2018/09/19

Wednesday, September 19, 2018 10:05 AM

Strain solution around the crack tip:

$$\sigma = C\,\varepsilon \longrightarrow \varepsilon = C^{-1}\sigma$$

$$\boxed{\sigma \propto \frac{1}{\sqrt{r}}}$$

known

stiffness

around the crack tip

$$\boxed{\varepsilon \propto \frac{1}{\sqrt{r}}}$$

Displacement field:

ε ✓

$$\varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

$$\left(1D \quad \varepsilon = u_{,x} \qquad u = \int \varepsilon\, dx\right)$$

$$\varepsilon = \frac{1}{\sqrt{x}} \rightarrow u \propto \sqrt{x}$$

Mode I: displacement field

$$Z(z) = \frac{K_I}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

$$Z(z) = \frac{K_I}{\sqrt{2\pi\xi}} \qquad \bar{Z} = \int Z(z)dz$$

Recall

$$\boxed{\begin{aligned} 2\mu u &= \frac{\kappa-1}{2}\mathrm{Re}\,\tilde{Z} - y\,\mathrm{Im}Z \\ 2\mu v &= \frac{\kappa+1}{2}\mathrm{Im}\,\tilde{Z} - y\,\mathrm{Re}Z \end{aligned}}$$

$$\tilde{Z}(z) = 2\frac{K_I}{\sqrt{2\pi}}\xi^{1/2} = 2K_I\sqrt{\frac{r}{2\pi}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad z = \xi + a, \quad \xi = re^{i\theta}$$

$$\boxed{\begin{aligned} u &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 + 2\sin^2\frac{\theta}{2}\right) \\ v &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 - 2\cos^2\frac{\theta}{2}\right) \end{aligned}}$$

Kolosov coef. κ

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

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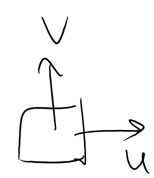
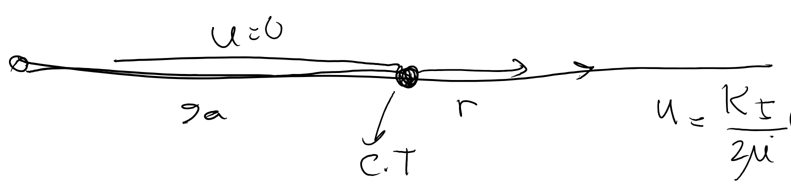
$$u, v \propto \sqrt{r}$$

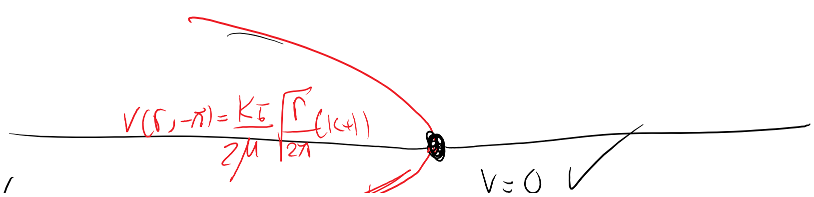


$\theta = \pi$ $\qquad$ $\theta = 0$

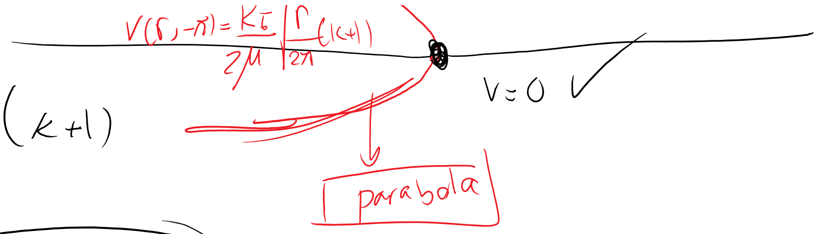


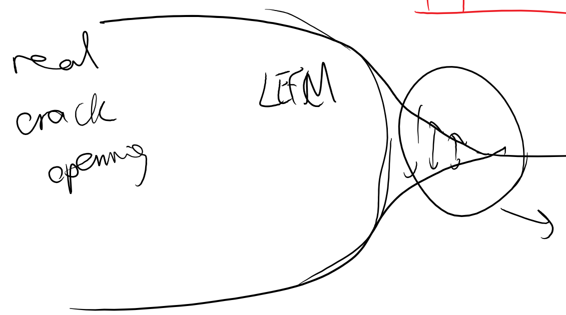
$$u = \frac{K_I}{2\mu}(\kappa - 1)\sqrt{\frac{r}{2\pi}}$$

$V(r, \theta = \pi) =$



$v(r, \theta = \pi) = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \sin\frac{\pi}{2}(\kappa+1)$



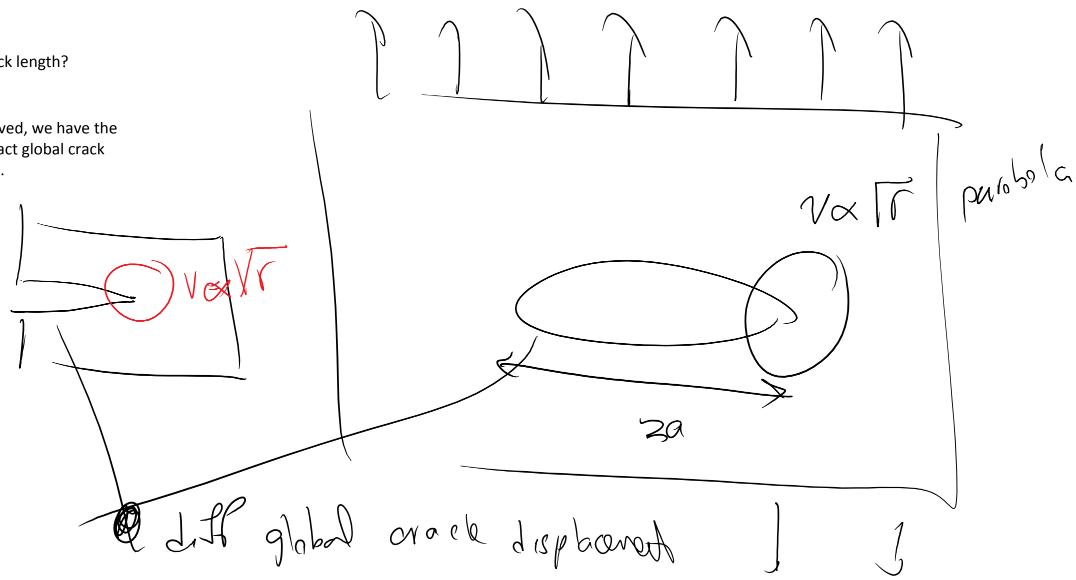


Contributing to $\sigma \propto \frac{1}{\sqrt{r}}$ & not being singular

This solution is **general** and **asymptotic** (r -> 0)

What if we want the displacement field across the whole crack length?

For the mid-crack geometry in an infinite domain that we solved, we have the exact stress function -> by integration of it we can get the exact global deformation (away from the crack tip) ONLY for this problem.



diff global crack displacement

Crack face displacement

$$y = 0, -a \leq x \leq a$$

$$2\mu v = \frac{\kappa+1}{2}\mathrm{Im}\tilde{Z} - y\mathrm{Re}Z \longrightarrow v = \frac{\kappa+1}{4\mu}\mathrm{Im}\tilde{Z}$$

$$Z(z) = \frac{\sigma x}{\sqrt{x^2-a^2}} \longrightarrow \tilde{Z}(z) = \sigma\sqrt{x^2-a^2}$$

$$-a \leq x \leq a \quad i = \sqrt{-1} \longrightarrow \tilde{Z}(z) = i(\sigma\sqrt{a^2-x^2})$$

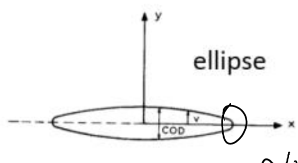
$$\boxed{v = \frac{\kappa+1}{4\mu}\sigma\sqrt{a^2-x^2}}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{ellipse}$$

$$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$$

Crack Opening Displacement

$$\mathrm{COD} = 2v = \frac{\kappa+1}{2\mu}\sigma\sqrt{a^2-x^2}$$



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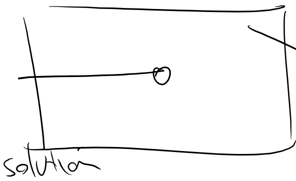
$\approx$ a parabola $\propto \sqrt{r}$

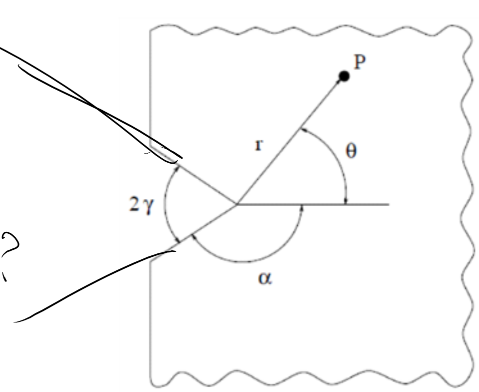
As shown in the course presentation file, we can get principal stresses and stress in polar coordinate from the universal solutions.

or $\gamma = 0$

$\alpha = \pi$

which we got the solution sharp crack





What is the stress solution for this?

are the stresses singular?

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

polar coordinate

$\lambda = ?$

stress function

$$\nabla^2 \nabla^2 \psi = 0 \qquad \psi = r^{\lambda+1} \underbrace{F(\theta)}_{?}$$

plug ψ in $\nabla^2\nabla^2\psi = 0 \longrightarrow$

$$\underbrace{\frac{\partial^4 F}{\partial \theta^4}}_{m^4 F} + \underbrace{2(\lambda^2+1)\frac{\partial^2 F}{\partial \theta^2}}_{2(\lambda^2+1)m^2} + (\lambda^2-1)^2 F = 0$$

$F = e^{m\theta} \longrightarrow$

$$\left[m^2 + (1-\lambda)^2\right]\left[m^2 + (1+\lambda)^2\right] = 0$$

$\longrightarrow \quad m = \pm i(1-\lambda)$

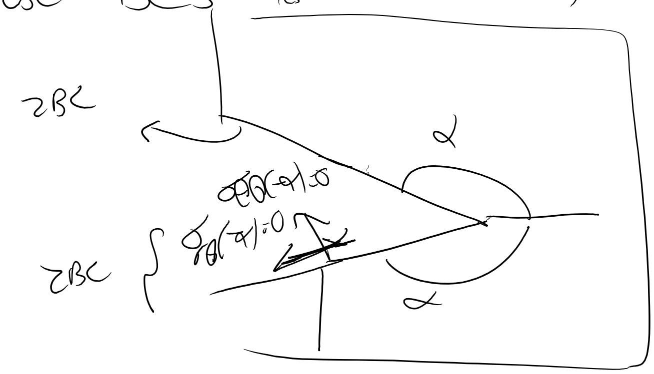
$\qquad\qquad\quad\;\; \pm i(1+\lambda)$

$\longrightarrow$ $F(\theta)$'s are $\cos(\lambda-1)\theta$, $\cos(1+\lambda)\theta$, $\sin(\lambda-1)\theta$, $\sin(\lambda+1)\theta$

$$\boxed{\Phi(r,\theta) = r^{\lambda+1} \underbrace{\left[A\cos(\lambda-1)\theta + B\cos(\lambda+1)\theta + C\sin(\lambda-1)\theta + D\sin(\lambda+1)\theta\right]}_{F(\theta,\lambda)}}$$

2 der. → σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$

& use BCs to obtain A, B, C, D



- Stress values

$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = r^{\lambda-1}\lambda(\lambda+1)F(\theta)$$
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = r^{\lambda-1}[-\lambda F'(\theta)]$$

- Boundary conditions ⇒

$$\sigma_{\theta\theta}|_{\theta=\pm\alpha} = 0$$
$$\sigma_{r\theta}|_{\theta=\pm\alpha} = 0 \Rightarrow$$

$$F(\alpha) = F(-\alpha) = F'(\alpha) = F'(-\alpha) = 0 \Rightarrow$$

$$\boxed{\begin{bmatrix} \cos(\lambda-1)\alpha & \cos(\lambda+1)\alpha & 0 & 0 \\ \omega\sin(\lambda-1)\alpha & \sin(\lambda+1)\alpha & 0 & 0 \\ 0 & 0 & \sin(\lambda-1)\alpha & \sin(\lambda+1)\alpha \\ 0 & 0 & \omega\cos(\lambda-1)\alpha & \cos(\lambda+1)\alpha \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0}$$

for [A, B, C, D] ≠ 0 det must be zero

- Eigenvalues and eigenvectors for nontrivial solutions

eq1 $\sin 2\lambda_n\alpha + \lambda_n \sin 2\alpha = 0$ ← **Mode I**

eq2 $\sin 2\xi_n\alpha - \xi_n \sin 2\alpha = 0$ ← **Mode II**

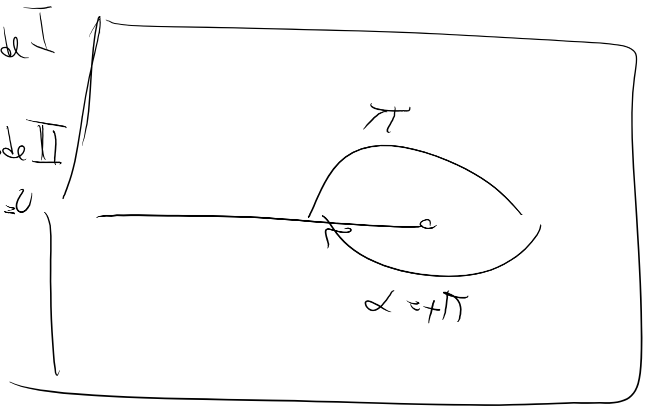
you'll use these eqns to obtain powers of singularity for mode I & II

$\alpha = \pi$: sharp crack

eq1 $\sin 2\lambda_n\pi + \lambda_n \cancel{\sin 2\pi} = 0$ mode I

eq2 $\sin 2\lambda_n\pi - \lambda_n \cancel{\sin 2\pi} = 0$ mode II

$\theta = 0$



in both cases we have

$\sin 2\lambda_n\pi = 0$ →

$2\lambda_n\pi = n\pi$, $n \in \mathbb{Z}$ → $\boxed{\lambda_n = \frac{n}{2}}$

$\lambda_n = \ldots, -\frac{3}{2}, -\frac{2}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots$

$\sigma \propto r^{\lambda-1}$

$\lambda - 1 = -\frac{5}{2}, -\frac{3}{2}, -1, \boxed{-\frac{1}{2}}, 1, \frac{1}{2}, \ldots$

$\sigma\alpha \rightarrow \frac{1}{\sqrt{r}}$

what about these terms?

$\frac{1}{\sqrt{r}}$ $\frac{1}{r}$ $\sqrt{r}$

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$\frac{5}{2}$

$r^{2.5}$

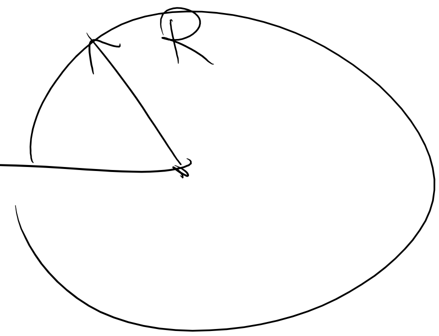
what about these terms?

$r^{2.5}$

The problem with stronger singularities is infinite internal energy!

$$U(R) = \int_0^R \underbrace{\sigma : (C^{-1} \sigma)}_{\varepsilon} \, \underbrace{dV}_{r\,dr\,d\theta\,dt}$$

thickness



$\sigma \propto r^{\lambda-1}$ $\quad \varepsilon \propto r^{\lambda-1}$ $\quad r$

$$\propto \int_0^R r^{2\lambda-1}$$

$2\lambda - 1 \geq 0$ $\qquad \lambda \geq \frac{1}{2}$ $\rightarrow$ $\sigma(r)$ is weaker than $\frac{1}{\sqrt{r}}$

higher order terms are added to get more accurate s/n around the crack tip

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$|x| < 1$

dominant term we keep for $|x| \ll 1$

added as $x \to 1$

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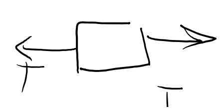
around the crack tip

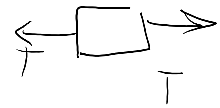
$$\sigma_{11}(r, \theta) = \frac{K_I}{\sqrt{2\pi r}} f_{11}^{\theta}(\theta) + T$$



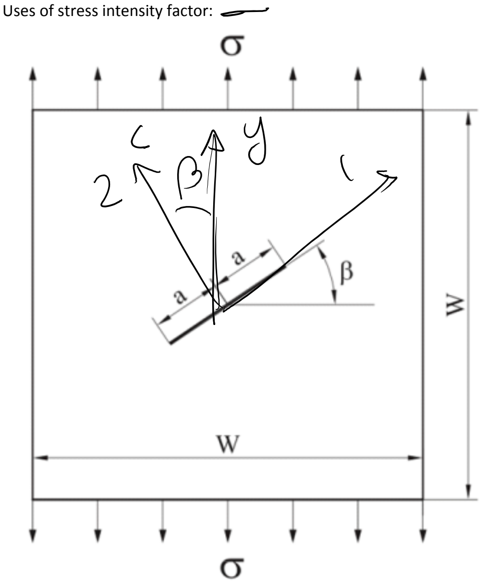
$$\sigma_{11}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{11}(\theta) + T$$

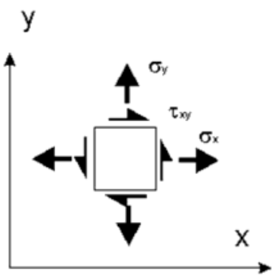
$\frac{K_I}{\sqrt{2\pi r}}$: 1st term

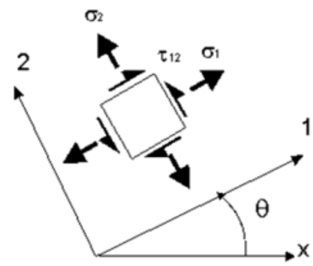
T stress: second term of the asymptotic expansion

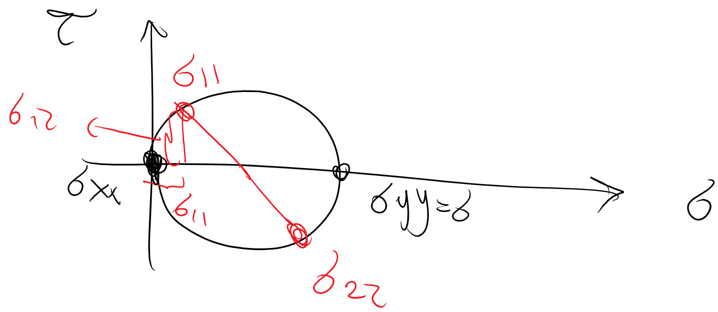


Uses of stress intensity factor:



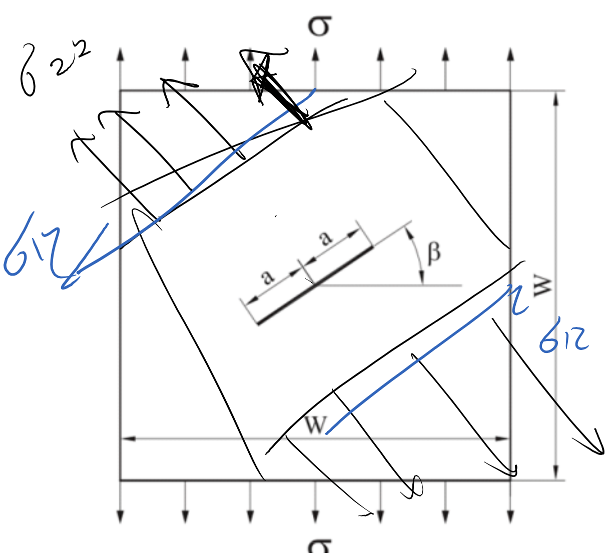






$$\sigma_{11} = (1-\cos 2\beta)\,\sigma/2$$
$$\sigma_{22} = (1+\cos 2\beta)\,\sigma/2 \longrightarrow$$
$$\sigma_{12} = \sigma/2 \,\sin 2\beta$$

$$\boxed{\begin{aligned}\sigma_{11} &= \sigma \sin^2\beta \\ \sigma_{22} &= \sigma \cos^2\beta \\ \sigma_{12} &= \sigma \sin\beta\cos\beta\end{aligned}}$$



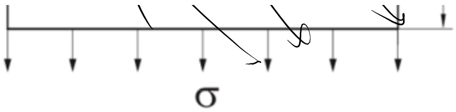
$K_I = ?$

$K_{II} = ?$

$$K_I = \sigma_{22}\sqrt{\pi a}$$

$$K_{II} = \sigma_{12}\sqrt{\pi a}$$

$$K_I = \sigma\sqrt{\pi a}\cos^2\beta$$

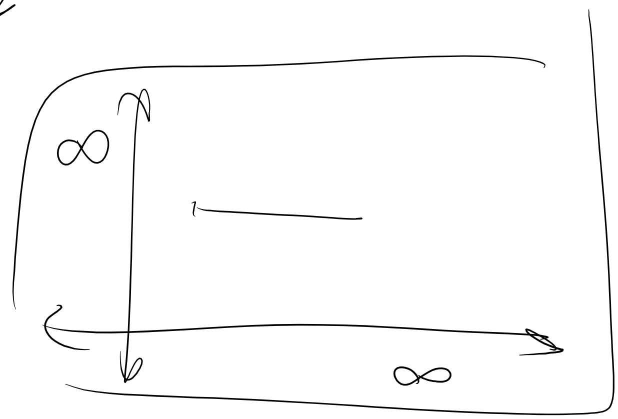


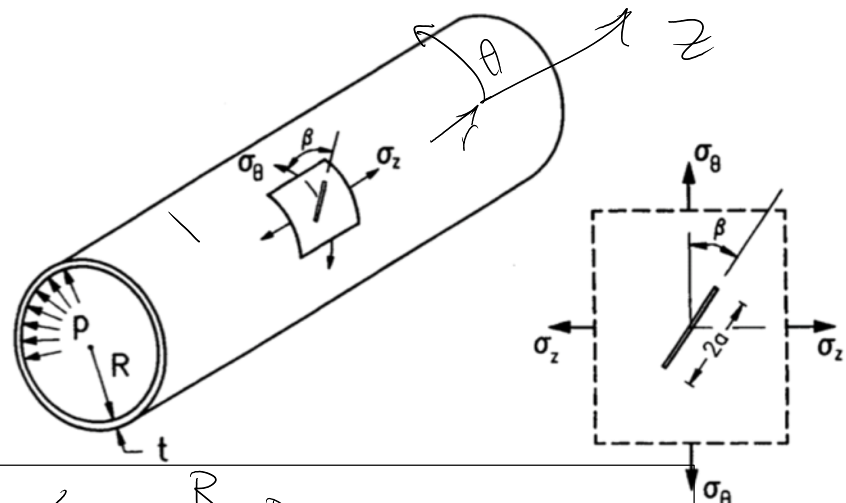
σ_{zz}

$$K_I = \sigma\sqrt{\pi a}\cos^2\beta$$
$$K_{II} = \sigma\sqrt{\pi a}\sin\beta\cos\beta$$

approximate

$K_I = \sigma\sqrt{\pi a}$ exact for



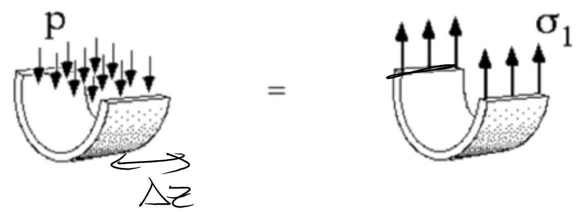


$p(\pi R^2) = \sigma_z(2\pi R t)$

$\rightarrow \boxed{\sigma_z = \frac{R}{2t}p}$

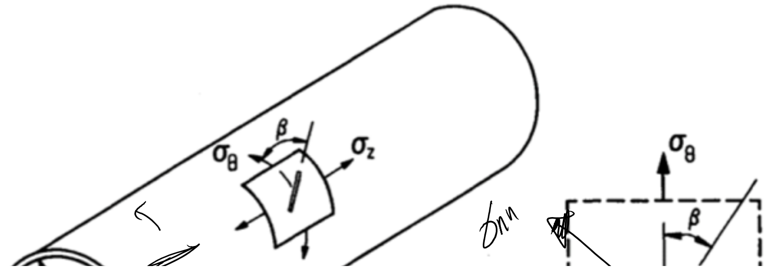
$$\sigma_z = \frac{R}{2t}p$$
$$\sigma_\theta = \frac{R}{t}p = 2\sigma_z$$
$$\sigma_r = [0 \rightarrow p] \ll \sigma_z, \sigma_\theta \qquad \frac{R}{t} \gg 1$$

ignored

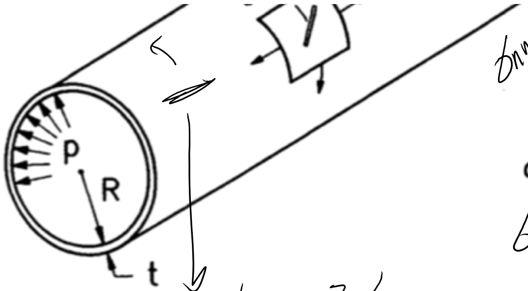


$p(2R\Delta z) = \sigma_\theta(2t\Delta z)$

$\rightarrow \boxed{\sigma_\theta = \frac{R}{t}p}$

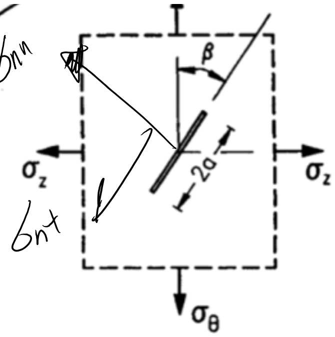


$$\sigma_{nn} = \sigma_z\cos^2\beta + \sigma_\theta\sin^2\beta$$
$$\sigma_{nt} = -\sigma_z\sin\beta\cos\beta + \sigma_\theta\sin\beta\cos\beta$$



$\sigma_z = 2\sigma_\theta$ since

this is the worst direction for the crack



σ_{nn}

σ_{nt}

$\sigma_{nn} = \sigma_z \sin^2\beta + \sigma_\theta \cos^2\beta$

$$\boxed{\sigma_z = \frac{R}{2t}p, \quad \sigma_\theta = \frac{R}{t}p}$$

$$\boxed{\begin{aligned}\sigma_{nn} &= \frac{PR}{2t}(1+\sin^2\beta)\\ \sigma_{nt} &= \frac{PR}{2t}\sin\beta\cos\beta\end{aligned}}$$

$$\boxed{\begin{aligned}K_I &\approx \sigma_{nn}\sqrt{\pi a}\\ K_{II} &\approx \sigma_{nt}\sqrt{\pi a}\end{aligned}}$$

Computation of SIFs

- Analytical methods (limitation: simple geometry)

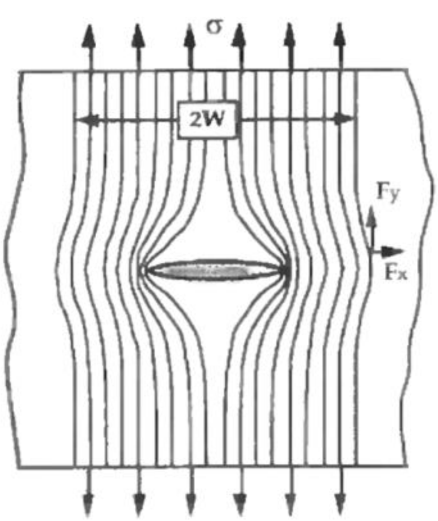
  - superposition methods
  - weight/Green functions

- Numerical methods (FEM, BEM, XFEM) → 2nd term project

  numerical solutions -> data fit -> **SIF handbooks**

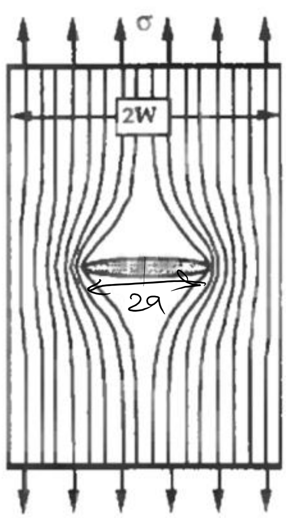
  pre computed SIFs & best fit { curves, eqns }

- Experimental methods
  - photoelasticity



(a) Infinite plate

or $\frac{a}{W} \to 0$



(b) Finite plate

W

$f(a/W, \text{geom})$

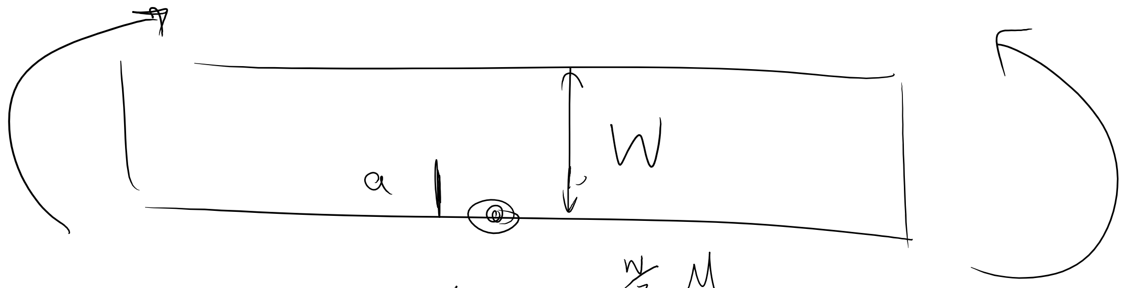
$$K = \frac{1}{\sqrt{\cos\frac{\pi a}{W}}}\sigma\sqrt{\pi a}$$

$$f\left(\frac{a}{W} \to 0\right) = 1 \quad \checkmark$$

$$f\left(\frac{a}{W} \to \frac{1}{2}\right) \to \infty$$

$K_I = \sigma\sqrt{\pi a}$

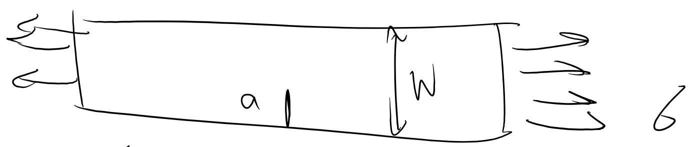
VY



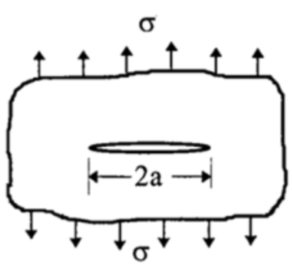
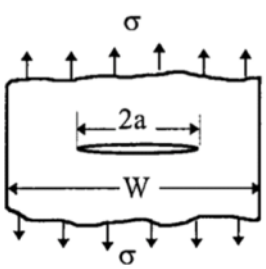
σ_{max}

$\sigma_{max} = \frac{\frac{W}{2} M}{EI}$

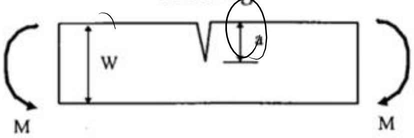
$$\boxed{K = f(geom)\,\sigma_{max}\sqrt{\pi a} \qquad geom: \frac{a}{W}}$$



$\sigma_{max} = \sigma$

Geometry	Stress Intensity Factor
1. Crack in an infinite body 	$K_I = \sigma\sqrt{\pi a}$
2. Centre crack in a strip of finite width 	f(geom) $K_I = \sqrt{\sec\frac{\pi a}{W}}\,\sigma\sqrt{\pi a}$ $\sigma_{max}\sqrt{\pi a}$ $K_I = \frac{1}{\cos\frac{\pi a}{W}}\,\sigma\sqrt{\pi a}$

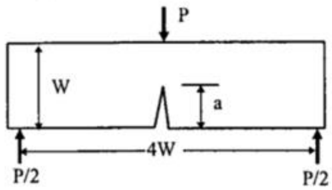
5. Edge crack in a beam of width B subjected to bending



$$K_I = f\left(\frac{a}{W}\right)\sigma\sqrt{\pi a} \quad \text{where } \sigma = \frac{6M}{BW^2}$$

a/W	$f(a/W)$
0.1	1.044
1.5 →	1.0495
0.2	1.055
0.3	1.125
0.4	1.257
0.5	1.500
0.6 ✓	1.915

9. Single-edge notch bend (SENB), thickness B

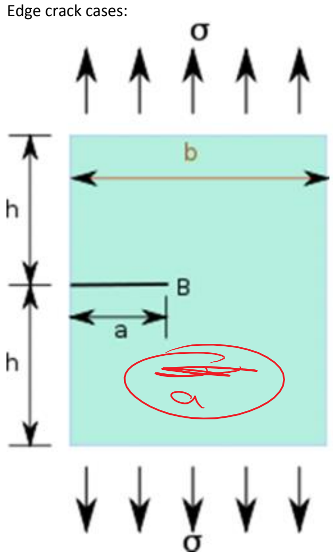


$$B = W/2$$

$$K_I = Y\frac{4P\sqrt{\pi}}{B\sqrt{W}}$$

$$Y = 1.63\left(\frac{a}{W}\right)^{1/2} - 2.6\left(\frac{a}{W}\right)^{3/2} + 12.3\left(\frac{a}{W}\right)^{5/2} - 21.3\left(\frac{a}{W}\right)^{7/2} + 21.9\left(\frac{a}{W}\right)^{9/2}$$

Edge crack cases:



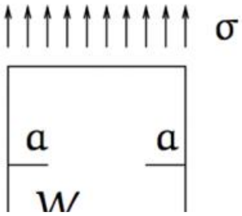
$h/b \geq 1$ and $a/b \leq 0.6$

$$K_I = \sigma\sqrt{\pi a}\left[1.12 - 0.23\left(\frac{a}{b}\right) + 10.6\left(\frac{a}{b}\right)^2 - 21.7\left(\frac{a}{b}\right)^3 + 30.4\left(\frac{a}{b}\right)^4\right]$$

$$\frac{a}{b} \to 0$$

$K_I = 1.12\,\sigma\sqrt{\pi a}$ — worse — edge crack

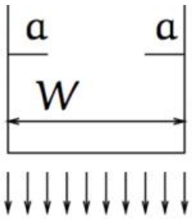
$K_I = 1\,\sigma\sqrt{\pi a}$ — Mid-crack



$$K_I = \sigma\sqrt{a}\left[1.12\sqrt{\pi} + 0.76\frac{a}{W} - 8.48\left(\frac{a}{W}\right)^2 + 27.36\left(\frac{a}{W}\right)^3\right]$$

$$\frac{a}{W} \to 0$$

$K_I = 1.12\,\sigma\sqrt{\pi a}$



$$8.48\left(\frac{a}{W}\right)^2 + 27.36\left(\frac{a}{W}\right)^3\Big]$$

$$\approx 1.12\sigma\sqrt{\pi a}$$

$$\boxed{K_I = 1.12\,\sigma\sqrt{\pi a}}$$

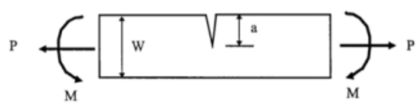
For more complicated situations we use superposition:

Superposition method

A sample in mode I subjected to tension and bending:

$$\sigma_{ij} = \frac{K_I^{\text{tension}}}{\sqrt{2\pi r}} f_{ij}(\theta) + \frac{K_I^{\text{bending}}}{\sqrt{2\pi r}} f_{ij}(\theta)$$

$$\sigma_{ij} = \frac{K_I^{\text{tension}} + K_I^{\text{bending}}}{\sqrt{2\pi r}} f_{ij}(\theta)$$

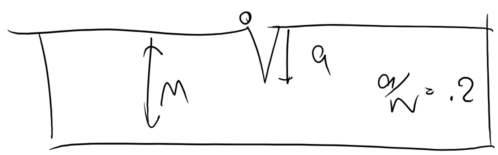


$$\boxed{K_I = K_I^{\text{tension}} + K_I^{\text{bending}}}$$

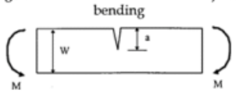
$$\sigma_{Max} = \frac{6M}{BW^2}$$

$$a/W = 0.2$$

$$K_I^{\text{bend}} = \underbrace{f_M(a/W)}_{1.055} \overbrace{\frac{6M}{BW^2}}^{\sigma_{max}} \sqrt{\pi a}$$



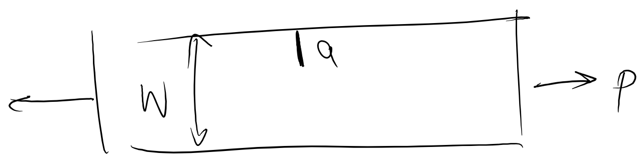
5. Edge crack in a beam of width B subjected to bending



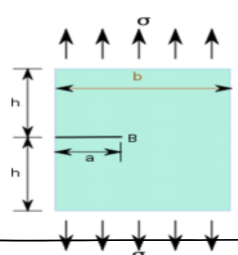
$K_I = f\left(\frac{a}{W}\right)\sigma\sqrt{\pi a}$ where $\sigma = \frac{6M}{BW^2}$

a/W	f(a/W)
0.1	1.044
0.2	1.055

$$K_I^{\text{tension}} = \underbrace{f_T\left(\frac{a}{W}\right)}_{1.12} \underbrace{\left(\frac{P}{BW}\right)}_{\sigma_{max}} \sqrt{\pi a}$$



Using:



$$K_I = \sigma\sqrt{\pi a}\left[1.12 - 0.23\left(\frac{a}{b}\right) + 10.6\left(\frac{a}{b}\right)^2 - 21.7\left(\frac{a}{b}\right)^3 + 30.4\left(\frac{a}{b}\right)^4\right]$$

$$\boxed{K = 1.055\,\frac{6M}{BW^2}\sqrt{\pi a} + 1.12\,\frac{\sigma}{BW}\sqrt{\pi a}}$$

$$\boxed{K = 1.055\,\frac{6M}{BW^2}\sqrt{\pi a} + 1.12\,\frac{6}{BW}\sqrt{\pi a}}$$