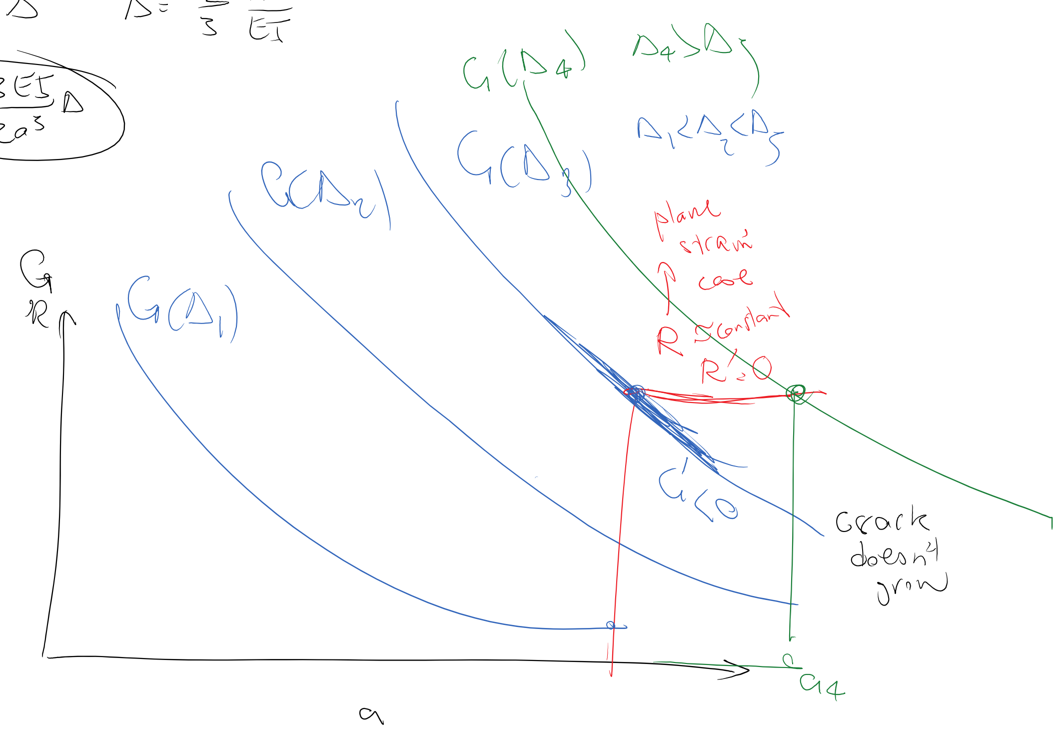
to increase $\Delta$

$$G = \frac{P^2 a^2}{BEI}$$

$$P \leftrightarrow \Delta \qquad \Delta = \frac{2}{3}\frac{Pa^3}{EI}$$

$$P = \frac{3EI}{2a^3}\Delta$$

$$\boxed{G = \frac{9EI}{4B}\frac{\Delta^2}{a^4}}$$



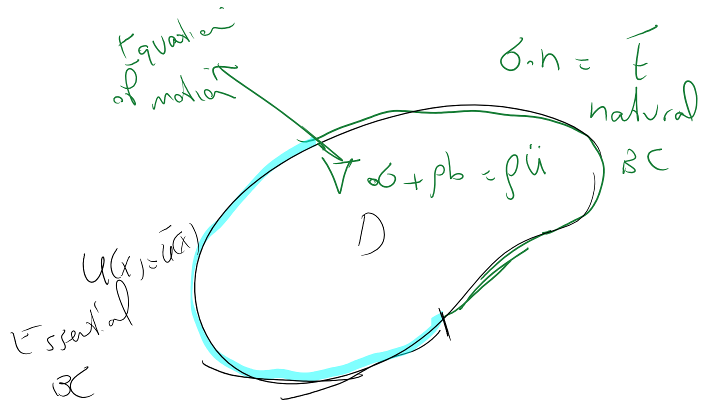

$\Delta \geq \Delta_3$ for the crack to start growing

For displacement-control loading, even if R is constant (e.g. plane strain condition), G' < R' so the crack does not grow in an unstable manner. We need to keep increasing $\Delta$ to have a continued crack growth.

# 4.2. Stress solutions, Stress Intensity Factor K (SIF)

Solving elastodynamic / elastostatic problem:

1. Displacement-based approach




$$\nabla \cdot \sigma + \rho b = \rho \ddot{u}$$

static $\ddot{u} = 0$

$$\nabla \cdot \sigma + \rho b = 0$$

Constitutive eqn $\quad \sigma = \mathbb{C}\,\varepsilon$

$\mathbb{C}$: 4th order elasticity tensor

Example: For isotropic material $\quad \sigma_{ij} = \underbrace{(\lambda\, \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}))}_{C_{ijkl} \text{ for isotropic material}} \epsilon_{kl}$

Lamé's parameters

**Isotropic**

**3D** $\quad T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij} \quad$ or $\quad \mathbf{T} = \lambda \mathbf{I}_E + 2\mu \mathbf{E}$

**2D** (plane strain)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ \nu & \nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

**2D** (plane stress)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$\sigma$ is a linear function of $\varepsilon$

---

strain $\quad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\boxed{\begin{aligned} &\text{2D} \quad && \epsilon_{11} = u_{1,1} \\ & && \epsilon_{22} = u_{2,2} \\ & && \epsilon_{12} = \tfrac{1}{2}(u_{1,2} + u_{2,1}) \end{aligned}}$$

1 derivative

$$\left.\begin{aligned} &\nabla \cdot \sigma + \rho b = 0 \\ &\sigma = C\varepsilon \quad \text{linear relation} \\ &\varepsilon = \tfrac{1}{2}(\nabla u + \nabla u^T) \end{aligned}\right\} \longrightarrow$$

another derivative

$\nabla \cdot C \nabla u + \rho b = 0 \qquad$ 2nd order elliptic PDE for displacement

For isotropic material this PDE is

$$\boxed{(\lambda + \mu)\nabla\nabla.\mathbf{u} + \mu\nabla.\nabla\mathbf{u} = 0} \; \star \quad \text{or} \quad (\lambda + \mu)u_{j,ji} + \mu u_{i,jj} = 0$$

$$\boxed{(\lambda+\mu)\nabla\nabla.\mathbf{u}+\mu\nabla.\nabla\mathbf{u}=0} \quad \text{or} \quad (\lambda+\mu)u_{j,ji}+\mu u_{i,jj}=0$$

1. Using BC & PDE we solve $u = \begin{cases} u_1 \\ u_2 \\ u_3 \end{cases}$

2. $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$

3. $\sigma = C\,\varepsilon$

---

Today, we cover a method that we first compute stress, then strain, and finally displacement.

# Stress function approach

## What are Airy stress function approach?

Use of stress function ⇨

Balance of linear momentum is automatically satisfied (no body force, static)

$$\boxed{\psi(x_1,x_2) \quad \rightarrow \quad \sigma_{ij}=-\psi_{,ij}+\delta_{ij}\psi_{,kk} \quad \rightarrow \quad \sigma_{ij,j}=0}$$

2D geometry

$\psi(x_1,x_2)$ is a scalar function

$\sigma_{ij} = -\psi_{,ij} + \delta_{ij}\,\psi_{,kk}$

$$\forall i,j \in \{1,2\} \quad \sigma_{ij} = -\psi_{,ij} + \delta_{ij}\left(\sum_{k=1}^{2}\psi_{,kk}\right)$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$i=1, j=1$ $\quad \sigma_{11} = -\psi_{,11} + \overset{1}{\delta_{11}}(\psi_{,11}+\psi_{,22}) = \psi_{,22}$

$i=2, j=2$ $\quad \sigma_{22} = -\psi_{,22} + \overset{1}{\delta_{22}}(\psi_{,11}+\psi_{,22}) = \psi_{,11}$

$i=1, j=2$ $\quad \sigma_{12} = -\psi_{,12} + \overset{0}{\delta_{12}} \quad " \quad = -\psi_{,12}$

$\sigma_{21} = -\psi_{,21} = -\psi_{,12} = \sigma_{12}$

$\boxed{\sigma_{11} = \psi_{,22}} \rightarrow$ Airy stress function

$$\boxed{\begin{aligned}\sigma_{11} &= \psi_{,22}\\ \sigma_{22} &= \psi_{,11}\\ \sigma_{21} = \sigma_{12} &= -\psi_{,12}\end{aligned}}$$

Airy stress function

$$\nabla\cdot\sigma = \begin{bmatrix}\sigma_{11,1} + \sigma_{12,2}\\ \sigma_{21,1} + \sigma_{22,2}\end{bmatrix}$$

$$\nabla\cdot\sigma = \begin{bmatrix}(\psi_{,22})_{,1} + (-\psi_{,12})_{,2}\\ (-\psi_{,12})_{,1} + (\psi_{,11})_{,2}\end{bmatrix} = \begin{bmatrix}\psi_{,221} - \psi_{,122}\\ -\psi_{,121} + \psi_{,112}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\boxed{\nabla\cdot\sigma = 0}$$

Starting from any Airy stress function, by using

$\sigma_{11} = \psi_{,22}$ we get a stress field that satisfies

$\sigma_{22} = \psi_{,11}$ static eq. of motion with no body

$\sigma_{12} = -\psi_{,12}$ force

---

Is it too good to be true that we don't need to solve a PDE?
Are there any potential problems?

Example $\psi(x_1, x_2) = x_1^4 \longrightarrow$

$$\boxed{\begin{aligned}\sigma_{11} &= \psi_{,22} = 0\\ \sigma_{22} &= \psi_{,11} = 12x_1^2\\ \sigma_{12} &= -\psi_{,12} = 0\end{aligned}}$$

$\sigma_{11,1} + \sigma_{12,2} = 0$ ✓ verified

$\sigma_{21,1} + \sigma_{22,2} = 0$ ✓

$$u \longrightarrow \varepsilon = \left(\frac{\nabla u + \nabla u}{2}\right) \longrightarrow \sigma = C\varepsilon$$

integrate $\varepsilon$ to get $u$ $\longleftarrow$ $\varepsilon = C^{-1}\sigma$ $\longleftarrow$ $\sigma$

Assume $\nu = 0$

$0 = \sigma_{11} = E\varepsilon_{11} \longrightarrow$

$12x_1^2 = \sigma_{22} = E\varepsilon_{22}$

(a) $\varepsilon_{11} = 0$ | $\varepsilon_{11} = u_{1,1}$

(b) $\varepsilon_{12} = 0$ | $\varepsilon_{12} = \frac{u_{1,2} + u_{2,1}}{2}$ (3)

(c) $\varepsilon_{22} = \frac{12x_1^2}{\ldots}$

$12x_1^2 = \sigma_{22} = E\varepsilon_{22}$

$0 = \sigma_{12} = G\sigma_{12}$

(b) $\varepsilon_{12} = 0$ | $\varepsilon_{12} = \frac{u_{1,2} + u_{2,1}}{2}$ (3)

(c) $\varepsilon_{22} = \frac{12x_1^2}{E}$ | $\varepsilon_{22} = u_{2,2}$

(3a) $\rightarrow$ $u_{1,1} = 0 \rightarrow$ $u_1 = f(x_2)$

(3c) $u_{2,2} = \frac{12x_1^2}{E} \rightarrow$ $u_2 = \frac{12x_1^2 x_2}{E} + g(x_1)$

plug them into (3b)

$\varepsilon_{12} = u_{1,2} + u_{2,1} = 0 \rightarrow$

only a function of $x_2$ | function of $x_1 x_2$ | only a function of $x_1$

$$\boxed{\underbrace{f'(x_2)}_{u_{1,2}} + \underbrace{\frac{24x_1x_2}{E} + g'(x_1)}_{u_{2,1}} = 0}$$

this cannot be satisfied :( why?

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The problem is that we have 3 equations and only 2 unknowns. That's not we cannot always solve for u

$$\boxed{\begin{array}{ll} \text{eq1} & \varepsilon_{11} = u_{1,1} \\ \text{eq2} & \varepsilon_{22} = u_{2,2} \\ \text{eq3} & \varepsilon_{12} = \frac{u_{1,2} + u_{2,1}}{2} \end{array} \quad u_1, u_2 \text{ unknowns } 1, 2}$$

$3 - 2 \longrightarrow$ we need ONE COMPATIBILITY EQUATION

$\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} =$

$u_{1,122} + u_{2,211} - (u_{1,212} + u_{2,112}) = 0$

Compatibility eqn is (4) $\boxed{\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0}$

The Airy stress function, should be chosen such that the compatibility eqn (4) is satisfied. Then, it can be shown that displacements u can be integrated from $\varepsilon_i$

3D compatibility equations:

## Can we always obtain u by integration? No

3 displacements (unknowns)
6 strains (equations)

Need to satisfy strain compatibility condition(s)

$$\boxed{\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0.}$$

2D (I just derived them)

$$2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = 0$$

Strains are obtained from stresses by using the inverse of stiffness matrix (compliance matrix) and stresses are obtained from the Airy stress function

$$\varepsilon_{ij} = \frac{1+\nu}{E}\{-\psi_{,ij} + (1-\nu)\delta_{ij}\psi_{,kk}\}$$

plug this into compatibility eqn

$$2\psi_{,1122} + \psi_{,2222} + \psi_{,1111} = 0 \quad \rightarrow$$

$$(\psi_{,11} + \psi_{,22})_{,11} + (\psi_{,11} + \psi_{,22})_{,22} = 0$$

Compatibility equation

$$\Delta\psi = \psi_{,11} + \psi_{,22}$$

$$(\Delta\psi)_{,11} + (\Delta\psi)_{,22} = 0$$

5 $$\boxed{\Delta(\Delta\psi) = 0}$$

the 4th order PDE for stress function

Biharmonic eqn.

the solution of this is a biharmonic function

---

stress approach

$$\Delta(\Delta\psi) = 0$$

$$\downarrow$$

$$\psi$$

displacement

$$\nabla\cdot(C\nabla u) = 0$$

$$\Delta(\Delta\psi)=0 \;\rightarrow\; \psi \;\rightarrow\; \begin{cases}\sigma_{11}=\psi_{,22}\\ \sigma_{22}=\psi_{,11}\\ \sigma_{12}=-\psi_{,12}\end{cases} \;\rightarrow\; \varepsilon=C^{-1}\sigma \;\rightarrow\; \frac{\nabla u+\nabla u^T}{2}=\varepsilon \rightarrow \text{integrate to get } u\text{'s}$$

$$\nabla\cdot(C\nabla u)=0 \;\rightarrow\; u \;\rightarrow\; \varepsilon=\frac{\nabla u+\nabla u^T}{2} \;\rightarrow\; \sigma=C\varepsilon$$

---

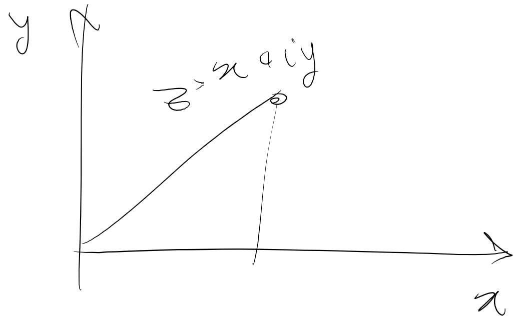
Although the PDE of stress function is more difficult there are a lot of biharmonic functions available

$$f(z)=z^2$$

$$=(x+iy)^2=$$

$$\underbrace{(x^2-y^2)}+i(2xy)$$

$$\underbrace{U(x,y)}_{\text{Real part of } f(z)}+i\,\underbrace{V(x,y)}_{\text{imaginary part}}$$




$$U(x,y)=x^2-y^2 \qquad U_{,xx}+U_{,yy}=2-2=0 \checkmark$$

$$V(x,y)=2xy \qquad V_{,xx}+V_{,yy}=0+0=0 \checkmark$$

Any complex function does this, meaning that its real & imaginary parts are harmonic

$$f(z) = U(x,y) + i\,V(x,y)$$

$$\boxed{\begin{aligned} \Delta U &= 0 \\ \Delta V &= 0 \end{aligned}}$$

both are harmonic $\longrightarrow$

$$\left.\begin{aligned} \Delta\Delta U &= 0 \\ \Delta\Delta V &= 0 \end{aligned}\right\}$$ they are biharmonic as well

$$f(z) = z^3$$

Next time

- Any biharmonic solution can be expressed by Kolonov-Muskhelishvili complex potentials, $\phi, \chi$:

$$\Psi(x_1, x_2) = \mathrm{Re}\,[\bar{z}\phi + \chi]$$

Conjugate of

$$z = x + iy$$

$$\bar{z} = x - iy$$

two complex functi

Why real and imaginary parts of a complex function are harmonic?

$$f(z) = U(x,y) + i\,V(x,y)$$

$$\frac{\partial f}{\partial x} = U_{,x} + i\,V_{,x}$$

$$\frac{\partial^2 f}{\partial x^2} = U_{,xx} + i\,V_{,xx}$$

Similarly

$$\frac{\partial^2 f}{\partial y^2} = U_{,yy} + i\,V_{,yy}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(U_{,xx} + U_{,yy}\right) + i\left(V_{,xx} + V_{,yy}\right)}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = f' \cdot 1 \qquad \frac{\partial z}{\partial x} = \frac{\partial (x+iy)}{\partial x} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = f'' \cdot 1 = f'' \qquad f'' = \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = f''\cdot 1 = f'' \qquad f'' = \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = f' i \qquad \frac{\partial z}{\partial y} = \frac{\partial (x+iy)}{\partial y} = i$$

$$\frac{\partial^2 f}{\partial y^2} = f''(i)(i) = -f'' \qquad \boxed{i^2 = -1}$$

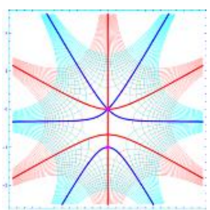
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f'' + (-f'') = 0$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(U_{,xx} + U_{,yy}\right) + i\left(V_{,xx} + V_{,yy}\right) \Rightarrow$$

$$\boxed{\begin{aligned} U_{,xx} + U_{,yy} &= 0 \\ V_{,xx} + V_{,yy} &= 0 \end{aligned}}$$

Real & Imaginary parts of a complex function are harmonic functions

$f(z) = z^2 \qquad U(x,y) = x^2 - y^2 \qquad V(x,y) = 2xy$



Use for us:

Real and imaginary parts of a complex function satisfy

$$\nabla^2 U = 0 \qquad \& \qquad \nabla^2 \psi = 0$$

We were looking for a bi-harmonic function =>

We were looking for Airy stress function satisfying $\nabla^2\nabla^2\psi = 0$

Obviously if a function is harmonic

$$\nabla^2\psi = 0$$

Then it is automatically a bi-harmonic funciton

$$\nabla^2\nabla^2\psi = 0$$

The real and imaginary parts of a complex function are both also bi-harmonic functions