2018/10/08

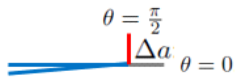
Monday, October 08, 2018 11:39 AM

# J-K relationship

In fact both J1 (J) and J2 are related to SIFs:

$$J_1 = \int_\Gamma \left( w dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_2 = \int_\Gamma \left( w dx - \mathbf{t}\frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

J1 & J2: crack advance for (θ = 0, 90) degrees




$$J = J_1 - iJ_2 = \frac{(1+\nu)(1+\kappa)}{4E}(K_I^2 + K_{II}^2 + 2iK_I K_{II})$$ Hellen and Blackburn (1975)

$$G = \boxed{\begin{aligned} J_1 &= \frac{K_I^2 + K_{II}^2}{E'} \\ J_2 &= \frac{-2K_I K_{II}}{E'} \end{aligned}}$$

$$E' = \begin{cases} E & \text{plane strain} \\ \frac{E}{1-\nu^2} & \text{plane stress} \end{cases}$$

Note that if $K_I = a, K_{II} = b$ is a general solution is:
$K_I = \pm a, K_{II} = \pm b$ and $K_I = \pm$
???

Calculating $J_1$ & $J_2$ we can calculate $K_I$ & $K_{II}$ from them

one sln

$K_I = a$ $\quad K_{II} = b$

$K_I = \pm a$ $\quad K_{II} = \pm b$

$K_I = \pm b$ $\quad K_{II} = \pm a$

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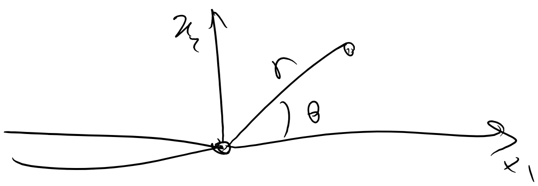
| LEFM | PFM |
|---|---|
| $G$ energy release rate | $J$ contour integral |
|  | $\sigma$ close to crack tip |
| $\sigma \propto \frac{K}{\sqrt{r}}$ | $\underbrace{\sigma \propto J^{\alpha} r^{\beta}}$ |
| $G = \frac{K^2}{E'}$ $\quad \sigma \propto G^{\frac{1}{2}} r^{-\frac{1}{2}}$ | we expect $\sigma$ to be "less singular" |



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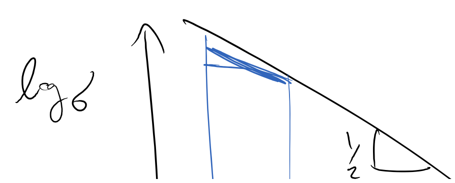
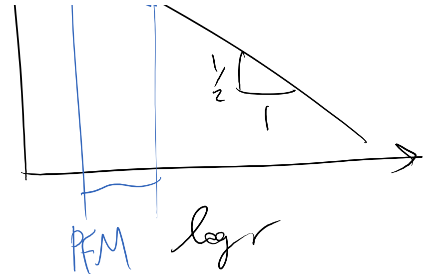
Goal: obtain $\alpha$ & $\beta$

## 5.3. 5. Plastic crack tip fields; Hutchinson, Rice and Rosengren (HRR) solution

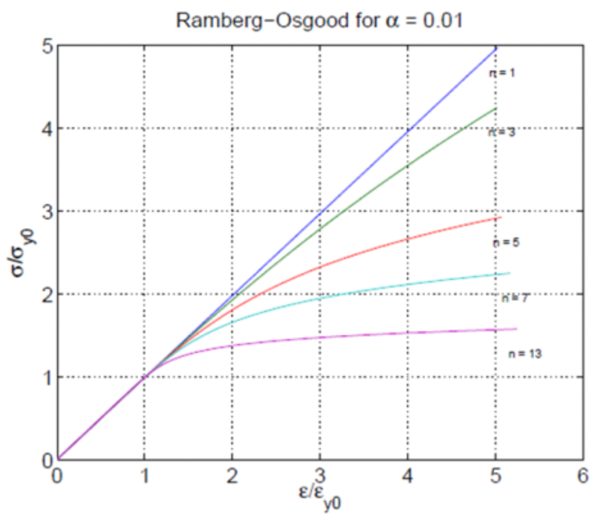


## 5.3. 5. Plastic crack tip fields; Hutchinson, Rice and Rosengren (HRR) solution

Even with this model $\sigma$ remains singular :(




# Ramberg–Osgood model




$$\frac{\varepsilon}{\varepsilon_{y0}} = \frac{\sigma}{\sigma_{y0}} + \alpha\left(\frac{\sigma}{\sigma_{y0}}\right)^n$$

typically we write $\sigma(\varepsilon)$

but for elastic material $\varepsilon \leftrightarrow \sigma$

we can go back and forth

$$\frac{\varepsilon}{\varepsilon_{y0}} = \frac{\sigma}{\sigma_{y0}} + \alpha\left(\frac{\sigma}{\sigma_{y0}}\right)^n$$

$$\varepsilon = \varepsilon_e + \varepsilon_p$$

Similar

$n \to \infty$

$n = 1$ $\quad \frac{\varepsilon}{\varepsilon_{y0}} = \frac{\sigma}{\sigma_{y0}}(1+\alpha)$ linear elastic

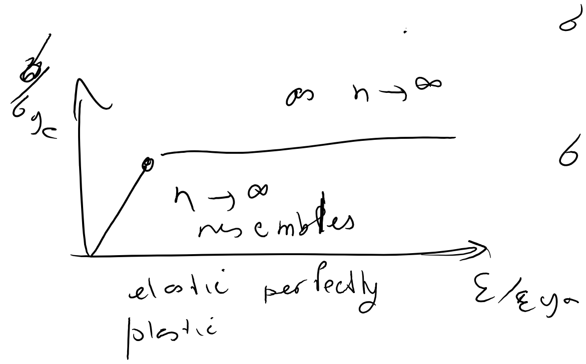
$n \to \infty$ $\quad \frac{\varepsilon}{\varepsilon_{y0}} = \frac{\sigma}{\sigma_{y0}} + \underbrace{\alpha\left(\frac{\sigma}{\sigma_{y0}}\right)^n}_{*}$

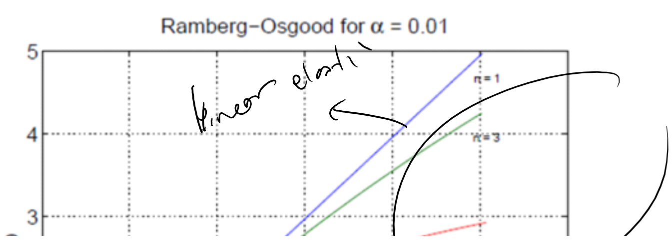
$\sigma < \sigma_{y0}$ for example 0.9999

$* \to 0$ $\quad \sigma = \left(\frac{\sigma_{y0}}{\varepsilon_{y0}}\right)\varepsilon$

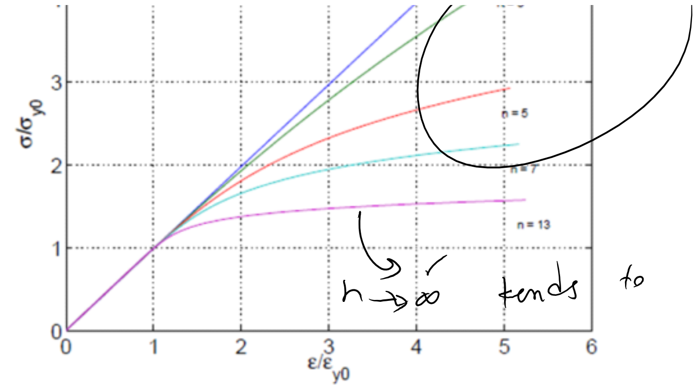
$\sigma > \sigma_{y0}$ for example $\varepsilon$ 1.001

$\varepsilon \to \infty$



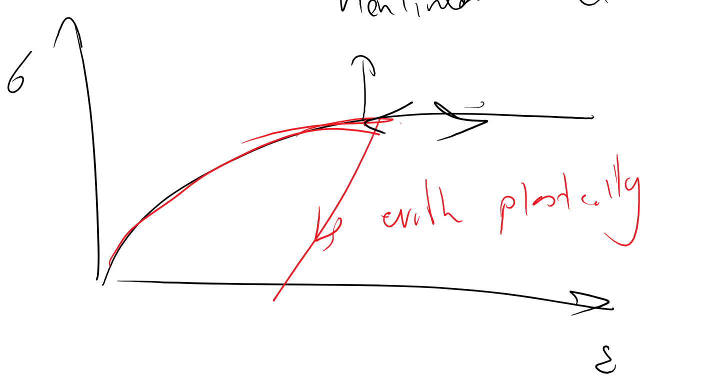




Some hardening




$n \to \infty$ tends to elastic perfectly plastic

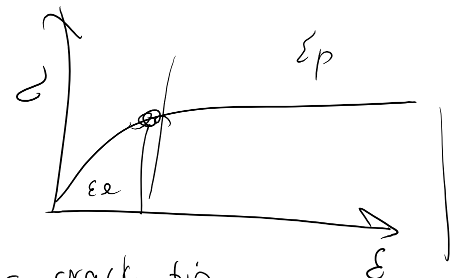
This model is much easier to use than plastic models. But, if there is unloading their responses are different and we cannot accurately use them!




Sketch of deriving the solution:

$$\frac{\varepsilon}{\varepsilon_{y_0}} = \underbrace{\left(\frac{\sigma}{\sigma_{y_0}}\right)}_{\sim\ \varepsilon_e} + \underbrace{\alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n}_{\varepsilon_p} \qquad n > 1$$

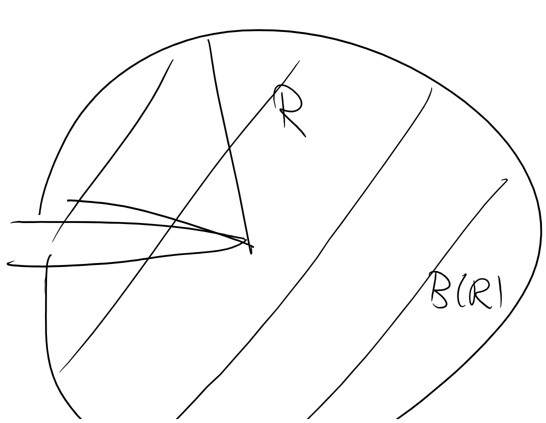
$\varepsilon_p$ should be the dominant term around a crack tip



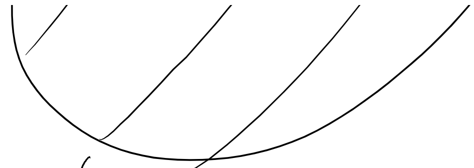

around a crack tip

$$\boxed{\varepsilon \propto \sigma^n} \qquad (i)$$

$$\left.\begin{aligned} \sigma &= \frac{c_1}{r^x} \\ \varepsilon &= \frac{c_2}{r^y} \end{aligned}\right\} \qquad \begin{aligned} x &= ? \\ y &= ? \end{aligned}$$



$$\varepsilon = \frac{\dots}{r^y}$$



$$\text{Elastic energy} = \int_{B(R)} \underbrace{w(\varepsilon)}_{\propto\, \sigma\varepsilon} \; \underbrace{dA}_{=\, r\,dr\,d\theta}$$

$$= \int_0^{2\pi} d\theta \int_0^R (\sigma \varepsilon) \; r\,dr\,d\theta$$

this term cannot be singular

$$= \int_0^{2\pi} d\theta \int_0^R (\sigma \varepsilon r)\,dr\,d\theta$$

$$\sigma \propto \frac{1}{r^x} \qquad \varepsilon = \frac{1}{r^y}$$

$$-x - y + 1 \geq 0 \qquad x + y \leq 1 \qquad \text{in fact } \boxed{x + y = 1} \quad (i)$$

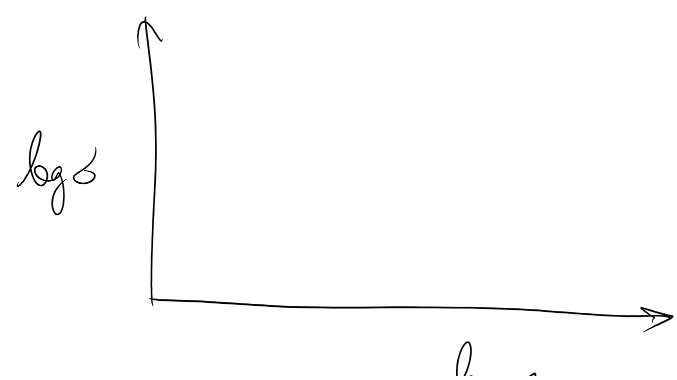
$$\varepsilon \propto \sigma^n \quad \left(\varepsilon \propto \frac{1}{r^x} \quad \sigma \propto \frac{1}{r^y}\right) \Rightarrow \frac{1}{r^x} = \frac{1}{(r^y)^n} \rightarrow$$

$$\boxed{x + y = 1} \qquad \boxed{x = ny}$$

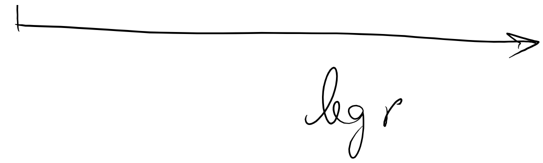
$$\boxed{\begin{aligned} y &= \frac{1}{n+1} \\ x &= \frac{n}{n+1} \end{aligned}}$$

$$\varepsilon \propto \frac{1}{r^{\frac{n}{n+1}}}$$

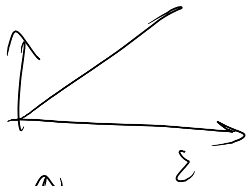
$$\sigma \propto \frac{1}{r^{\frac{1}{n+1}}}$$



$\sigma \propto \frac{1}{r^{\frac{1}{n+1}}}$

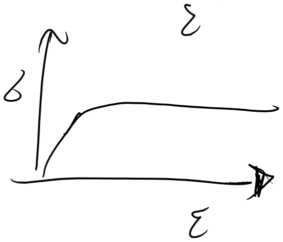



$\boxed{n=1}$




$\varepsilon, \sigma \propto \frac{1}{\sqrt{r}}$ LEFM

$n \to \infty$




$\varepsilon \to \frac{1}{r}$

$\sigma \to$ constant $\quad r^0$

↓ it will be bounded by yield stress of a ~ elastic-perfectly plastic model

plays the role of $\sqrt{K}$ in LEFM

powers of singularity

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n,\theta)$$

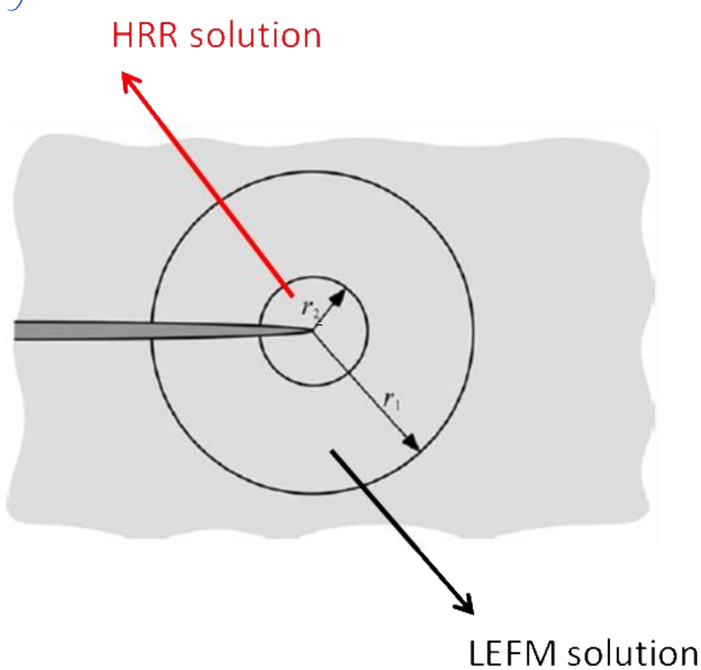
$$\epsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r}\right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(n,\theta)$$

$$\left[\frac{EJ}{\sigma_0^2 r}\right] = \frac{[\sigma]\,[\sigma][L]}{[\sigma]^2\,[L]} = 1$$

LEFM $\sigma, \varepsilon \propto \frac{K}{r^{1/2}} = \left(\frac{G}{r}\right)^{\frac{1}{2}}$

matches HRR for $n=1$

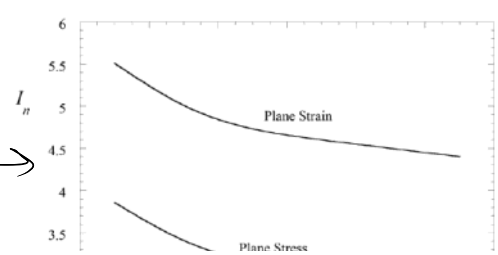
$$\sigma_{ij}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$$



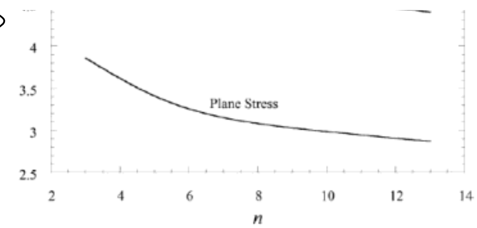

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n,\theta)$$

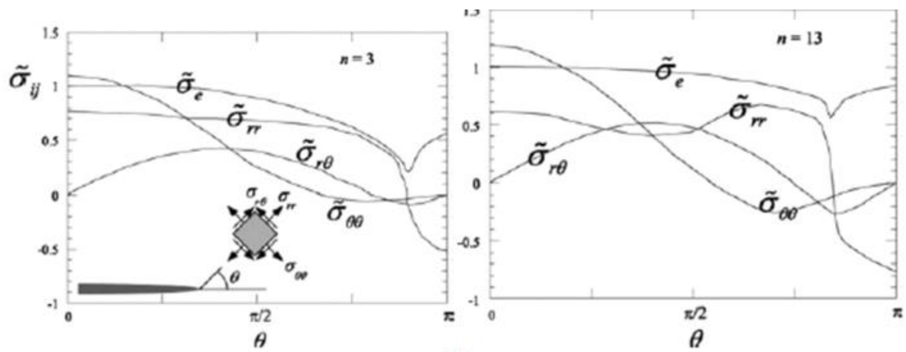
$$\epsilon_{ij} = \frac{\alpha \sigma_0}{E} \underbrace{\left(\frac{EJ}{\alpha \sigma_0^2 I_n r}\right)^{\frac{n}{n+1}}}_{r \text{ dependent part}} \tilde{\epsilon}_{ij}(n,\theta)$$

angle-dependent part

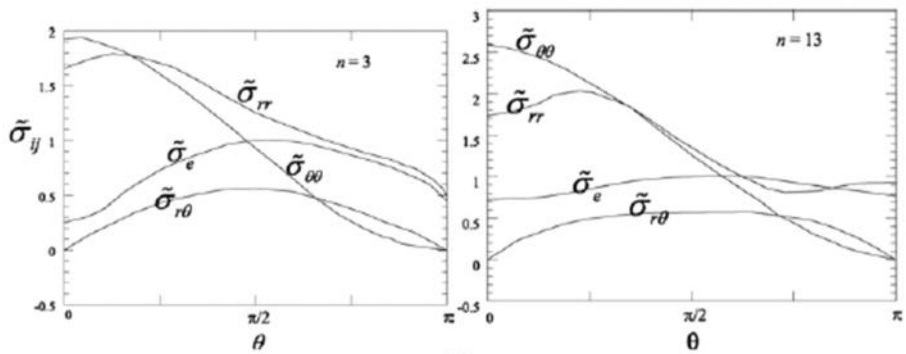


r dependent part



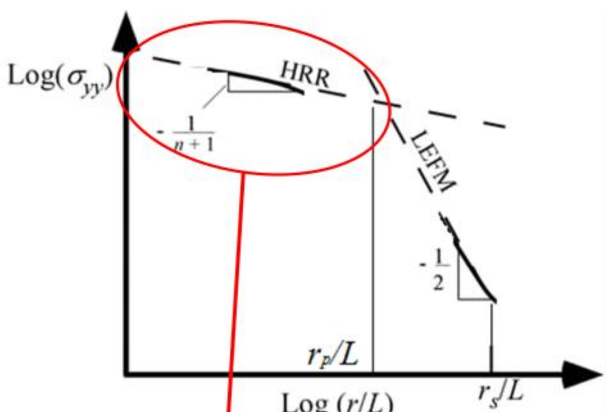




(a)

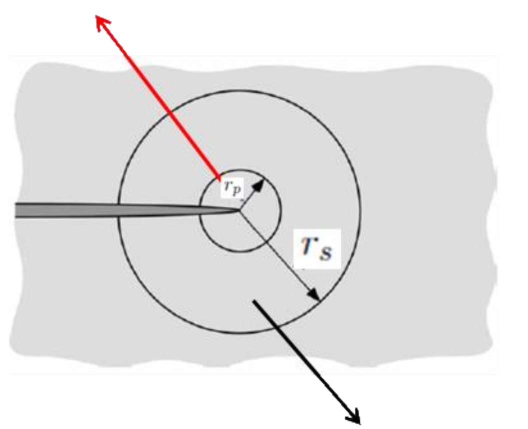



(b)

$$\sigma_{ij} = \sigma_0 \left( \frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n, \theta)$$




HRR solution




LEFM solution

Stress is still singular but with a weaker power of singularity!

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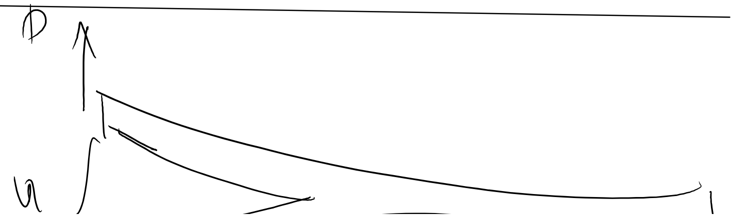
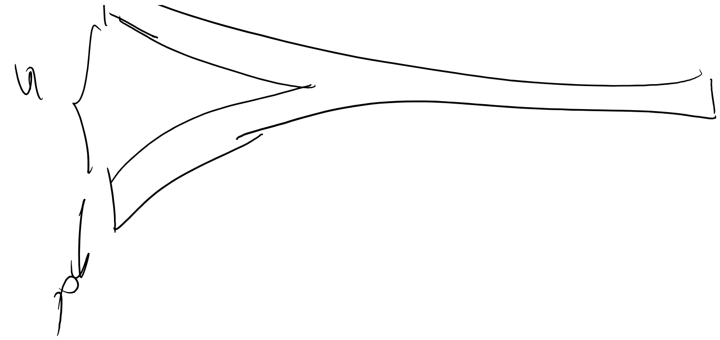
Next topic:

How to compute G (or J) from P-u (delta) systems: systems that we apply load and have some global displacement

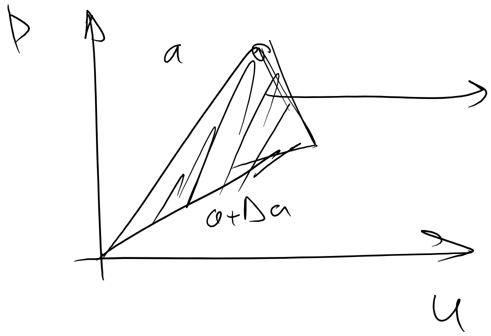


have some global displacement

want to calculate energy release rate




linear analysis




energy release rate $= \lim_{\Delta a \to 0} \frac{\text{Shaded area}}{B}$

$$= \frac{P^2}{2B}\frac{dC}{da} = \frac{u^2}{2B}\frac{dK}{da}$$

$C = \frac{u}{P}$ → compliance

$K = \frac{P}{u}$ → stiffness

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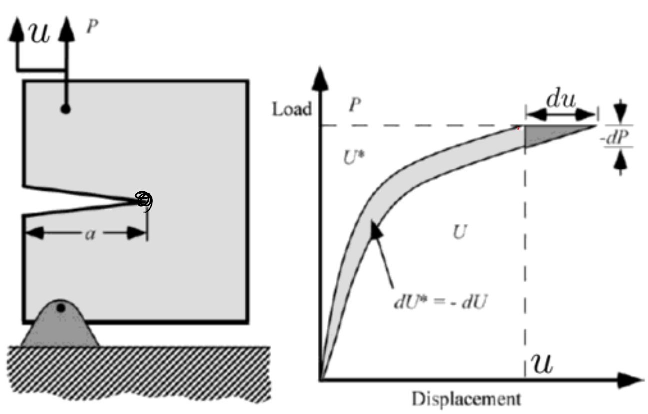
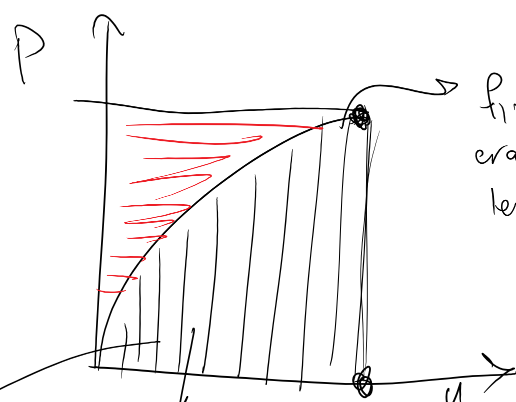
we want to generalize these relations to

**NONLINEAR ELASTICITY**



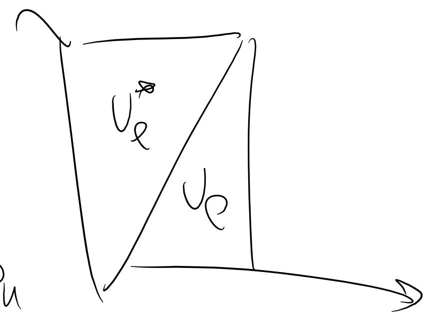

internal stored elastic energy ← $U_e(u)$

external work for a ~~fixed~~ crack length

"All elastic response with no plasticity or crack growth

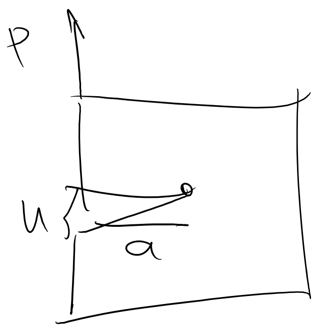
$$U_e = \int_0^u P(u)\,du$$ internal energy

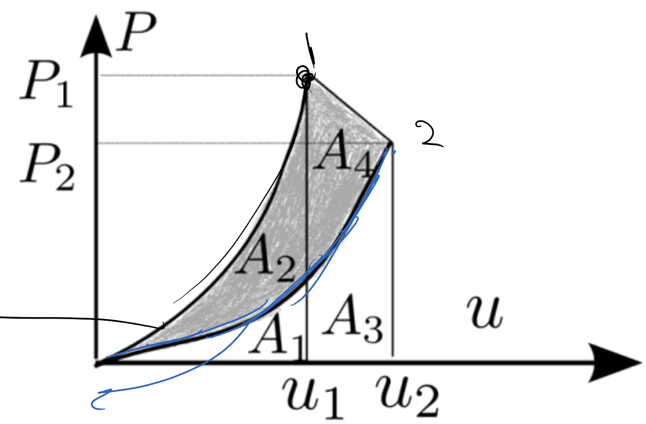
$$U_e^* = \int_0^P u(P)\,dP = Pu - U_e$$ complementary internal energy

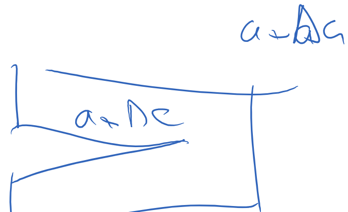



LINEAR RESPONSE

$$U_e = U_e^* = \frac{1}{2}Ku^2 = \frac{1}{2}Pu$$

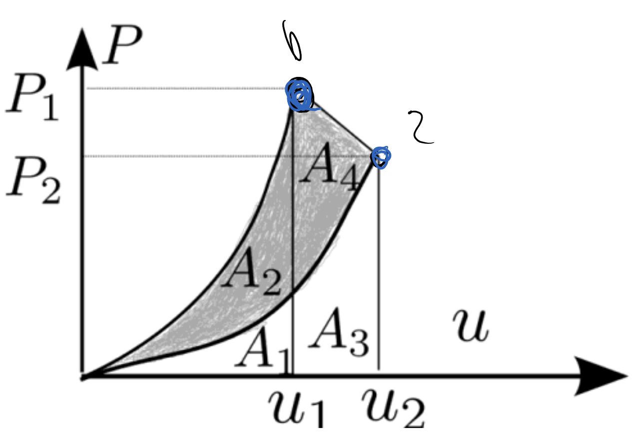



energy release rate

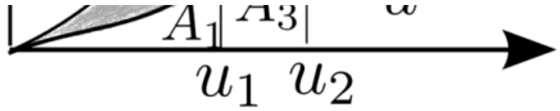
$$G = \lim_{\Delta a \to 0} -\frac{\Delta \Pi_e}{B\,\Delta a} = \lim_{\Delta a \to 0} \frac{(\Delta U_e - W_{12})}{B\,\Delta a}$$




$$\left.\begin{aligned}(U_e)_1 &= A_1 + A_2 \\ (U_e)_2 &= A_1 + A_3\end{aligned}\right\} \Rightarrow \Delta U_e = (U_e)_2 - (U_e)_1 = A_3 - A_2$$

$$W_{12} = A_3 + A_4$$

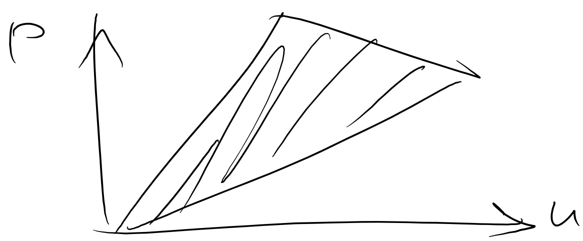
$$G = \lim_{\Delta a \to 0} \frac{-(A_3 - A_2) + (A_3 + A_4)}{B\,\Delta a}$$




$$G = \lim_{\Delta a \to 0} \frac{-(A_3 - A_2) + (A_3 + A_4)}{B\,\Delta a}$$

$$G = \lim_{\Delta a \to 0} \frac{A_2 + A_4}{B\,\Delta a}$$

$$\boxed{G = \lim_{\Delta a \to 0} \frac{\text{shaded area}}{B\,\Delta a} \quad \text{like the linear elastic case}}$$

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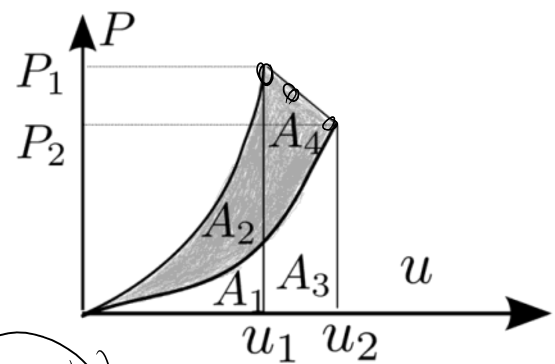
Can we have analytical eqns like

$$G = \frac{P^2}{2B}\frac{dC}{da} = -\frac{u^2}{2B}\frac{dK}{da} \quad ?$$




$$G = \lim_{\Delta a \to 0}\left(\frac{\Delta \Pi}{B\,\Delta a}\right) = \lim\left(\frac{-(U_{e2} - U_{e1}) + W_{12}}{B\,\Delta c}\right)$$

$$= \frac{1}{B}\left(-\lim_{\Delta a \to 0}\left\{\frac{U_e(a+\Delta a) - U_e(a)}{\Delta a}\right\} + \lim_{\Delta a \to 0}\frac{P_1 + P_2}{2}\left(\frac{u_2 - u_1}{\Delta a}\right)\right)$$

$$= \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right), \quad \text{(i)}$$

often $U_e(u)$

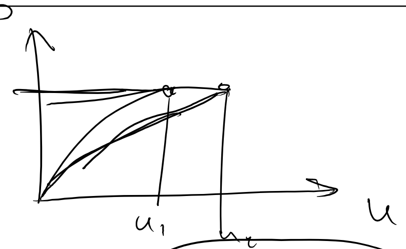
$$= \frac{1}{B}\left(-\frac{dU_e(u)}{du} + P\right)\frac{du}{da} \quad \text{(ii)}$$

$$\frac{dU_e}{da} = \frac{dU_e}{du}\frac{du}{da}$$

special cases:

case 1 $\quad P = \text{constant}$

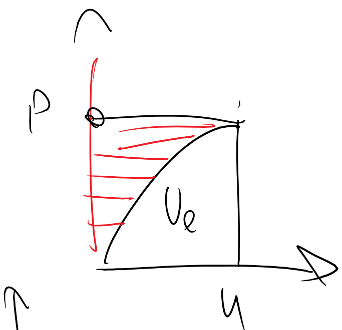
"dead load"



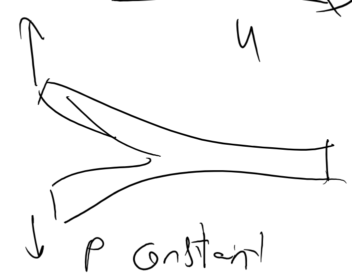

(i)

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + \frac{d\overset{\text{constant}}{P}u}{da}\right) = \frac{1}{B}\frac{d(Pu - U_e)}{da}$$

$U_e^*$

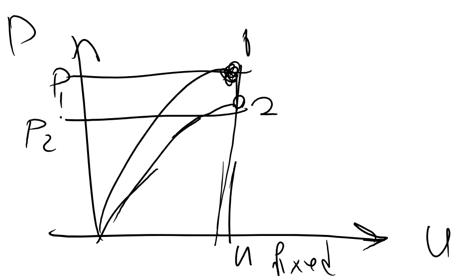



dead load $\quad \boxed{G = \frac{1}{B}\frac{dU_e^*}{da}\Big|_{P-\text{constant}}}$




case 2 $\quad$ Fixed grip

$u = \text{constant}$




eqn (i)

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\cancel{\frac{du}{da}}\right), \quad u = \text{constant}$$

$$\boxed{G = -\frac{1}{B}\frac{dU_e}{da}\Big|_{u-\text{constant}}} \quad \text{fixed grip}$$

Summary:

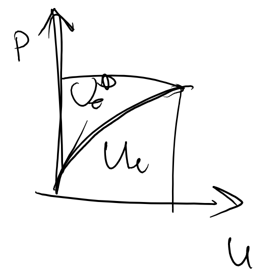
$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right), \quad \text{(i)} \quad U_e(u)$$

$$= \frac{1}{B}\left(-\frac{dU_e(u)}{du} + P\right)\frac{du}{da} \quad \text{(ii)} \quad \frac{dU_e}{da} = \frac{dU_e}{du}\frac{du}{da}$$

special cases

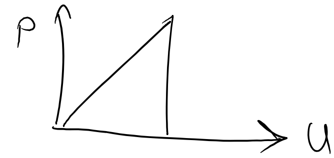
Load control P constant $\quad G = \frac{1}{B}\frac{dU_e^*}{da}$

Displacement " u " $\quad G = -\frac{1}{B}\frac{dU_e}{da}$




Do they match the linear case eqns:

$$U_e = \frac{1}{2}Pu = \frac{1}{2}\underbrace{K}_{\text{stiffness}}u^2$$




Use

$$G = \frac{1}{B}\left(-\frac{d\overset{\frac{1}{2}Ku^2}{U_e}}{da} + P\frac{du}{da}\right) = \frac{1}{B}\left(\frac{-d\left(\frac{1}{2}Ku^2\right)}{da} + P\frac{du}{da}\right)$$

$$= \frac{1}{B}\left(-\frac{1}{2}\frac{dK}{da}u^2 - \frac{1}{2}\underbrace{\left(2\times u\frac{du}{da}\right)}_{\frac{du^2}{da}}K + P\frac{du}{da}\right)$$

$$= -\frac{u^2}{2B}\frac{dK}{da} \cancel{-(uK)\frac{du}{da}} \cancel{+\underset{\downarrow\, uK}{P}\frac{du}{da}}\Bigg)$$

$$G = -\frac{u^2}{2B}\frac{dK}{da}$$

we derived this for linear case

so it matches the linear solution

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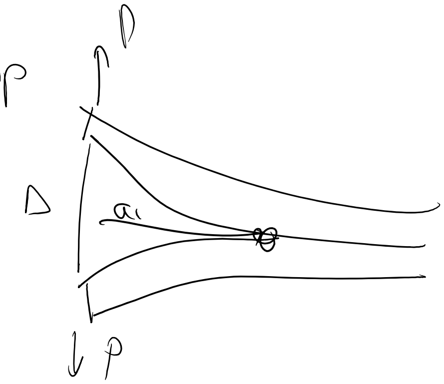
How to use this experimentally

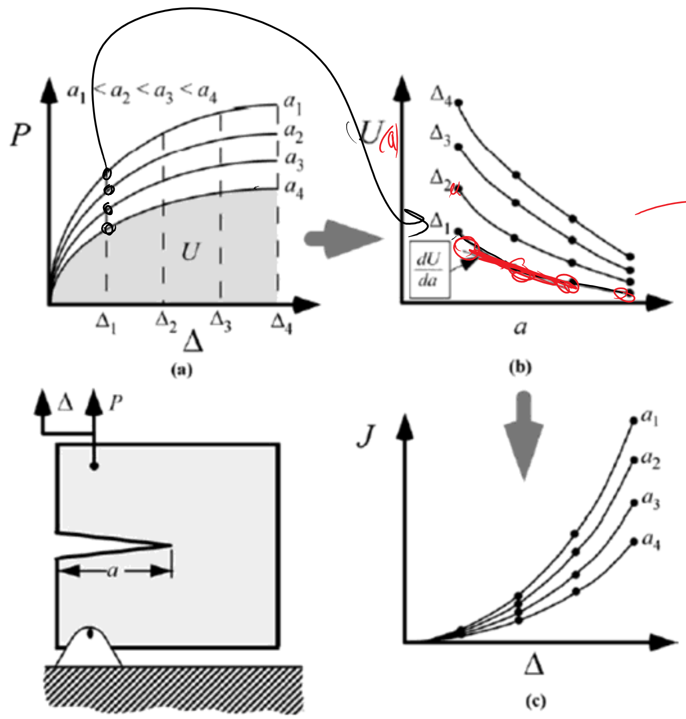
$$G = \frac{-1}{B}\frac{dU_e}{da}\Big|_{u\text{-fixed}}$$

fixed grip

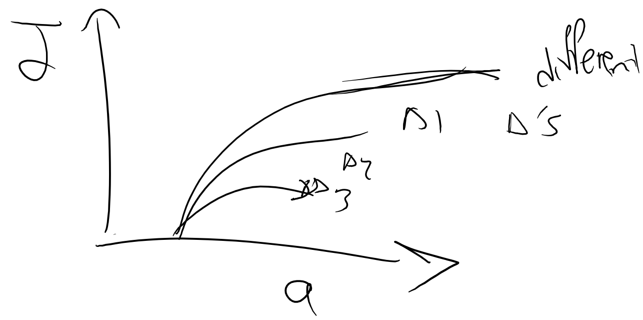



$u \to \Delta$

$$G = -\frac{1}{B}\frac{dU_e(a)}{da}\Big|_{\Delta\text{-fixed}}$$




$\to G(a_i, \Delta_j)$



- P-$\Delta$ curves for different crack lengths $a$
    - $J$ as a function of $\Delta$
- Rice proposes a method to obtain J with <u>only one test</u> for certain geometries

cf. Anderson 3.2.5 for details