2018/09/17

Monday, September 17, 2018 11:40 AM

From last time, we obtained the following PDE for stress function

$$\Delta(\Delta\psi) = 0$$

$\psi \rightarrow$ stress function

- Note: real & imag. parts of complex functions are harmonic ($\Delta^2 U = 0$, $\Delta^2 V = 0$)

$f(z) = U(x,y) + iV(x,y)$

Stress function approach

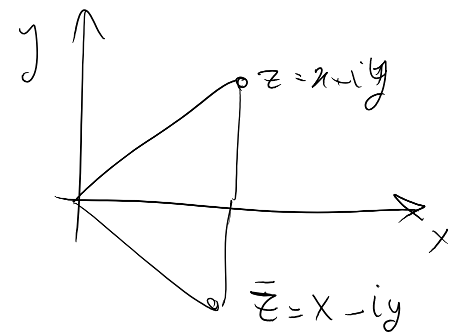
- Any biharmonic solution can be expressed by Kolonov-Muskhelishvili complex potentials, ϕ, χ:

$$\Psi(x_1, x_2) = \mathrm{Re}\left[\bar{z}\phi + \chi\right]$$

$\longrightarrow$ Any solution of biharmonic eqn $\Delta(\Delta\psi) = 0$ can be written in this form

$$\psi(x_1, x_2) = \mathrm{Re}(\bar{z}\phi + \chi)$$

$\phi, \chi \rightarrow$ two complex functions



$$\sigma_{11} = \psi_{,22}$$

$$\sigma_{22} = \psi_{,11}$$

$$\sigma_{12} = -\psi_{,12}$$

$\phi, \chi \longrightarrow \sigma, u, \varepsilon$

$$\Psi(x_1, x_2) = \mathrm{Re}\left[\bar{z}\phi + \chi\right] \Longrightarrow$$

- Stresses are obtained differentiation,

$$\sigma_{11} = \Psi_{,22} = \mathrm{Re}\left[\phi' - \frac{1}{2}\bar{z}\phi'' - \frac{1}{2}\chi''\right]$$

$$\sigma_{22} = \Psi_{,11} = \mathrm{Re}\left[\phi' + \frac{1}{2}\bar{z}\phi'' + \frac{1}{2}\chi''\right]$$

$$\sigma_{12} = -\Psi_{,12} = \frac{1}{2}\mathrm{Re}\left[\bar{z}\phi'' + \chi''\right]$$

- Displacements are obtained by integration of strains:

$$u_1 = \mathrm{Re}\left[\kappa\phi - \bar{z}\phi' - \chi'\right]$$

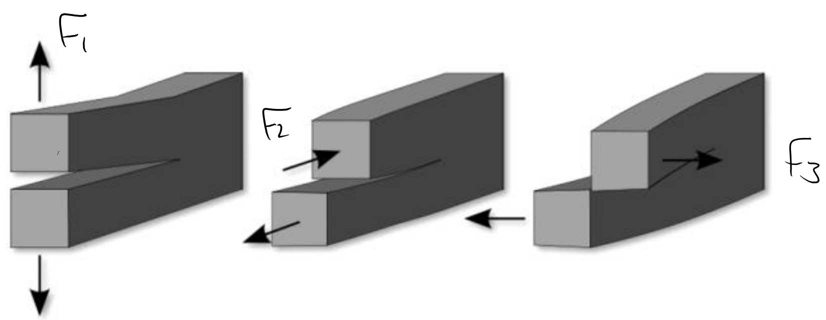
$$u_2 = \mathrm{Im}\left[\kappa\phi + \bar{z}\phi' + \chi'\right]$$

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

We'll use this approach to get the solution for a problem with a sharp crack

Modes of fracture:

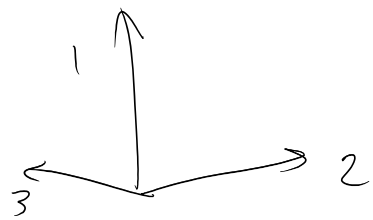
Modes of fracture:



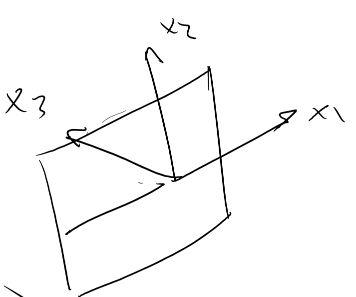
Mode I: Opening

Mode II: In-plane shear

Mode III: Out-of-plane shear

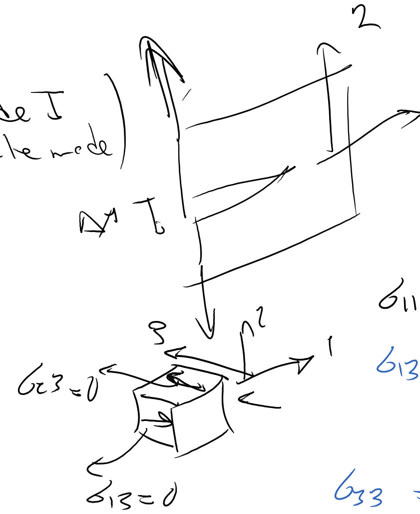


Mode I & II



Mode I & mode II are in-plane modes

(mode I tensile mode)



$u_1, u_2 \neq 0$

$u_3 = 0$ p. strain

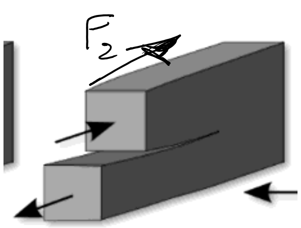
$\sigma_{11}, \sigma_{22}, \sigma_{12} \neq 0$

$\sigma_{13} = \sigma_{23} = 0$ out of plane shear stresses

$\sigma_{33} = 0$ for plane stress

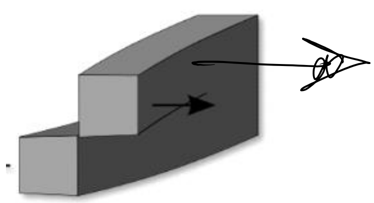
$\epsilon_{33} = 0$ " " strain

Mode 2



in-plane mode

Mode 3



easiest

out-of-plane

in-planes

$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0$

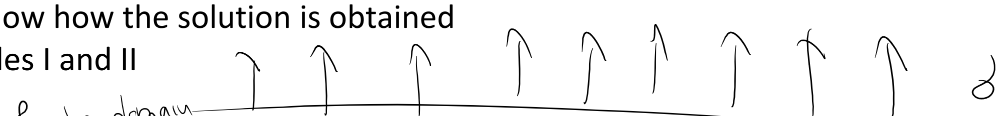
$u_1 = u_2 = 0$

out of plane $\sigma_{13}, \sigma_{23} \neq 0$

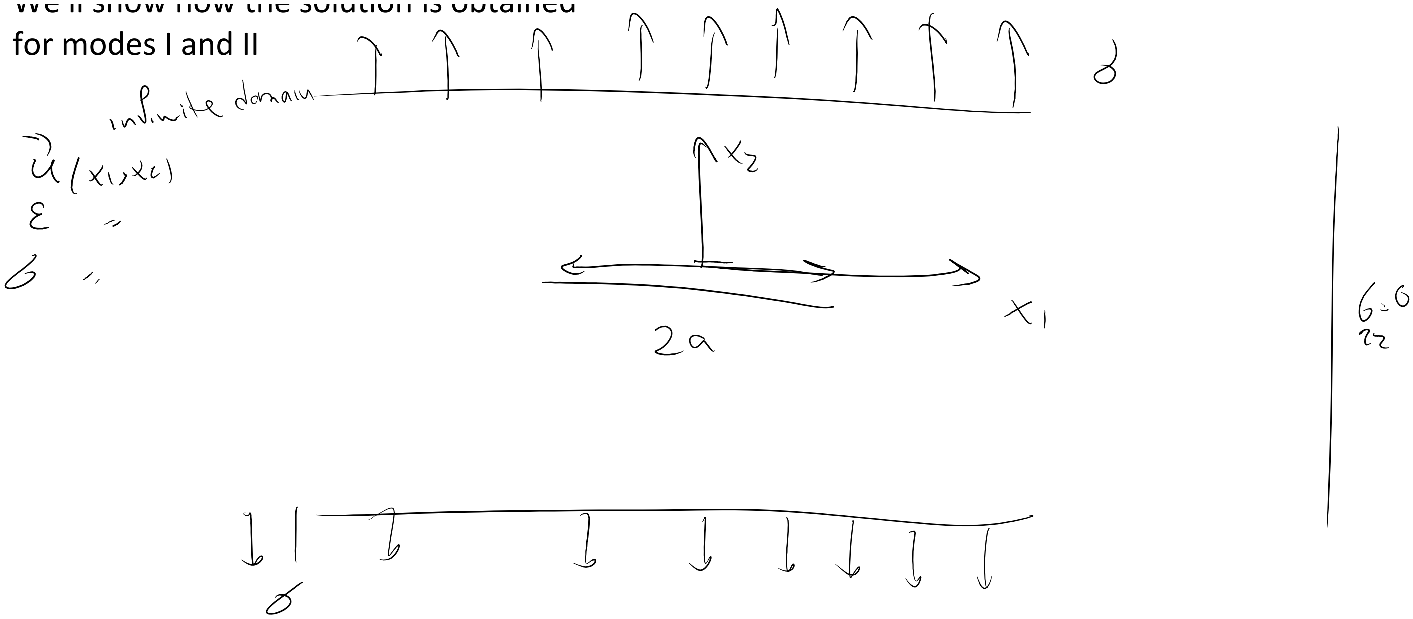
$u_3(x_1, x_2) \neq 0$

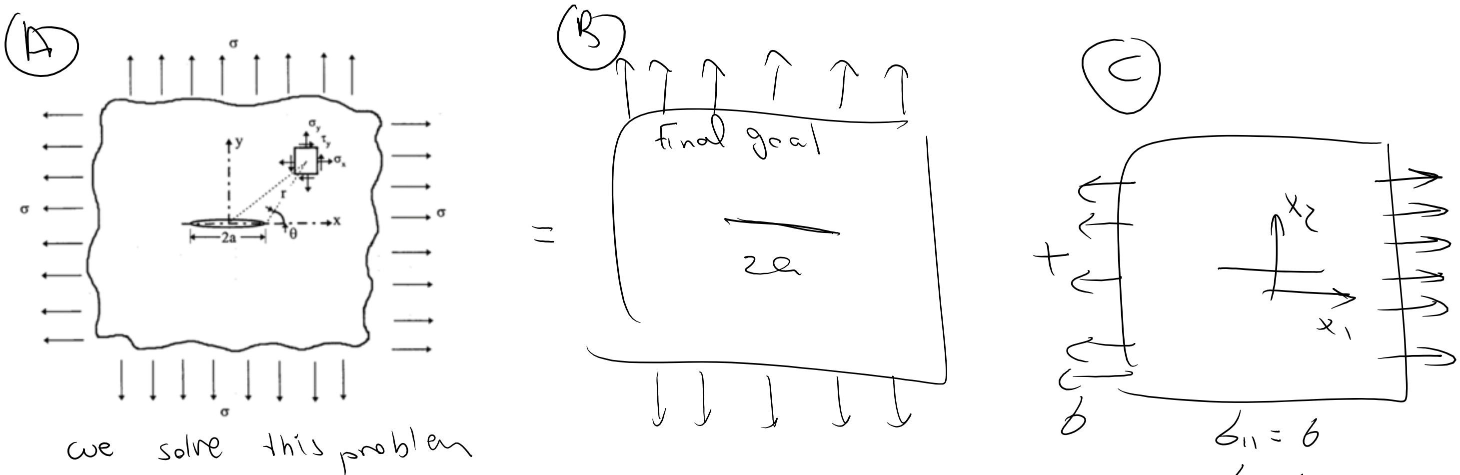
Mode III: Out-of-plane shear

We'll show how the solution is obtained for modes I and II



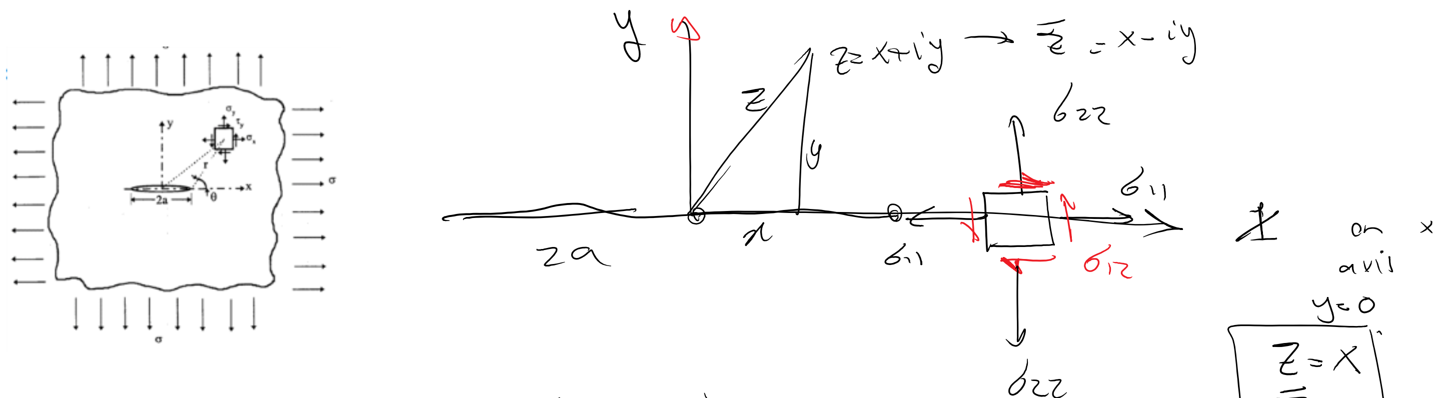
We'll show how the solution is obtained for modes I and II





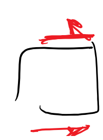
(A) & (B) are the same except a constant ($\sigma_{11} = \sigma$)

so by solving (A) we basically have the solution to (B)



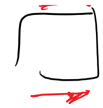
$$\boxed{\begin{aligned} z &= x \\ \bar{z} &= x \end{aligned}}$$

$\sigma_{12} = 0$ on $y = 0$ line (x axis) because of symmetry



$$\sigma_{12} = -\psi_{,} = \frac{1}{\ } \mathrm{Re}\left(\bar{z}\phi'' + \chi''\right)$$

$\sigma_{12} = -\psi_{,12} = \frac{1}{2} \mathrm{Re}(\bar{z}\phi'' + \chi'')$



on x axis $\bar{z} = z$

$\sigma_{12} = 0 \qquad \frac{1}{2}\mathrm{Re}(z\phi'' + \chi'') = 0$

what if $\quad z\phi'' + \chi'' = 0 \longrightarrow$ integrate

$$\chi' = -z\phi' + \phi \longrightarrow \text{''}$$

$$\boxed{\chi = -z\phi + 2\tilde{\phi}} \qquad \tilde{\phi}' = \phi$$

anti-derivative of ϕ

plug into

$$\underbrace{\psi}_{\text{stress function}} = \mathrm{Re}(\bar{z}\phi + \chi) = \mathrm{Re}\Big((x-yi)\phi - \overbrace{(x+yi)}^{z}\phi + 2\tilde{\phi}\Big)$$

$$\Longrightarrow \qquad \psi = 2\left[y\,\mathrm{Im}\,\phi + \mathrm{Re}\,\tilde{\phi}\right]$$

Let $\quad 2\phi = \tilde{Z} \qquad (\tilde{Z}' = 2\phi)$

$$\boxed{\psi = y\,\mathrm{Im}\,\tilde{Z} + \mathrm{Re}\,\tilde{\tilde{Z}}}$$

$Z(x,y)$ arbitrary complex function $\quad$ (I)

$\tilde{\tilde{Z}}' = \tilde{Z}$

$\tilde{Z}' = Z$

$\sigma_{11} = \psi_{,22}$

$\sigma_{22} = \psi_{,11}$

$\sigma_{12} = -\psi_{,12}$

use eqn (I) to get stresses

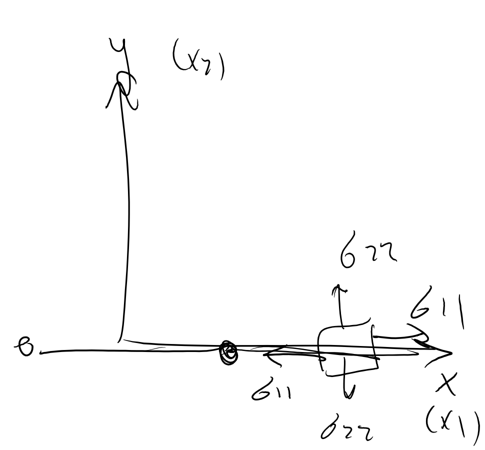
$\quad D \rightarrow \quad T \rightarrow' \qquad \psi\ (x,y$

II

$$\sigma_{11} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z'$$

$$\sigma_{22} = \mathrm{Re}\, Z + y\, \mathrm{Im}\, Z'$$

$$\sigma_{12} = -y\, \mathrm{Re}\, Z'$$

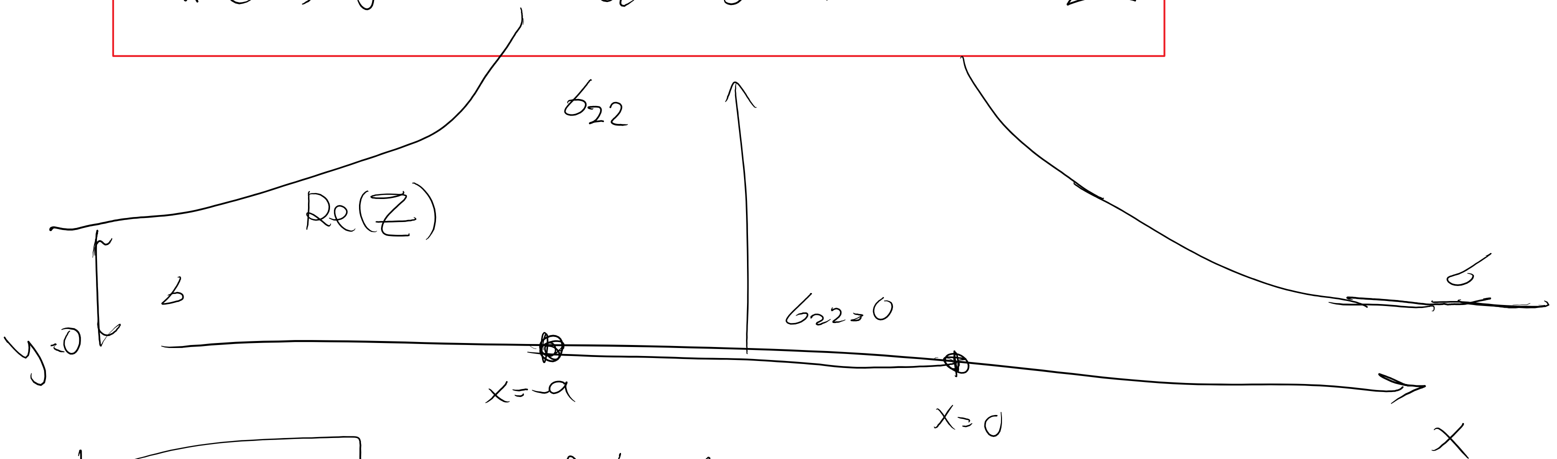


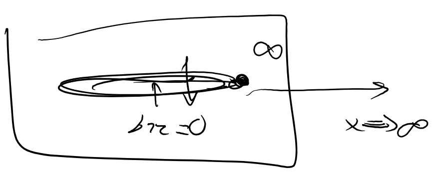
Q1: what is $\sigma_{12}(x, y=0)$ =

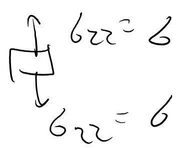
Q2: what are σ_{11} & σ_{22} on $y=0$ (crack line)

III

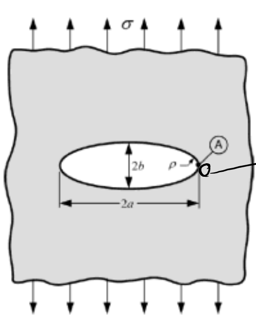
$$\boxed{\sigma_{11}(x, y=0) = \sigma_{22}(x, y=0) = \mathrm{Re}\, Z}$$







for field stress



$$\sigma_{22} = \sigma\left(1 + \frac{2a}{b}\right)$$

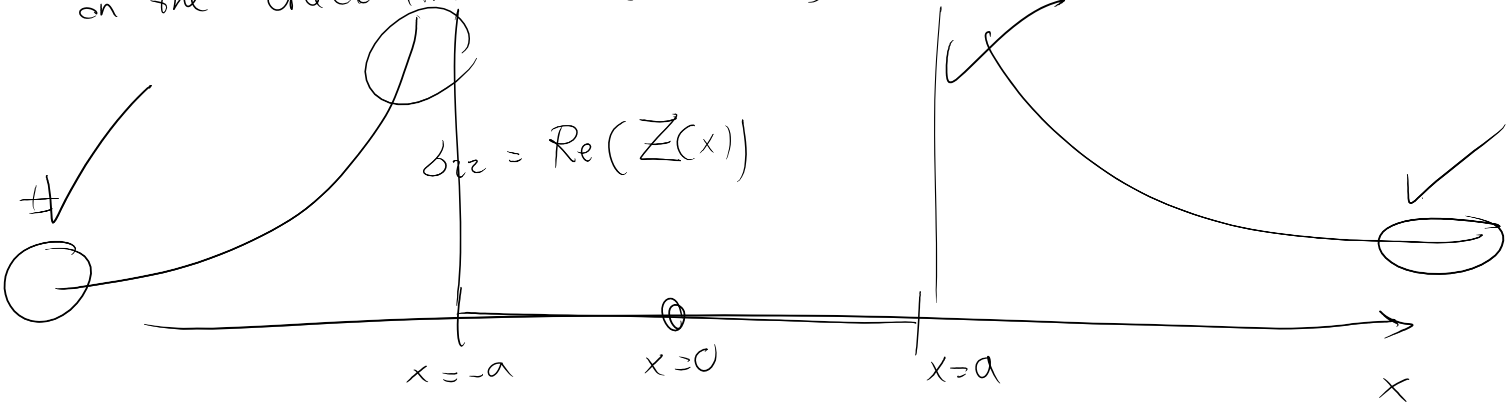
$b \to 0$

$z = x + iy$

$$Z(z) = Z(x, y) \qquad z = x$$

on crack line
$y=0$

on the crack line we're working $Z(x)$

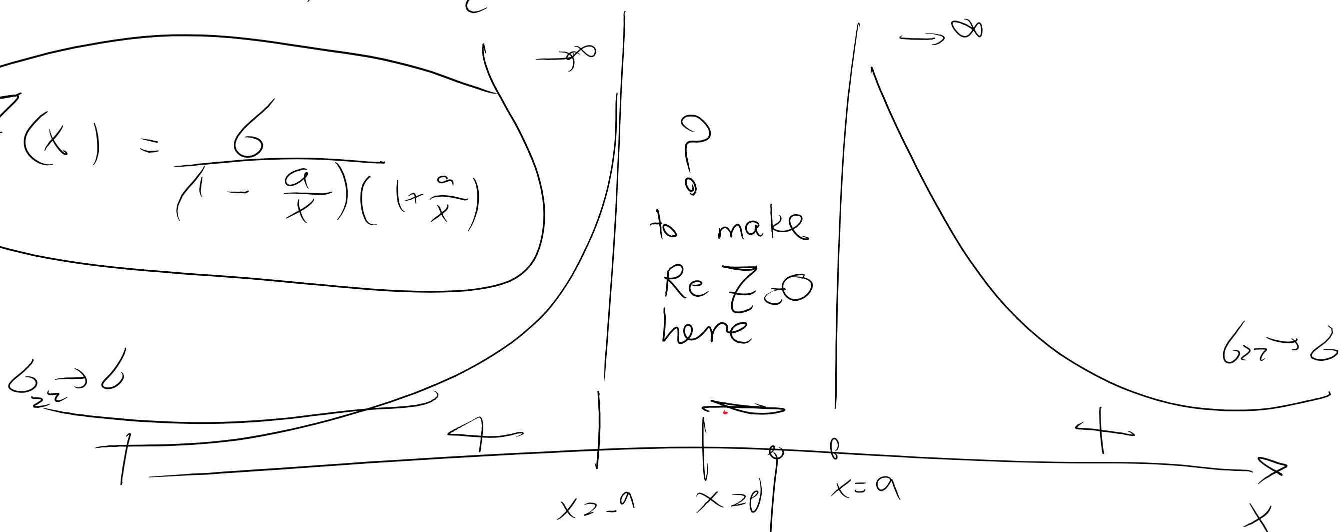


$$Z(x) = \frac{\sigma}{1-\frac{a}{x}} \qquad x \to \pm\infty \quad \mathrm{Re}\, Z = Z \to \sigma$$

$x \to a^+ \qquad \sigma(a^+) \to \infty$ ✓

$\sigma(-a) = \frac{\sigma}{2}$ ✗

$$Z(x) = \frac{\sigma}{\left(1-\frac{a}{x}\right)\left(1+\frac{a}{x}\right)}$$



$\sigma_{zz} = \mathrm{Re}(Z)$

$$\sigma_{zz} = \mathrm{Re}\, \frac{\sigma}{1-\left(\frac{a}{a/2}\right)^2} = -\frac{\sigma}{3}$$

take square root

$$Z(x) = \frac{\sigma}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \qquad x \to z$$

$$\text{(III)} \quad \boxed{Z(z) = \frac{\sigma}{\sqrt{1-\left(\frac{a}{z}\right)^2}}}$$

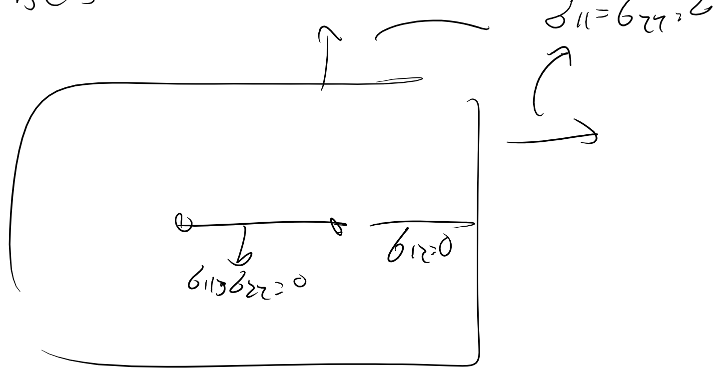
suggested function

$$\sigma_{11} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z'$$

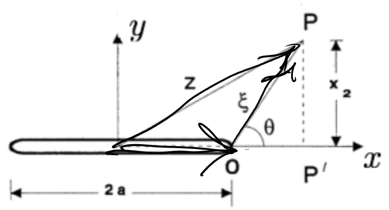
$$\sigma_{22} = \text{''} + \text{''}$$

$$\sigma_{12} = -y\, \mathrm{Re}\, Z'$$

it's easy to check that all BC's are satisfied



Asymptotic fields around a crack tip:



$$z = a + \xi$$

ξ: coordinate w.r.t crack tip

$$Z(z) = \frac{\sigma z}{\sqrt{z^2 - a^2}} = \frac{\sigma a\left(1 + \xi/a\right)}{\sqrt{(a+\xi)^2 - a^2}}$$

$$\rightarrow \boxed{Z(\xi) = \frac{\sigma a\left(1 + \xi/a\right)}{\sqrt{2\xi a\left(1 + \frac{\xi}{2a}\right)}}}$$

$$\frac{\xi}{a} \ll 1$$

$$Z(\xi) \simeq \frac{\sigma\sqrt{a\pi}}{\sqrt{2\pi\xi}}$$

Stress intensity factor

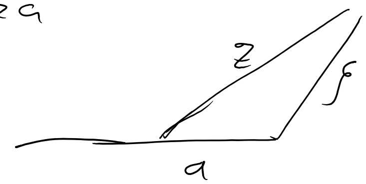
K_I

$$\boxed{Z(\xi) = \frac{K_I}{\sqrt{2\pi\xi}} \qquad K_I = \sigma\sqrt{\pi a}}$$

very close to the crack $\left|\frac{\xi}{a}\right| \ll 1$

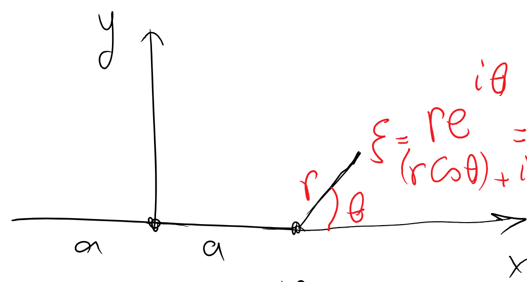
$$\sigma_{11} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z'$$

$$Z' = \frac{\partial Z}{\partial z} = \frac{\partial Z}{\partial \xi}\frac{\partial \xi}{\partial z} = \frac{\partial Z}{\partial \xi}\frac{\partial (z-a)}{\partial z}$$



$$Z' = \frac{\partial Z}{\partial \xi}$$

$$\sigma_{11} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, \frac{\partial Z}{\partial \xi}$$



$\xi = re^{i\theta} = (r\cos\theta) + i(r\sin\theta)$

$$Z(\xi) = \frac{K_I}{\sqrt{2\pi\xi}}$$

$$\frac{\partial Z}{\partial \xi} = -\frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\xi^{-3/2}$$

$\xi = re^{i\theta} \rightarrow$

$\xi^{-\frac{1}{2}} = \frac{1}{\sqrt{r}} e^{-i\frac{\theta}{2}}$

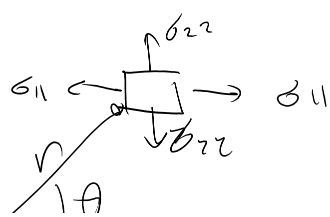
$\frac{1}{\sqrt{r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$

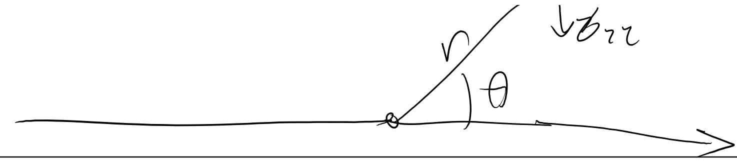
$$\sigma_{11} = \mathrm{Re}\, \frac{K_I}{\sqrt{2\pi\xi}} - \underbrace{y}_{r\sin\theta}\, \mathrm{Im}\left(-\frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\xi^{-3/2}\right)$$

$\xi^{-\frac{3}{2}} = \frac{1}{r\sqrt{r}} e^{-\frac{3i\theta}{2}}$

$= \frac{1}{\sqrt{r}\,r}\left(\cos\frac{3\theta}{2} - i\sin\frac{3\theta}{2}\right)$

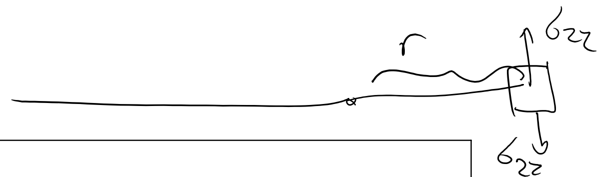
$$\boxed{\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)}$$



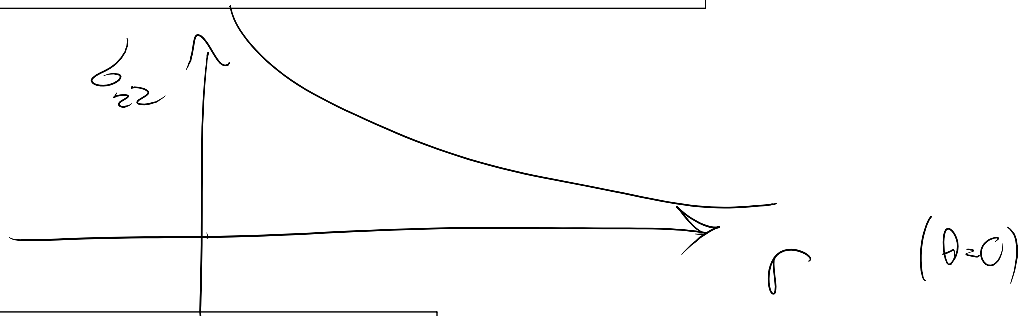


$$\boxed{\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)}$$

$\sigma_{22}\ (r, \theta = 0)$



$$\boxed{\sigma_{22}(r, \theta = 0) = \frac{K_I}{\sqrt{2\pi r}} \quad \text{on the crack line}}$$



$$\boxed{K_I = \lim_{r \to 0} \sigma_{22}(r, \theta = 0)\sqrt{2\pi r}}$$

Mode II solution looks very much the same

Mode II problem

Boundary conditions

$$\boxed{\begin{array}{l} (x, y) \to \infty : \sigma_{xx} = \sigma_{yy} = 0, \tau_{xy} = \tau \\ |x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0 \end{array}}$$

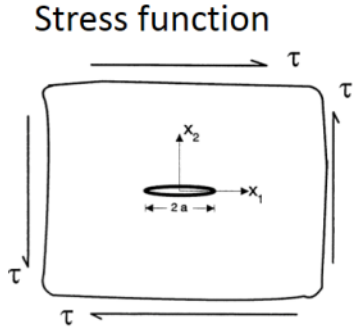
Stress function

$$\underbrace{\boxed{Z = -\frac{i\tau z}{\sqrt{z^2 - a^2}}}}_{\text{mode II}}$$

Z

Mode I

$$Z = \frac{\sigma z}{\sqrt{z^2 - a^2}}$$



Check BCs

Universal nature of the asymptotic stress field

Westergaards, Sneddon etc.

$$\sigma_{xx}^{I} = \frac{K_I}{\sqrt{2\pi r}}\underbrace{\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]}_{f_{11}^{I}}$$

$$\sigma_{yy}^{I} = \frac{K_I}{\sqrt{2\pi r}}\underbrace{\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]}_{f_{22}^{I}}$$

$$\tau_{xy}^{I} = \frac{K_I}{\sqrt{2\pi r}}\underbrace{\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}_{f_{12}^{I}}$$

(mode I)

$$\sigma_{xx}^{II} = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

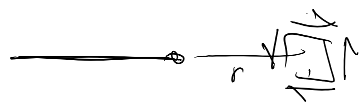
$$\sigma_{yy}^{II} = \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \qquad f_{22}^{II}(\theta)$$

$$\tau_{xy}^{II} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

(mode II)

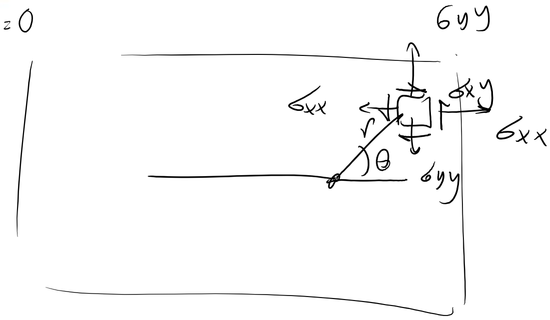
$\tau_{xy} = 0$

$\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$



$\sigma_{xx} = \sigma_{yy} = 0$

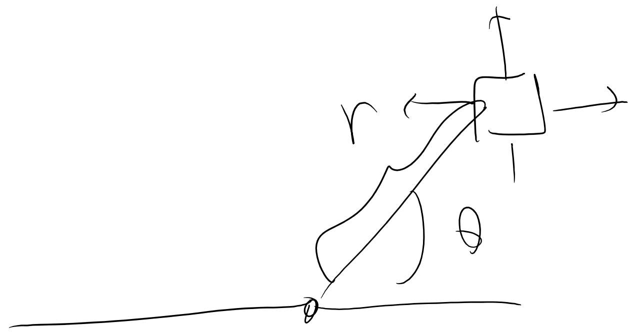
$\sigma_{12} = \frac{K_{II}}{\sqrt{2\pi r}}$



Mode III

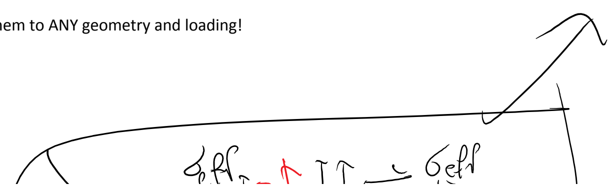
$\sigma_{13}^{III}(r,\theta)$

$\sigma_{23}^{III}(r,\theta)$



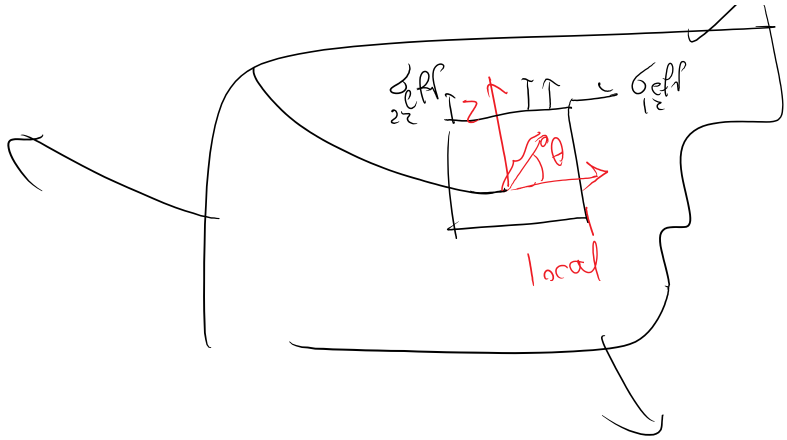
$$\boxed{\sigma_{ij}(r,\theta) = \frac{K^{I}}{\sqrt{2\pi r}} f_{ij}^{I}(\theta) + \frac{K^{II}}{\sqrt{2\pi r}} f_{ij}^{II}(\theta) + \frac{K^{III}}{\sqrt{2\pi r}} f_{ij}^{III}(\theta)}$$

We got the solutions for an infinite domain, but generalized them to ANY geometry and loading!



$\sigma_{eff} \uparrow \uparrow\uparrow \; \sigma_{eff}$

$K_I = \sqrt{\pi a}\, \sigma_{eff}^{22}$



$K_I = \sqrt{\pi a}\,\sigma^{eff}_{22}$

Similitude argument: solution fields around all crack tips are the same, at very close distance to them.

Basically by having KI, KII, and KIII we have the complete stress, strain, and displacement fields around the crack tip.