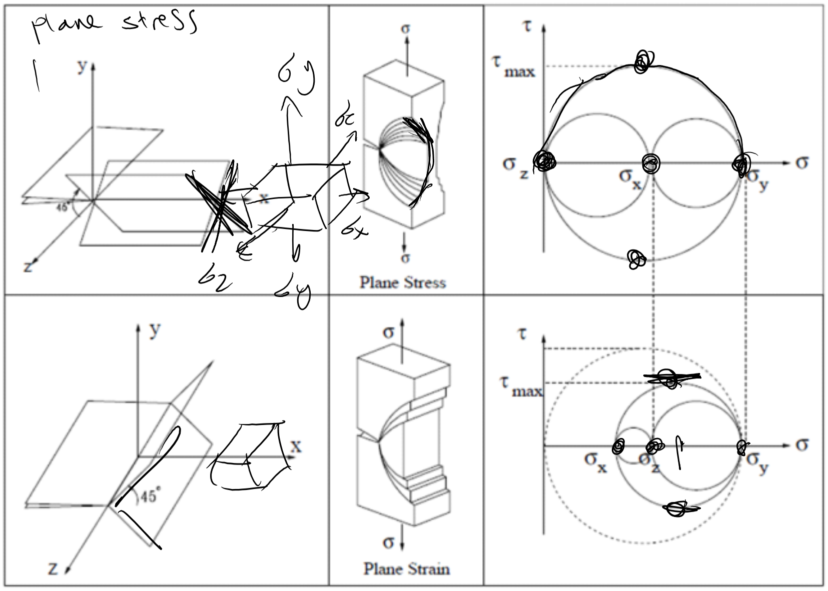
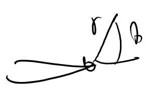
2018/10/03

Wednesday, October 03, 2018 11:37 AM



$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

J integral

	LEFM	PFM (plastic)
Global concept "Energy"	$G = -\dfrac{d\Pi}{dA}$ energy release rate $G = R_{es} \rightarrow$ crack can grow	J — J integral gives energy release rate even when we don't have linear elastic response — J integral $J = G$ for LEFM
Local view "stress"	 $\sigma_{yy}^{I} = \dfrac{K^{I} f_{yy}^{I}(\theta)}{\sqrt{2\pi r}}$ $G = \dfrac{K_I^2}{E'} + \dfrac{K_{II}^2}{E'}$	 $\sigma_{yy} = J^{\alpha} r^{-\beta}$

$G :=$ energy release rate
↳ also determines asymp stress field

J plays the role of G in PFM

Eshelby, Cherepanov, 1967, ~~Ric~~

- Components of J integral vector

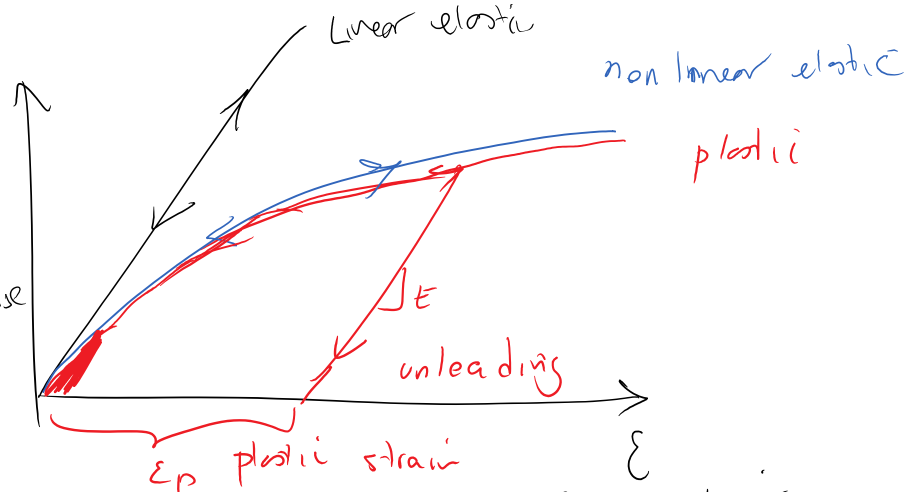
$$J_k = \int_{\Gamma} \left(W n_k - t_i \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$ → For nonlinear elastic

Rice, 1968

J of nonlinear elastic material

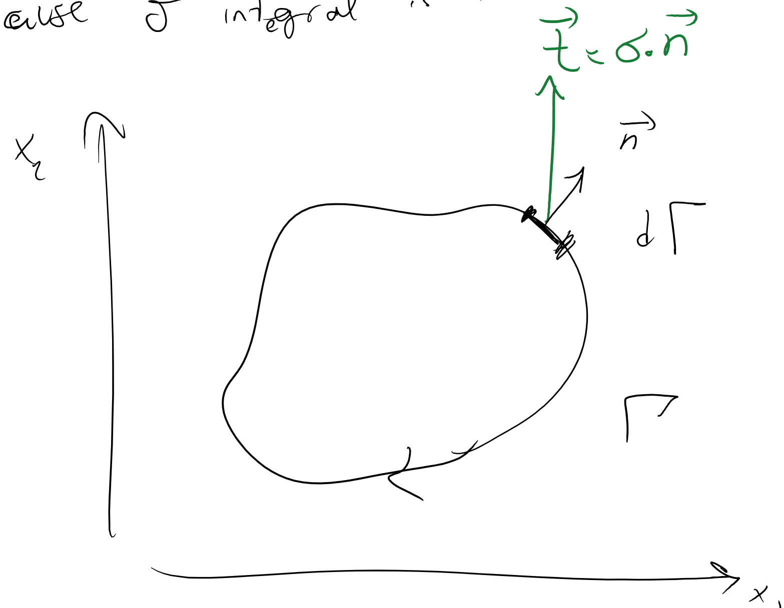
Rice, 1968

When no unloading, a plastic model can be represented by a nonlinear elastic response



2D:

We care about this because J integral is for nonlinear elastic material

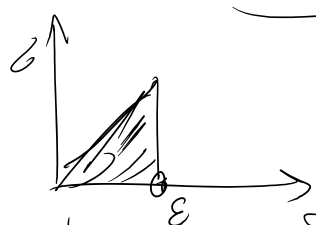


$$J_k = \int \left(W n_k - t_i \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ displacement vector

W = strain energy density

1D, linear elastic

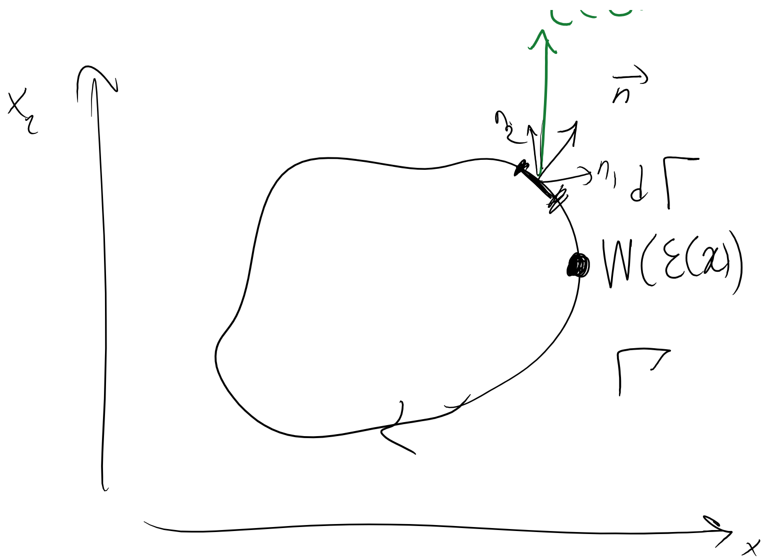


$$\sigma_{ij} = \frac{\partial W(\varepsilon)}{\partial \varepsilon_{ij}}$$

$W(\varepsilon) = \frac{1}{2} E\varepsilon^2 \longrightarrow \sigma = \dfrac{\partial \frac{1}{2} E \varepsilon^2}{\partial \varepsilon} = E\varepsilon$

Example 1D linear

$t = \sigma \cdot \vec{n}$ $\vec{n}$ $J_k = \int \left(W n_k - t_i \frac{\partial u_i}{\ldots} \right) d\Gamma$



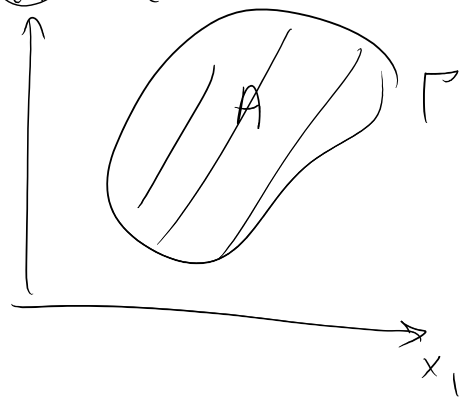
$$J_k = \int \left(W n_k - t_i \cdot \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$$= \int_\Gamma \left(W n_k - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_k} \right) d\Gamma$$

Showing that $J = 0$ around a closed curve Γ

$t_i \frac{\partial u_i}{\partial x_k} = \sum_{i=1}^{2} t_i \frac{\partial u_i}{\partial x_k}$

not written for repeated indicies



$$J = \int_\Gamma \left(W n_k - t_i \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$$= \int_\Gamma \left(W n_k - (\sigma_{ij} n_j) \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$$= \int_\Gamma W n_k d\Gamma - \int_\Gamma \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_j \, d\Gamma$$

$$J = \int_A \frac{\partial W}{\partial x_k} dA - \int_A \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_j} \right) dA \qquad \star$$

$$\frac{\partial W}{\partial x_k} = \underbrace{\frac{\partial W(\varepsilon)}{\partial \varepsilon_{mn}}}_{\sigma_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_k} = \sigma_{mn} \frac{\partial}{\partial x_k} \left(\frac{u_{m,n} + u_{n,m}}{2} \right)$$

sym. of σ_{mn}

$$= \underbrace{\sigma_{mn}}_{\text{sym.}} \left(u_{m,nk} \delta_{mk} + u_{n,mk} \delta_{nk} \right) = \sigma_{mn} \left(u_{n,nm} + u_{n,mn} \right) = \sigma_{mn} u_{m,nk}$$

$$J = \int_A \frac{\partial W}{\partial x_k}\, dA - \int_A \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) dA$$

$$\sigma_{mn}(u_{m,nk}) \qquad \sigma_{ij,j}\frac{\partial u_i}{\partial x_k} + \sigma_{ij}\frac{\partial^2 u_i}{\partial x_j \partial x_k}$$

$$(\nabla\cdot\sigma)_i = 0$$

static

$\vec{b} = 0$

$\nabla\cdot\sigma + \rho b = \rho a$

body force

accel.

$$J = \int_A \left(\sigma_{mn} u_{m,nk} - \sigma_{ij} u_{i,jk}\right) dA \qquad i \to m,\ j \to n$$

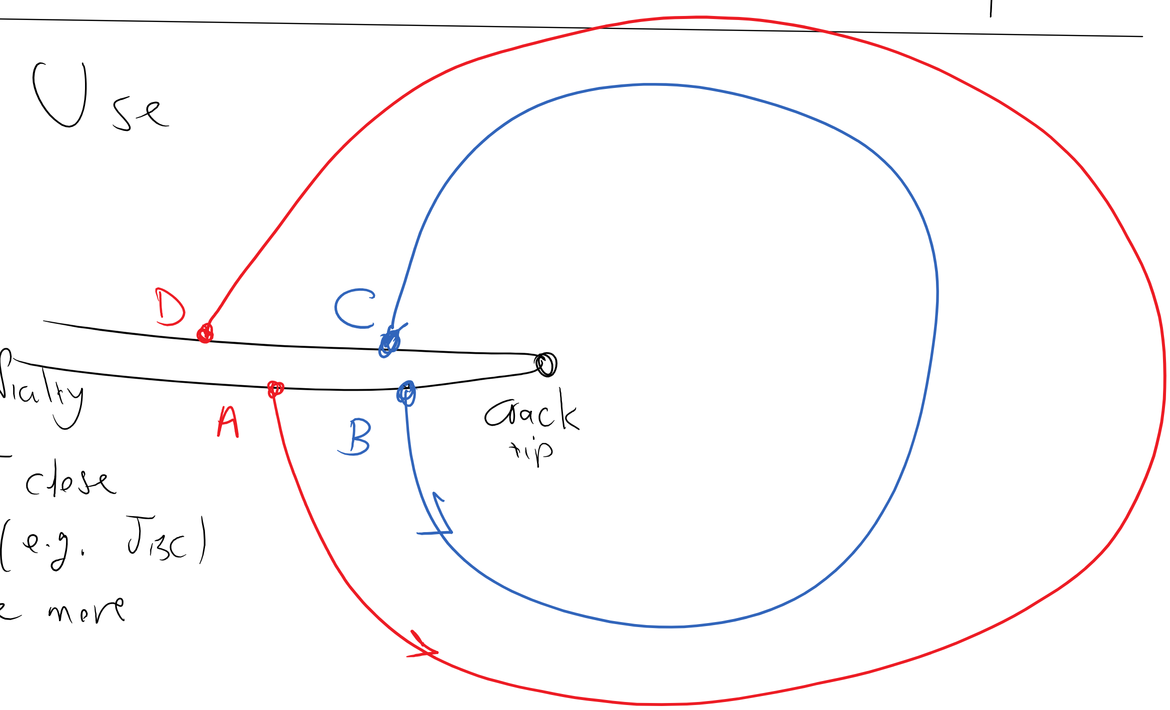
$$J = \int_A \left(\sigma_{mn} u_{m,nk} - \sigma_{mn} u_{m,nk}\right) dA = 0$$

$\boxed{J = 0 \text{ over any closed curve}}$

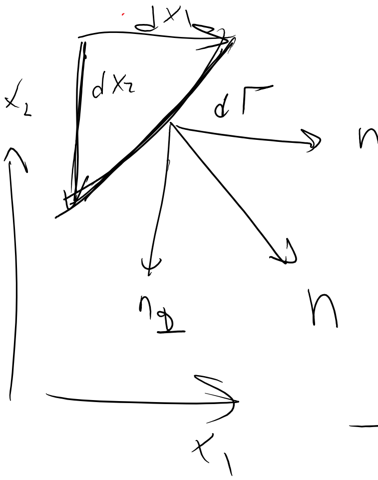
Pratical Use

$J_{AD} = J_{BC}$

numerically we'll have difficulty calculating J close to the crack (e.g. J_{BC}) but J_{AD} will be more accurate

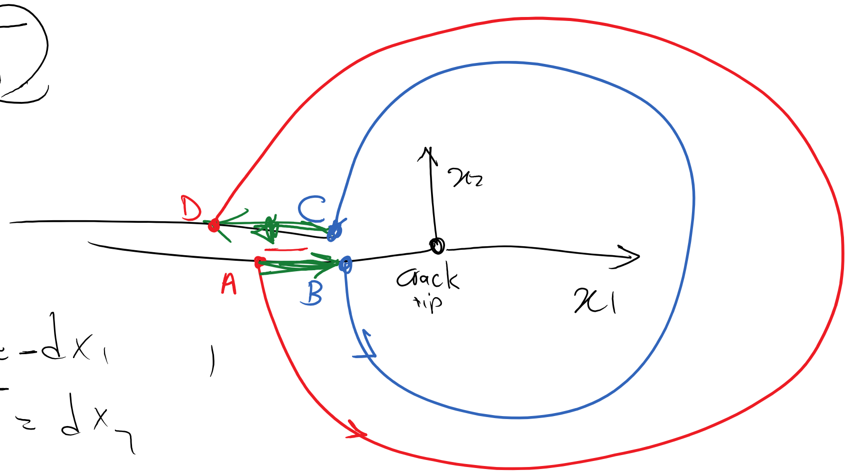


$$J_1 = \int \left(W n_1 - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_1} \right) d\Gamma$$



$n_2 \, d\Gamma = -dx_1$

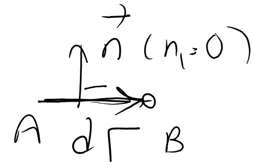
$n_1 \, d\Gamma = dx_2$



$$\rightarrow \boxed{J_1 = \int_\Gamma \left(W dx_2 - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_1} d\Gamma \right)}$$

J_{AB}

J_{CD}



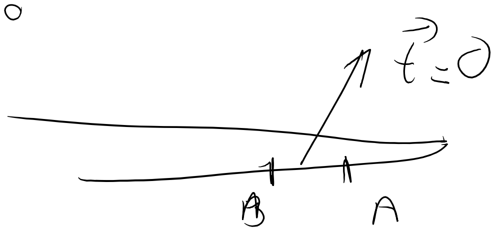
$= dx_1$

$dx_2 = 0$

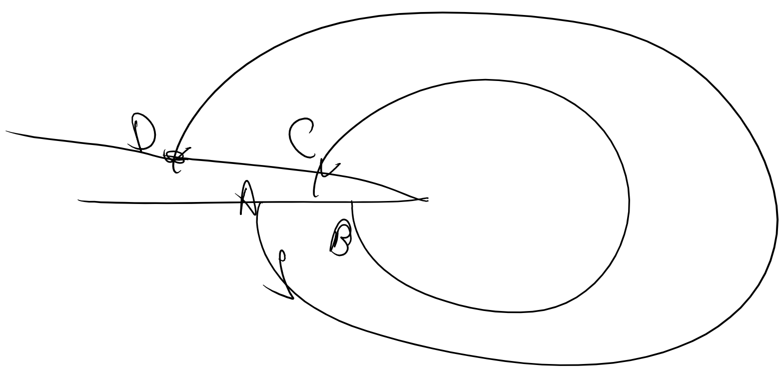
$W dx_2 = 0$ on $A \to B$

$$J_{ABCDA} = 0 = \underbrace{J_{AB}}_{0} + J_{BC} + \underbrace{J_{CD}}_{0} + \cancel{J_{DA}} - J_{AD}$$

$\boxed{\text{traction free crack surface}}$



$$\boxed{J_{BC} = J_{AD}}$$



$$J = G$$

energy release rate

$J = G$ rate

Energy release rate of J integral: Assumptions

1. Homogeneous body
2. Linear or non-linear elastic solid
3. No inertia, or body forces; no initial stresses
4. No thermal loading
5. 2-D stress and deformation field
6. Plane stress or plane strain
7. Mode I loading
8. Stress free crack

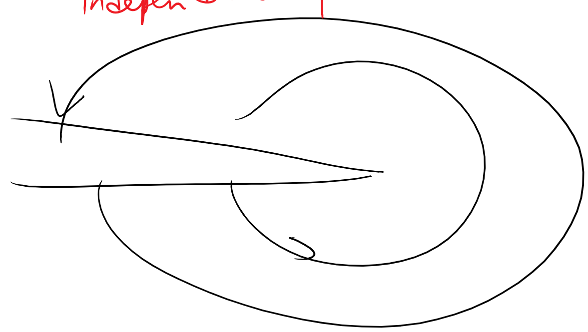
$v = \dot{u}$, $a = \dot{v} = \ddot{u} = 0$ $b = 0$

J integral is written for an elastic material

$\nabla \cdot \sigma = 0$ (correct only if $v = 0$ & $b = 0$)

$t = 0$

because of path independence proof



Proof that $J = G$ energy release per unit length (~~area~~)

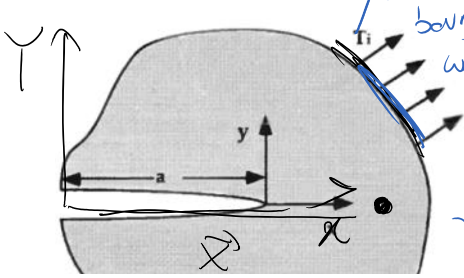
$$G = \frac{-\partial \Pi}{\partial \underbrace{B}_{\text{crack width}} \underbrace{a}_{\text{crack length}}}$$

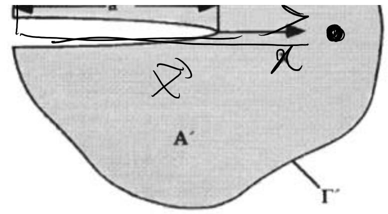
set $B = 1$

$$\boxed{G = \frac{-\partial \Pi}{\partial a}}$$

$$\Pi = U_e - W$$

natural boundary where we specify tractions





tractions

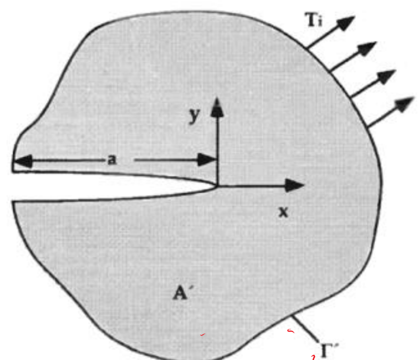
A' is the whole domain

$$\Pi = \underbrace{U_e}_{\text{internal energy}} - \underbrace{W}_{\text{external work}}$$

$$U_e = \int_{A'} \underbrace{\omega(\varepsilon)}_{\text{strain energy density}} dA$$

$$W = \int_{\partial D_f} u\bar{t}\, ds$$

$$\Pi = \int_{A'} \omega(\varepsilon)\, dA - \int_{\partial D_f} u\vec{t}\cdot ds$$

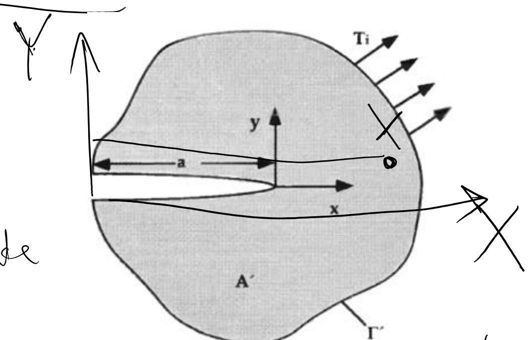


boundary of A'

$$G = \frac{-d\Pi}{d\underset{\text{crack length}}{a}} = -\int_{A'} \frac{d\omega(\varepsilon)}{da}\, dA - \int_{\partial D_f} \frac{du\vec{t}}{da}\, ds$$

skipping some steps

$$\boxed{G = -\int \frac{d\omega(\varepsilon)}{da}\, dA + \int_{\Gamma'} \frac{du}{da}\cdot\vec{t}\, ds}$$



$$X = a + x$$

$$\frac{d(\,)}{} \Big|$$

$$\frac{df}{} = \frac{\partial f}{} \frac{\partial x}{} + \frac{\partial f}{} \left(\frac{\partial a}{}\right)$$

$$\left.\frac{d(\,)}{da}\right| \quad X \text{ must be fixed}$$

$$\frac{df}{da} = \frac{\partial f}{\partial x}\left.\frac{\partial x}{\partial a}\right|_{X\text{-fixed}} + \frac{\partial f}{\partial a}\left(\frac{\partial a}{\partial a}\right)$$

$$= \frac{\partial f}{\partial x}\left.\frac{\partial (X-a)}{\partial a}\right|_{X\text{-fixed}} + \frac{\partial f}{\partial a}$$

$$\boxed{\left.\frac{df}{da}\right|_{X\text{-fixed}} = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial x}} \quad ✩$$

x → local coordinate attached to the crack tip

$$G = -\int_{A'} \frac{dw(\varepsilon)}{da}\, dA + \int_{\Gamma} \frac{du}{da}\cdot \vec{t}\, ds$$

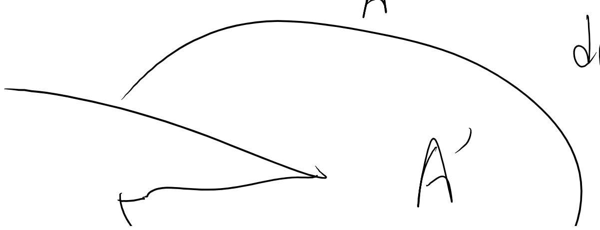
use eqn ✩

$$\frac{d\Pi}{da} = -G = \int_{A'} -\frac{\partial w}{\partial x}\, dA + \int_{\Gamma} \vec{t}\cdot\frac{\partial u}{\partial x}\, d\Gamma$$

$$+\int_{A'} \frac{\partial w}{\partial a}\, dA - \int_{\Gamma} \vec{t}\,\frac{\partial u}{\partial a}\, d\Gamma$$

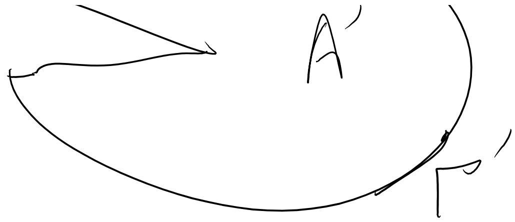
can show that this term goes to zero (see my notes)

$$\Longrightarrow \quad G = \int_{A'} \frac{\partial w(\varepsilon)}{\partial x}\, dA - \int_{\Gamma'} \vec{t}\cdot\frac{\partial \vec{u}}{\partial x}\, d\Gamma$$



divergence theorem

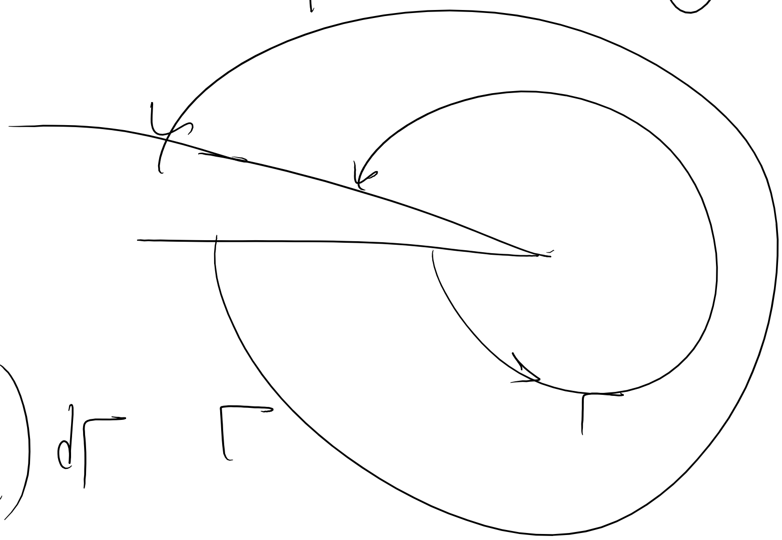
$$G = \int \left(w\, n_1 - \vec{t}\,\frac{\partial \vec{u}}{\partial x}\right) d\Gamma$$



$$G_0 = \int_{\Gamma'} \left(w_0 n_1 - t \frac{\partial u}{\partial x} \right) d\Gamma$$

$$J_i = \int_{\Gamma} \left(W n_i - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_i} \right) d\Gamma$$

$$\boxed{J_1 = G}$$



$$J_1 = \int_{\Gamma} \left(W n_1 - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x} \right) d\Gamma$$

$$= G$$

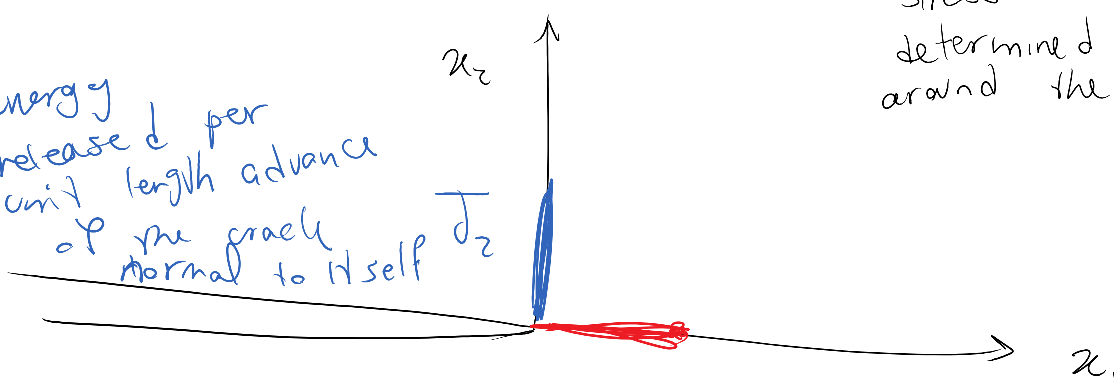
$$G = \frac{K_I^2 + K_{II}^2}{E'}$$

G ✓ $(G = J_1)$

$\underbrace{K_I = ?\ \ K_{II} = ?}$ stresses are determined around the crack tip

we need another eqn.

energy released per unit length advance of the crack normal to itself



J_1 energy released per unit length advance of crack in x_1 direction

$$\boxed{\begin{aligned} J_1 &= \frac{K_I^2 + K_{II}^2}{E'} \\ J_2 &= \frac{-2K_I K_{II}}{E'} \end{aligned}}$$

J_1, J_2 ✓ $\rightarrow K_I, K_{II}$