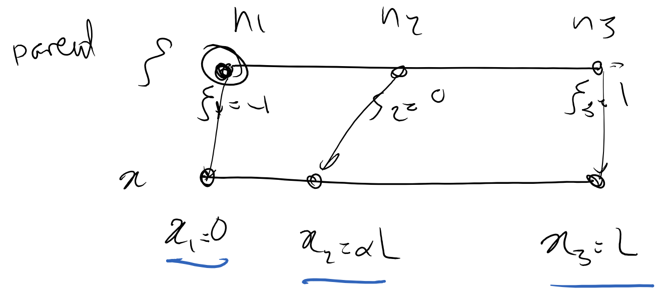
From last time:

$$\underset{\text{solution}}{u(\xi)} = \underbrace{u_1\left(\frac{\xi(\xi-1)}{2}\right)}_{N_1(\xi)} + u_2\underbrace{(1-\xi^2)}_{N_2(\xi)} + u_3\overbrace{\left(\frac{\xi(\xi+1)}{2}\right)}^{N_3(\xi)}$$

Goal $\quad \varepsilon \to \infty$ as $\xi \to -1$



$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\cdot\frac{d\xi}{dx} = \frac{du}{d\xi}\,\frac{1}{\frac{dx}{d\xi}}$$

$\xi \to x$

$$x(\xi) = \cancel{x_1}\, N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi)$$

$$= \alpha L\,(1-\xi^2) + L\left(\frac{\xi(\xi+1)}{2}\right)$$

$$x = L\left\{\alpha + \frac{\xi}{2} + \xi^2\left(\frac{1}{2}-\alpha\right)\right\} \quad (1a)$$

$$u = u_1\left(\frac{\xi(1-\xi)}{2}\right) + u_2(1-\xi^2) + u_3\frac{\xi(\xi+1)}{2} \quad (1b)$$

(1)

$$\varepsilon = \frac{\frac{du}{d\xi}}{\frac{dx}{d\xi}}$$ $\rightarrow$ 1st order polynomial in ξ

$$(1) \Rightarrow \boxed{\frac{dx}{d\xi} = L\left\{\frac{1}{2} + \xi(1-2\alpha)\right\}} \quad (2)$$

$\varepsilon \to \infty$ as $\xi \to -1$: $\frac{dx}{d\xi} \to 0$ as $\xi = -1$

$$\frac{dx}{d\xi} = 0 \text{ @ } \xi = -1 \;:\; \frac{1}{2} - 1(1-2\alpha) = 0 \implies \boxed{\alpha = \frac{1}{4}} \quad (3)$$

Let's plug $\alpha = \frac{1}{4}$ in (1a)

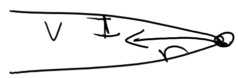
for $\alpha = \frac{1}{4}$ $\quad \boxed{x = \frac{L}{4}(\xi+1)^2 \iff \xi = 2\sqrt{\frac{x}{L}} - 1} \quad (4)$

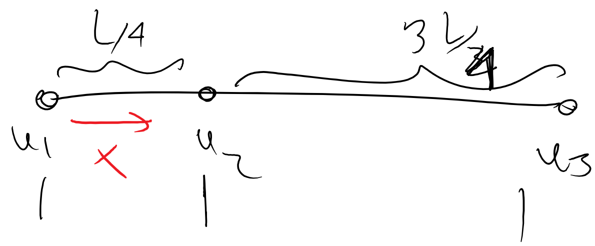
In 1b plug eqn (4) to get

$$u = u_1\left(\frac{\xi(1-\xi)}{2}\right) + u_2(1-\xi^2) + u_3\frac{\xi(\xi+1)}{2}$$

$$U = u_1\left(\frac{\xi(1-\xi)}{2}\right) + u_2(1-\xi^2) + u_3\frac{\xi(\xi+1)}{2}$$

(5)
$$\boxed{u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}(-3u_1 - u_2 + 4u_3) + \frac{2x}{L}(u_1 + u_2 - 2u_3)}$$





Recall

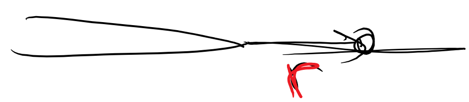
$$v(r) = \frac{4K_I}{\sqrt{2\pi}\,E'}\sqrt{r}$$

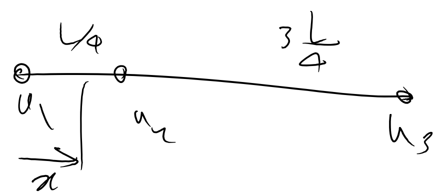
(5) $\varepsilon = \dfrac{du}{dx}$

$\sigma = E\varepsilon$

$$\boxed{\epsilon = \frac{1}{\sqrt{xL}}\left(-\frac{3}{2}u_1 - \frac{1}{2}u_2 + 2u_3\right) + \frac{1}{L}(2u_1 + 2u_2 - 4u_3)}$$

$$\sigma = \frac{E}{\sqrt{xL}} \quad '' \quad ''$$





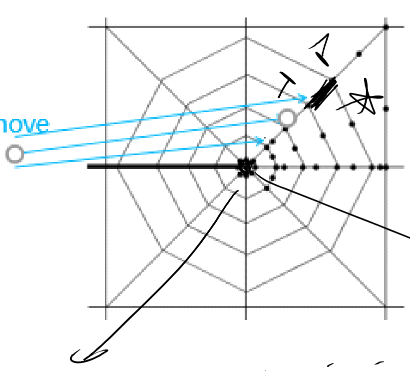
$$\sigma_y(r, \theta=0) = \frac{K_I}{\sqrt{2\pi r}}$$

By using 1/4, 3/4 positioning of the middle node not only we get singularity, but also we get the right power of singularity ? for LEFM

In practice, we'll have a spider web mesh around the crack tip:

- Transition elements: According to this analysis mid nodes of next layers move to ½ point from ¼ point

Lynn and Ingraffea 1977)



as elements get away from the crack tip, the "mid point" moves toward the center of the edge

ξ for CT for segment ☆ goes to $-\infty$

are at $\frac{1}{4}$ position

Recall

$$\underline{\frac{dx}{d\xi}} = L\left(\frac{1}{2} + \xi(1-2\alpha)\right)$$

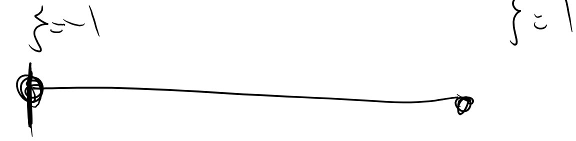
ξ is given

location of crack tip

want to make $\frac{dx}{d\xi}(\xi) = 0$ $\rightarrow$

$$\frac{1}{2} + \xi(1-2\alpha) = 0 \longrightarrow$$

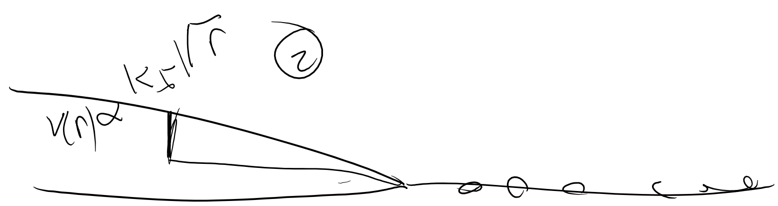
(6) $$\boxed{\alpha = \frac{1}{4\xi} + \frac{1}{2}}$$



$\alpha = \frac{1}{4} \leftarrow$ (6) - CT here $\xi = -1$

CT $(\xi \rightarrow -\infty)$ $\quad \alpha \rightarrow \frac{1}{2}$

6.1.3. Extraction of K (SIF), G



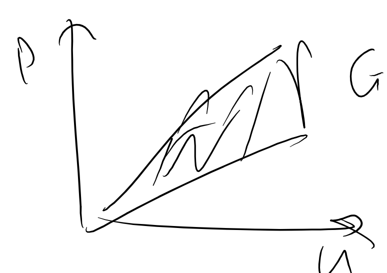
$$G(r, \theta=0) = \frac{K_I}{\sqrt{2\pi r}}$$ ✓ (1)

Approach (3) $\quad G = \frac{K_I^2}{E'}$ $\quad$ mode I problem

$$K_I = \sqrt{E' G}$$

$$E' = \begin{cases} E & \text{p. strain} \\ \frac{E}{1-\nu^2} & \text{" stress} \end{cases}$$

how do we calculate G?



$$G = \frac{\text{shaded Area}}{B} = \frac{P^2}{2B}\frac{dC}{da}$$

$$G = -\frac{d\Pi}{da}$$

$$G = J$$

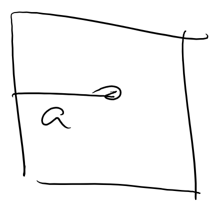
calculate

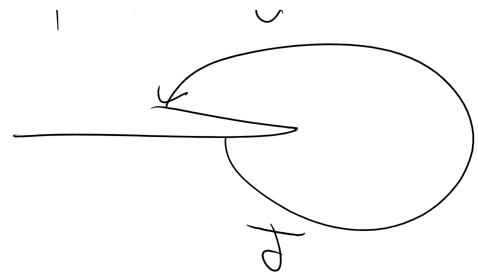
calculate

$\Pi(a)$

&

$\Pi(a+\Delta a) \Rightarrow G \approx -\frac{\Delta \Pi}{\Delta a}$





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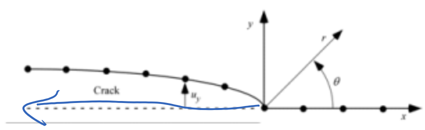
1. K from local fields

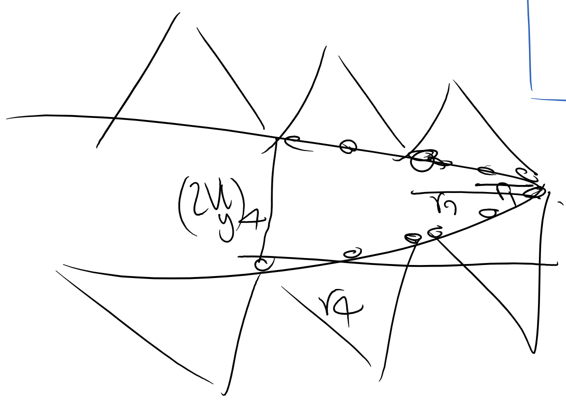
1. Displacement

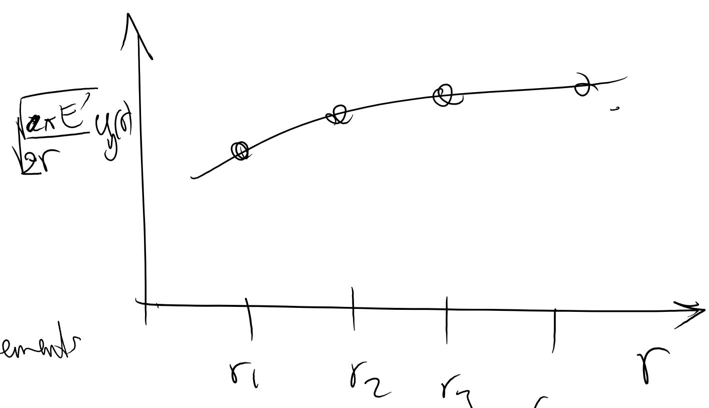


$u_y(r, \theta=\pi) = \frac{4 K_I \sqrt{r}}{\sqrt{2\pi} E'}$

$r \to 0$

$$⑥ \quad \boxed{K_I = \lim_{r \to 0} \left(\frac{\sqrt{2\pi} E' \, u_y(r,\pi)}{4\sqrt{r}} \right)}$$





Alternative approach, if we use $\frac{1}{4}$ elements

or alternatively from the first quarter point element:

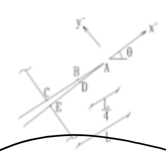
$$\left.\begin{aligned} v &= K_I \frac{\kappa+1}{2G}\sqrt{\frac{r}{2\pi}} \\ u' &= \overline{u}'_A + \left(-3\overline{u}'_A + 4\overline{u}'_B - \overline{u}'_C\right)\sqrt{\tfrac{r}{L}} + \left(2\overline{u}'_A + 2\overline{u}'_C - 4\overline{u}'_B\right)\tfrac{r}{L} \\ v' &= \overline{v}'_A + \left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)\sqrt{\tfrac{r}{L}} + \left(2\overline{v}'_A + 2\overline{v}'_C - 4\overline{v}'_B\right)\tfrac{r}{L} \end{aligned}\right\}$$

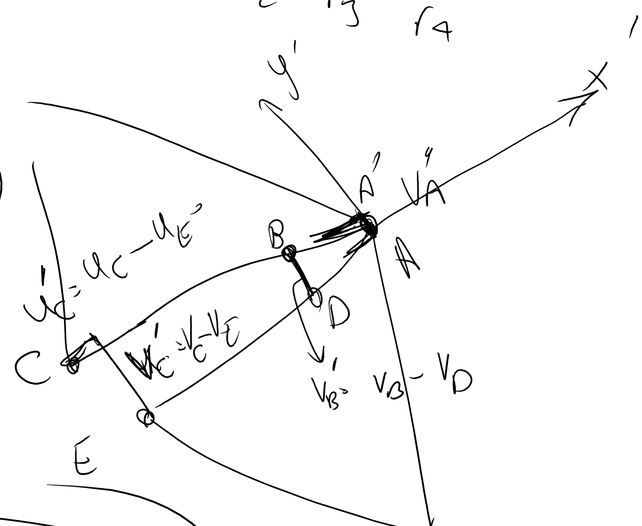
Recall for 1D

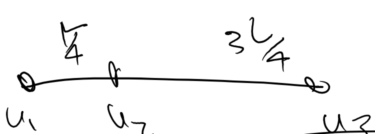
$$u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}(-3u_1 - u_2 + 4u_3) + \frac{2x}{L}(u_1 + u_2 - 2u_3)$$

$$\boxed{K_I = \frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)}$$

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \frac{1}{2}\frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3\overline{u}'_A + 4\left(\overline{u}'_B - \overline{u}'_D\right) - \left(\overline{u}'_C - \overline{u}'_E\right) \\ -3\overline{v}'_A + 4\left(\overline{v}'_B - \overline{v}'_D\right) - \left(\overline{v}'_C - \overline{v}'_E\right) \end{bmatrix}$$





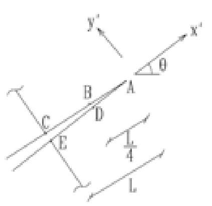


$$u(x) = u_1 + \frac{\sqrt{x}}{\sqrt{L}}\left(-3u_1 + 4u_2 - u_3\right)$$

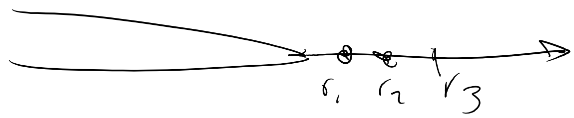
$$\left\{\begin{matrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{matrix}\right\} = \frac{1}{2}\frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3\overline{u}'_A + 4\left(\overline{u}'_B - \overline{u}'_D\right) - \left(\overline{u}'_C - \overline{u}'_E\right) \\ -3\overline{v}'_A + 4\left(\overline{v}'_B - \overline{v}'_D\right) - \left(\overline{v}'_C - \overline{v}'_E\right) \end{bmatrix}$$

275

Mixed mode generalization:



1. K from local fields

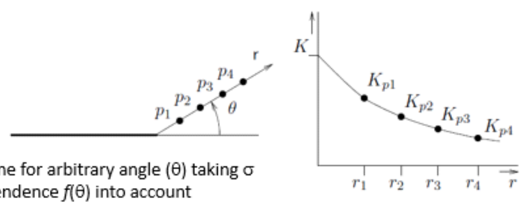


2. Stress

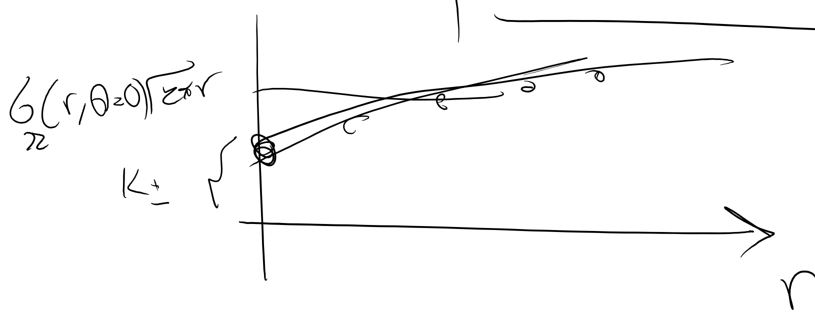
$$\boxed{K_I = \lim_{r\longrightarrow 0}\left(\sqrt{2\pi r}\;\sigma_{22}|_{\theta=0}\right) \quad ; \quad K_{II} = \lim_{r\longrightarrow 0}\left(\sqrt{2\pi r}\;\sigma_{12}|_{\theta=0}\right)}$$

$$\sigma(r, \theta=0) = \frac{K_I}{\sqrt{2\pi r}}$$

$$\boxed{K_I = \lim_{r\to 0} \sigma(r, \theta=0)\sqrt{2\pi r}}$$



or can be done for arbitrary angle (θ) taking σ angular dependence $f(\theta)$ into account



Why the stress approach is very bad:

1. Stress tends to infinity as we get close to the crack tip, whereas displacement goes to zero. FE shape functions have a much easier time to capture bounded displacement than a stress solution going to infinity (even with using 1/4 nodal position elements).
2. Displacement is the interpolated field in FE methods so it's one order more accurate than strains and stresses that are obtained from the gradient of displacement field.
3. Stress method performs poorly when there are tractions on crack surface

Calculating K from G: Mode I problem

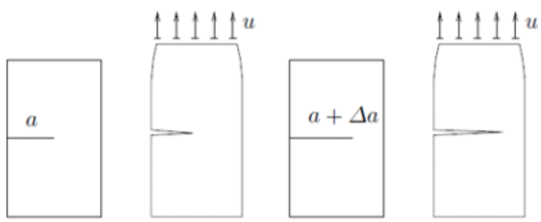
$$G = -\frac{d\Pi}{da}$$

Idea: solve to FEM problems with

a & $a+\Delta a$ crack lengths, use FD

$$G \simeq -\frac{\Delta \Pi}{\Delta a}$$

2.1 Elementary crack advance



⇒ 2 FEM slns needed

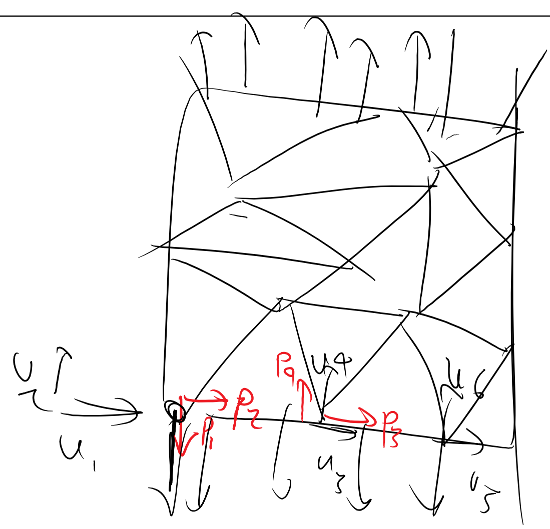
— FD errors :(

2.2 Virtual crack extension

$$\Pi = U_e - W$$

$U_e \rightarrow \frac{1}{2} u^T K u \quad - \quad P u$

vector of nodal unknowns



$U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ # of dofs

$P =$ nodal forces $\begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}$

$$\Pi = \frac{1}{2} u^T K u \quad - Pu$$

$$\frac{d\Pi}{da} = \frac{d\left(\frac{1}{2} u^T K u\right)}{da} - \frac{dPu}{da}$$

$$= \frac{1}{2}\left(\frac{du}{da}\right)^T K u + \frac{1}{2} u^T \frac{dK}{da} u + \frac{1}{2} u^T K \frac{du}{da} - P \frac{du}{da} - \frac{dP}{da} u$$

equal

$$\frac{1}{2}\left(\frac{du}{da}\right)^T K u = \frac{1}{2} u K \frac{du}{da}$$

K is sym.

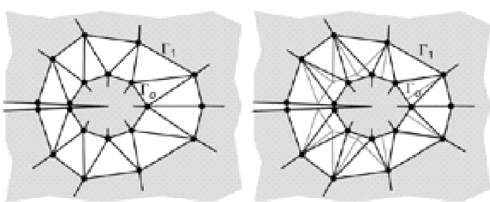
$$= \frac{du}{da}(Ku) + \frac{1}{2} u^T \frac{dK}{da} u - \frac{du}{da}(P) - \frac{dP}{da} u$$

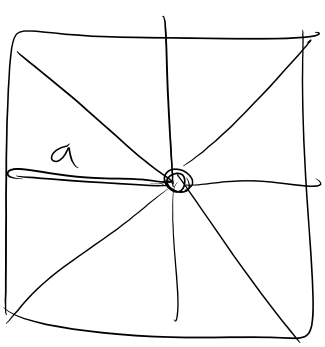
$$\frac{d\Pi}{da} = \frac{du}{da}\underbrace{\cancel{(Ku - P)}}_{0} + \frac{1}{2} u^T \frac{dK}{da} u - \frac{dP}{da} u$$

@ the FEM solution $Ku = P$

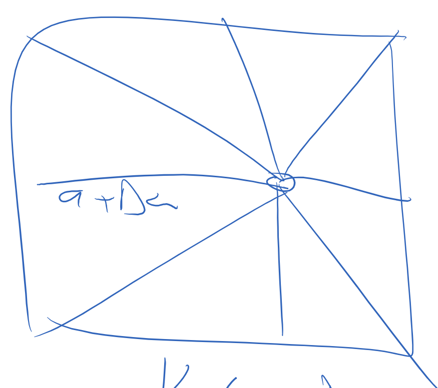
$$\boxed{G = -\frac{d\Pi}{da} = -\frac{1}{2} u^T \frac{dK}{da} u + \frac{dP}{da} u}$$

often these are prescribed loads independent of crack length $\Longrightarrow \frac{dP}{da}$ is often zero





$K(a)$



$K(a+\Delta a)$

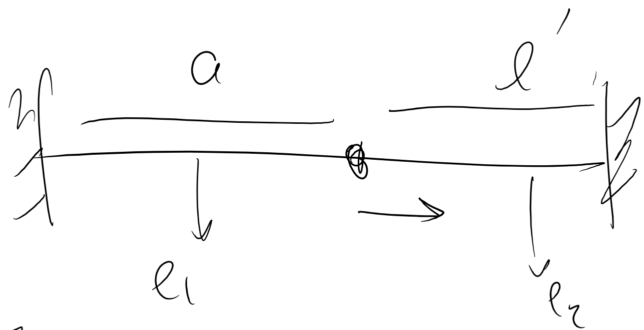
$dK \qquad K(a+\Delta a) - K(a)$

$$\frac{dK}{da} \approx \frac{K(a+\Delta a) - K(a)}{\Delta a}$$

1 Finite Element solution

1 " " K evaluations

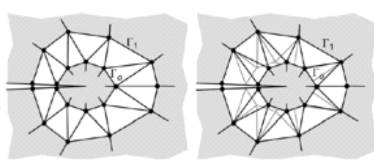
All we need for $\frac{dK}{da}$ is to know how around the crack tip local element stiffnesses change as a changes & assemble $\frac{dk}{da}$ of elements around the crack tip



$$k^{e_1} = \frac{(AE)_e}{a}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\boxed{\frac{dk^{e_1}}{da} = -\frac{(AE)_e}{a^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}$$ → Assemble this to $\frac{dK}{da}$

- Only the few elements that are distrorted contribute to $\frac{\partial K}{\partial a}$
- We may not even need to form elements and assemble K for a and $a + \Delta a$ to obtain $\frac{\partial K}{\partial a}$. We can explicitly obtain $\frac{\partial k^e}{\partial a}$ for elements affected by crack growth by computing derivatives of actual geometry of the element to parent geometry.



  - This method is equivalent to J integral method (Park 1974)

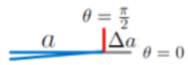
2.2 Virtual crack extension: Mixed mode

- For LEFM energy release rates G_1 and G_2 are given by

$$J_1 = G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$
$$J_2 = G_2 = \frac{-2K_I K_{II}}{E'}$$

- Using Virtual crack extension (or elementary crack advance) compute G_1 and G_2 for crack lengths a, $a + \Delta a$

$$J_1 = G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$
$$J_2 = G_2 = \frac{-2K_I K_{II}}{E'}$$



- Obtain K_I and K_{II} from:

$$\boxed{\begin{aligned} K_I &= \frac{s \pm \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4} \\ K_{II} &= \frac{s \mp \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4} \end{aligned}}$$

Note that there are two sets of solutions!

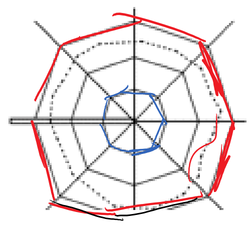
$$s = 2\sqrt{\frac{G_1 - G_2}{\alpha}} \text{ and } \alpha = \frac{(1+\nu)(1+\kappa)}{E}$$

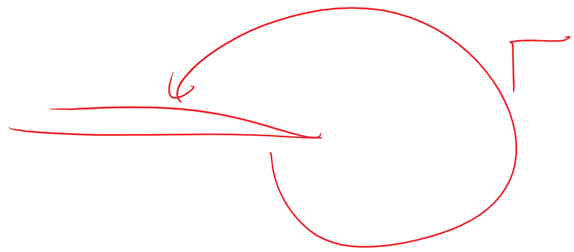
6.1.4. J integral

Methods to evaluate J integral:

1. Contour integral:

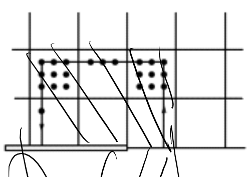
$$J_1 = \int_\Gamma \left(w dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_2 = \int_\Gamma \left(w dx - \mathbf{t} \frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

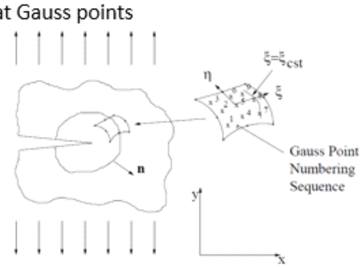


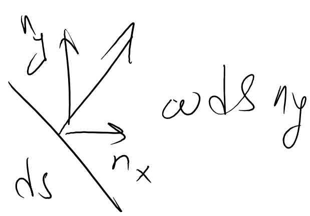


J integral: 1.Contour integral

- Stresses are available and also more accurate at Gauss points
- Integral path goes through Gauss points







$$J = \int_\Gamma w dy - \mathbf{t} \frac{\partial \mathbf{d}}{\partial x} ds \Rightarrow J^e = \int_{-1}^{1} \left\{ \underbrace{\frac{1}{2}\left[\sigma_x \frac{\partial u}{\partial x} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \sigma_y \frac{\partial v}{\partial y}\right]}_{w} \underbrace{\frac{\partial y}{\partial \eta}}_{dy} - \underbrace{\left[(\sigma_x n_1 + \tau_{xy} n_2)\frac{\partial u}{\partial x} + (\tau_{xy} n_1 + \sigma_y n_2)\frac{\partial v}{\partial x}\right]}_{\mathbf{t}\cdot\frac{\partial \mathbf{d}}{\partial x}} \underbrace{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}}_{ds} \right\} d\eta = \int_{-1}^{1} I \, d\eta$$

Cumbersome to formulate the integrand, evaluate normal vector, and integrate over lines (2D) and surfaces (3D)

Not commonly used

283