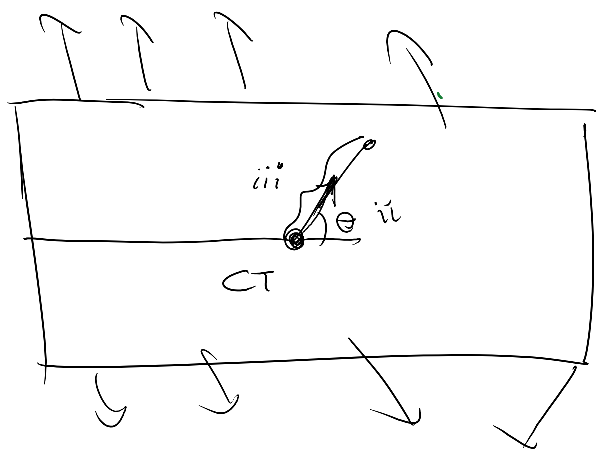
2018/11/05

Monday, November 05, 2018 11:35 AM

Mixed mode fracture (crack nucleation & propagation)



3 questions about crack propagation:

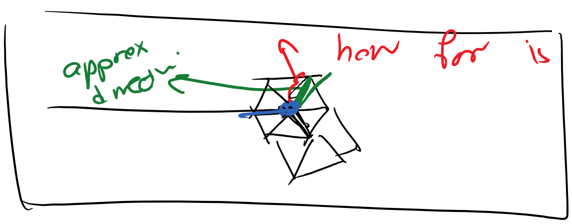
i. Does the crack propagate?

ii. In what direction does the crack propagate

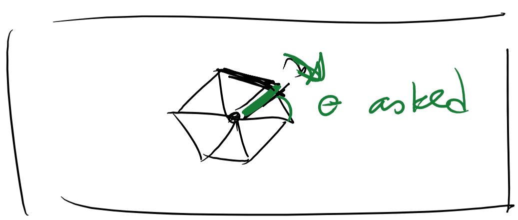
iii. What is the extent of the crack propagation

computational setting

typically CT is on a node



goes is one element edge at a time



Mesh Adaptive

θ is exact

still the extent can be inaccurate because crack advances 1 element at a time

XFEM

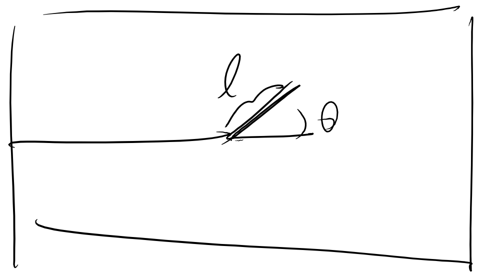
θ & extent can be exact

How do we answer these questions from the physics of the problem?

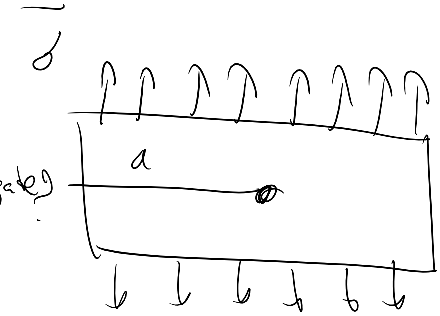
i. Does the crack propagate

ii. What angle

iii. How far



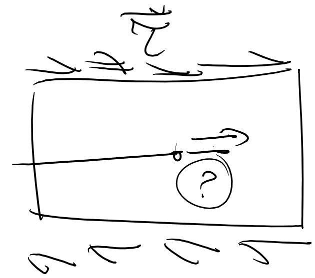
Mode I



i. Does the crack propagate?

$K(\bar{\sigma}, a, \text{geom}) = K_{Ic}$ then it can propagate

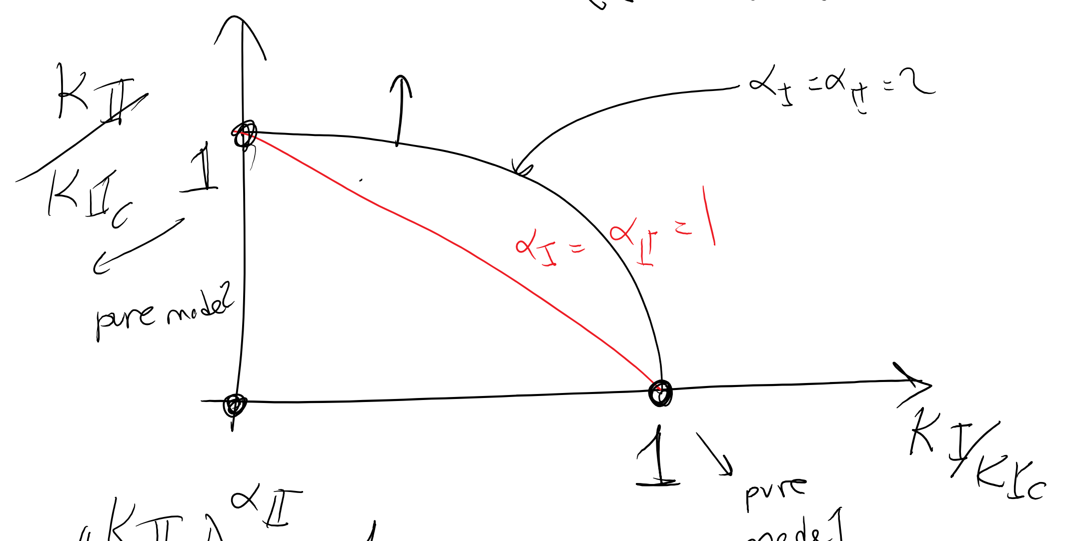
Mode II



$K(\bar{\tau}, a, \text{geom}) = K_{IIc}$

crack can propagate

i. can the crack propagate?

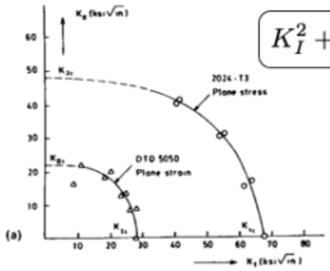


$$\left(\left|\frac{K_I}{K_{Ic}}\right|\right)^{\alpha_I} + \left(\left|\frac{K_{II}}{K_{IIc}}\right|\right)^{\alpha_{II}} = 1$$

examples of crack propagation criteria

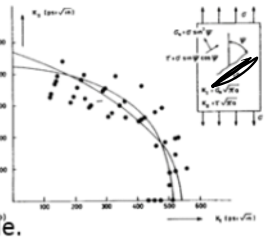
example

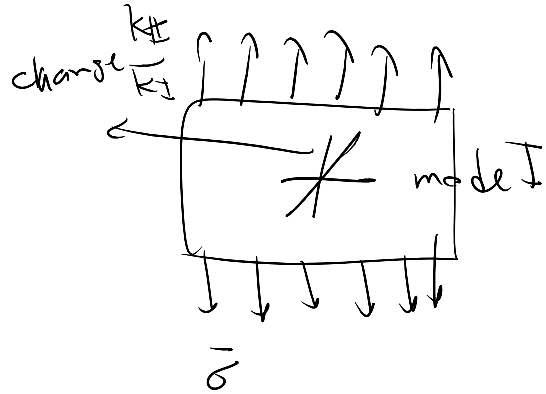
These criteria should be calibrated experimentally



$$\boxed{K_I^2 + K_{II}^2 = K_{Ic}^2}$$

a circle in
KI, KII plane





Data points do not fall exactly on the circle.

$$\underbrace{\left(\frac{K_I}{K_{Ic}}\right)^{2}_{\alpha_I} + \left(\frac{K_{II}}{K_{IIc}}\right)^{2} = 1}_{b}$$

α_{II}

self-similar growth

$$\boxed{G = \frac{(\kappa+1)K_I^2}{8\mu}}$$

326

with $\alpha_I = 2$ $\alpha_{II} = 2$ & $K_{IIc} = K_{Ic}$ we have

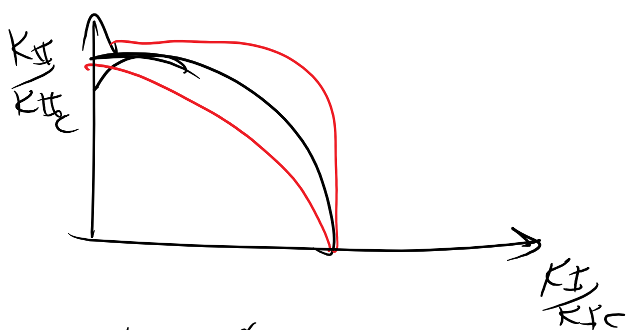
$$\frac{K_I^2 + K_{II}^2}{K_{Ic}^2} = 1 \qquad \text{crack propagates}$$

$$G = \frac{K_I^2 + K_{II}^2}{E'} = \frac{K_{Ic}^2}{E'} = G_c$$

↳ energy release rate

$$G = \frac{K_I^2 + K_{II}^2}{E'} = G_c$$

is a special case of failure criterion for



$$\left(\frac{K_I}{K_{Ic}}\right)^{\alpha_I} + \left(\frac{K_{II}}{K_{IIc}}\right)^{\alpha_{II}} \geq 1$$

for $\alpha_I = \alpha_{II} = 2$ & $K_{Ic} = K_{IIc}$

we need to calibrate failure criterion by experiment

There are three popular criteria that are often used to answer questions i to iii
(Does it propagate, what direction, how far)?

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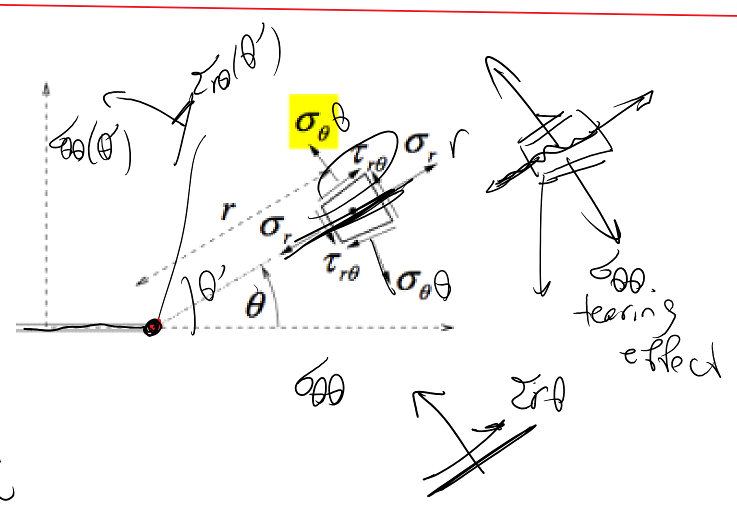


Maximum circumferential stress criterion

Erdogan and Sih

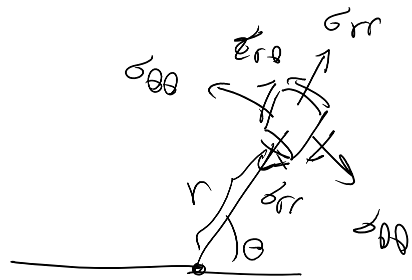
We are looking for a θ for which $\sigma_{\theta\theta}$ is maximum

If we use LEFM solution

$$\sigma_r = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right) \tag{7.35a}$$

$$\sigma_\theta = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right) \tag{7.35b}$$

$$\tau_{r\theta} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right). \tag{7.35c}$$



σ_θ is Max $\equiv$ $\underline{\tau_{r\theta} = 0}$

to find crack propagation direction for maximum circumferential (loop stress) criterion, we find θ for which $\tau_{r\theta}(\theta) = 0$

(7.35b)

$$\tau_{r\theta} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right). \tag{7.35c}$$

$$\tau_{r\theta} = 0 \longrightarrow \quad K_I\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + K_{II}\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right) = 0$$

$$\longrightarrow \quad \boxed{\theta_c = 2\arctan\frac{1}{4}\left(K_I/K_{II} \pm \sqrt{(K_I/K_{II})^2 + 8}\right)}$$

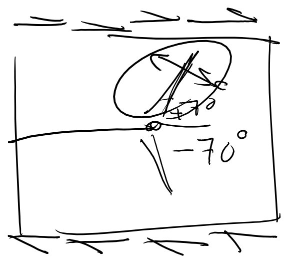
330

$K_{II} = 0 \qquad \theta_c = 0$

$K_I = 0 \qquad \theta_c \approx 70^\circ$

$K_I = 0$

$$\theta_c = 2\arctan \frac{1 \pm \sqrt{8}}{4} \simeq \pm 70^\circ$$



$\sigma_{nn} = ?$

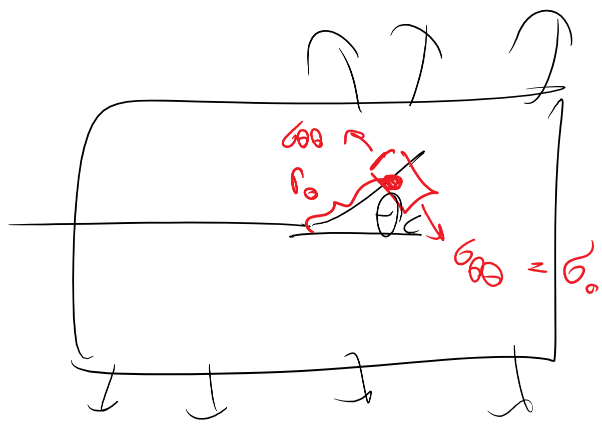
the correct solution

We answered Q #1: What direction.

How about Q #2: Does it even propagate?

K_I, K_{II}

θ_c



Plug θ_c in eqn for $\sigma_{\theta\theta}$ to obtain

$$\sigma_{\theta max} = \frac{1}{\sqrt{2\pi r_0}}\left(\cos\frac{\theta_0}{2}\left(K_I \cos^2\frac{\theta_0}{2} - \frac{3}{2} K_{II} \sin\theta_0\right)\right)$$

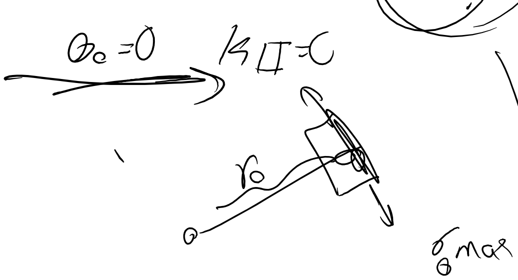
$$\sigma_{\theta\theta} = \frac{K_{\rm I}}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{\rm II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\theta_c = 2\arctan\frac{1}{4}\left(K_I/K_{II} \pm \sqrt{(K_I/K_I I)^2 + 8}\right)$$

$$\sigma_{\theta max}\sqrt{2\pi r_0} = K_{\rm Ic} = \cos\frac{\theta_0}{2}\left[K_{\rm I}\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{\rm II}\sin\theta_0\right]$$

assume $K_{II} = 0 \Rightarrow \theta_c = 0$

$$? = \sigma_{\theta max}\sqrt{2\pi r_0} = \cos\frac{\theta_0}{2}\left(K_I \cos^2\frac{\theta_0}{2} - \frac{3}{2} K_{II} \sin\theta_0\right)$$



$$\boxed{\sigma_{\theta max}\sqrt{2\pi r_0} = K_I} = K_{IC}$$

from propagation we should have $K_I = K_{IC}$

For general loading ? must be K_{IC}

$$\cos\frac{\theta_0}{2}\left(K_I\cos^2\frac{\theta_0}{2}-\frac{3}{2}K_{II}\sin\theta_0\right)\geq K_{IC}$$

find θ_0

$$\tan\frac{\theta_0}{2}=\frac{1}{4}\frac{K_I}{K_{II}}+\frac{1}{4}\sqrt{\left(\frac{K_I}{K_{II}}\right)^2+8}$$

This is summarized here:

Fracture criterion $\quad K_{eq} \geq K_{Ic}$

$$\sigma_\theta=\frac{K_{\rm I}}{\sqrt{2\pi r}}\cos\frac{\theta_0}{2}\left(1-\sin^2\frac{\theta_0}{2}\right)+\frac{K_{\rm II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta_0}{2}-\frac{3}{4}\sin\frac{3\theta_0}{2}\right) \tag{7.9}$$

must reach a critical value which is obtained by rearranging the previous equation

$$\sigma_{\theta max}\sqrt{2\pi r}=K_{\rm Ic}=\cos\frac{\theta_0}{2}\left[K_{\rm I}\cos^2\frac{\theta_0}{2}-\frac{3}{2}K_{\rm II}\sin\theta_0\right] \tag{7.10}$$

which can be normalized as

$$\boxed{\frac{K_{\rm I}}{K_{\rm Ic}}\cos^3\frac{\theta_0}{2}-\frac{3}{2}\frac{K_{\rm II}}{K_{\rm Ic}}\cos\frac{\theta_0}{2}\sin\theta_0=1} \tag{7.11}$$

11 This equation can be used to define an equivalent stress intensity factor K_{eq} for mixed mode problems

$$\boxed{K_{eq}=K_{\rm I}\cos^3\frac{\theta_0}{2}-\frac{3}{2}K_{\rm II}\cos\frac{\theta_0}{2}\sin\theta_0} \tag{7.12}$$

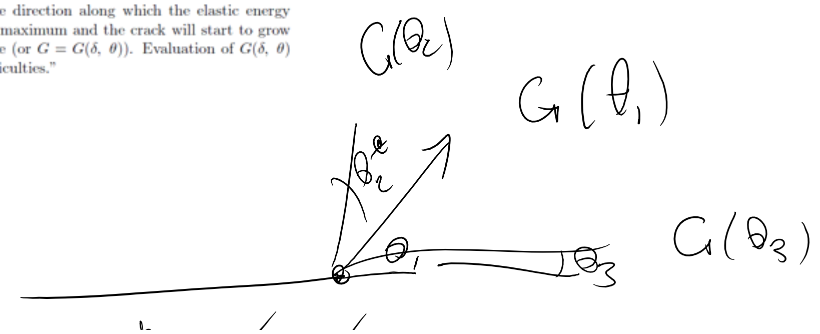
Other two criteria

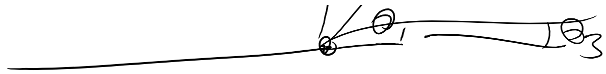
2. Max energy release rate:

G: crack driving force -> crack will grow in the direction that G is maximum

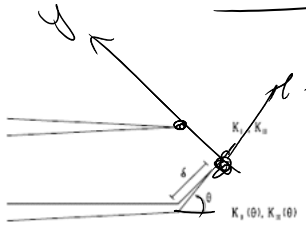
Erdogan, F. and Sih, G.C. 1963

"If we accept Griffith (energy) theory as the valid criteria which explains crack growth, then the crack will grow in the direction along which the elastic energy release per unit crack extension will be maximum and the crack will start to grow when this energy reaches a critical value (or $G = G(\delta,\ \theta)$). Evaluation of $G(\delta,\ \theta)$ poses insurmountable mathematical difficulties."





$\omega(\theta_3)$



K_I, K_{II}

θ in kink angle

Stress intensity factors for **kinked crack extension**:
Hussain, Pu and Underwood (Hussain et al. 1974)

$\theta = 0$

$$\begin{Bmatrix} K_I(\theta) \\ K_{II}(\theta) \end{Bmatrix} = \left(\frac{4}{3+\cos^2\theta}\right)\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{2\pi}} \begin{Bmatrix} K_I\cos\theta + \frac{3}{2}K_{II}\sin\theta \\ K_{II}\cos\theta - \frac{1}{2}K_I\sin\theta \end{Bmatrix}$$

w.r.t. x', y' axis

$$\begin{bmatrix} K_I(\theta) \\ K_{II}(\theta) \end{bmatrix} = \left(\frac{4}{3+1}\right)\left(\frac{1-0}{1+0}\right)^0 \begin{Bmatrix} K_I \times 1 + 0 \\ K_{II} \times 1 + 0 \end{Bmatrix}$$

$$\begin{bmatrix} K_I(\theta = 0) \\ K_{II}(\theta = 0) \end{bmatrix} = \begin{bmatrix} K_I \\ K_{II} \end{bmatrix}$$

plug these in energy release rate equation

$$G(\theta) = \frac{K_I^2(\theta) + K_{II}^2(\theta)}{E'}$$

$$G(\theta) = \frac{1}{E'}\left(K_I^2(\theta) + K_{II}^2(\theta)\right)$$

$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}}$$
$$[(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2]$$

find θ that maximizes $G(\theta)$

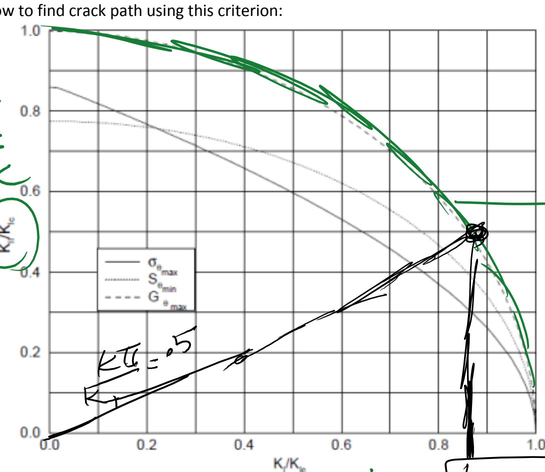
Maximization condition

$$\frac{\partial G(\theta)}{\partial \theta} = 0$$
$$\frac{\partial^2 G(\theta)}{\partial \theta^2} < 0$$

$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}} [(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2]$$

$$\boxed{4\left(\frac{1}{3+\cos^2\theta_0}\right)^2\left(\frac{1-\frac{\theta_0}{\pi}}{1+\frac{\theta_0}{\pi}}\right)^{\frac{\theta_0}{\pi}} \frac{}{2}\left[\left(1+3\cos^2\theta_0\right)\left(\frac{K_I}{K_{Ic}}\right)^2 + 8\sin\theta_0\cos\theta_0\left(\frac{K_I K_{II}}{K_{Ic}^2}\right) + \left(9-5\cos^2\theta_0\right)\left(\frac{K_{II}}{K_{Ic}}\right)^2\right] = 1}$$

How to find crack path using this criterion:

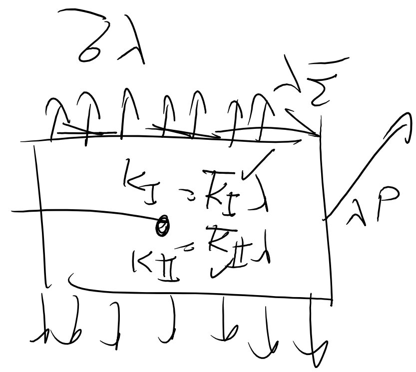


$\frac{K_{II}}{K_{Ic}}$

max energy release rate

K_I/K_{Ic}

$\boxed{\frac{K_I}{K_{Ic}} = .87}$



$\bar\sigma$

$\frac{K_I}{K_{II}}$ ✓ $\quad\lambda$ increases until the crack propagates

$$\frac{K_{II}}{K_I} = \frac{\bar K_{II}\lambda}{\bar K_I\lambda} = \frac{\bar K_{II}}{\bar K_I}$$

assume $\frac{\bar K_{II}}{\bar K_I} = .5$

$$\bar K_I \lambda = .87\, K_{Ic}$$

$\lambda = ?$

Last criterion: energy density in bulk

..... Energy Density

Strain Energy Density (SED) criterion

Sih 1973

$$U_i = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} \quad \boxed{U_i = \frac{1}{4\mu}\left[\frac{\kappa+1}{4}(\sigma_x^2+\sigma_y^2) - 2(\sigma_x\sigma_y - \tau_{xy}^2)\right]}$$

elastic energy density per unit volume

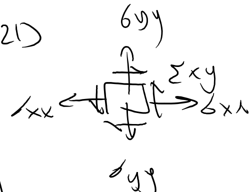
$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (7.13)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right).$$

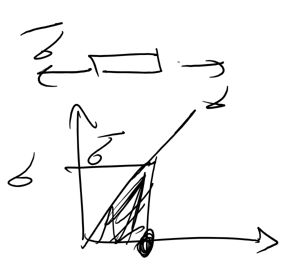
$$S = rU_i$$

2D



1D

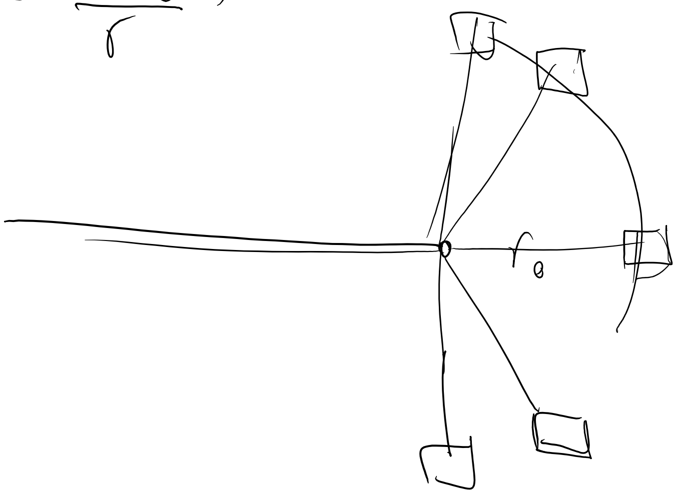
$\bar\varepsilon = \frac{\bar\sigma}{E}$



U_i internal energy density $= \frac{\bar\sigma\bar\varepsilon}{2} = \frac{\bar\sigma^2}{2E}$

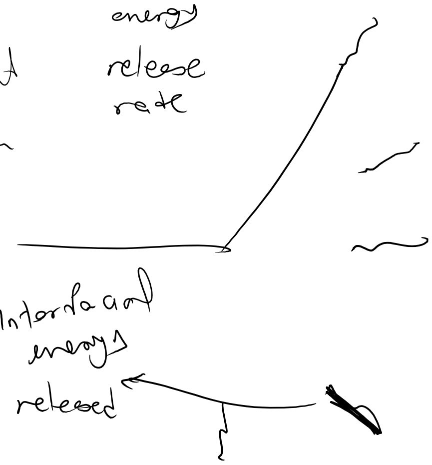
$S = rU_i$

$$U_i = \frac{S(\theta)}{r} \longrightarrow \text{only a function of } (\theta)$$



Different from

max energy release rate



Interfacial energy released

Crack propagates in a direction that Si is MINIMUM not maximum:

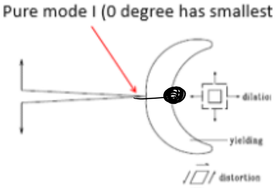
Strain Energy Density (SED) criterion

- Crack direction θ_0 which **minimizes** the strain energy density S
- Crack Extends when S reaches a critical value at a distance r_0

Minimization condition

$$\frac{\partial S}{\partial \theta} = 0$$
$$\frac{\partial^2 S}{\partial \theta^2} > 0$$

Pure mode I (0 degree has smallest S)



Strain Energy Density (SED) criterion

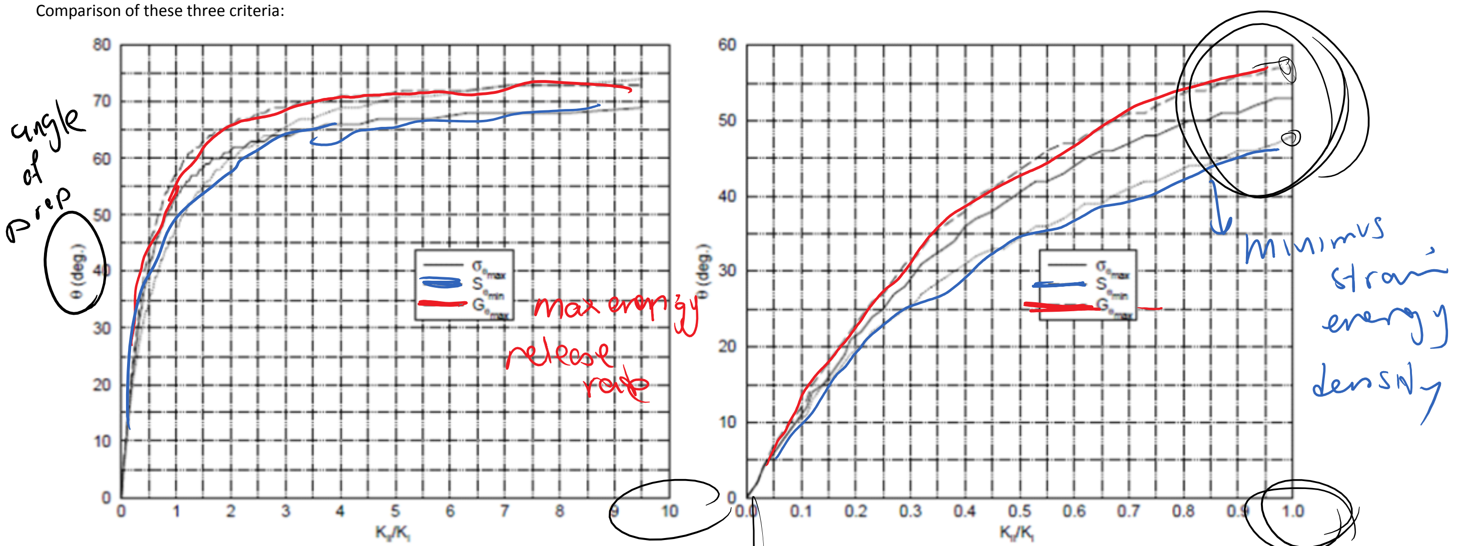
$$\boxed{\frac{8\mu}{(\kappa-1)}\left[a_{11}\left(\frac{K_I}{K_{Ic}}\right)^2+2a_{12}\left(\frac{K_IK_{II}}{K_{Ic}^2}\right)+a_{22}\left(\frac{K_{II}}{K_{Ic}}\right)^2\right]=1}$$

$$\begin{aligned}
a_{11} &= \frac{1}{16\mu}\left[(1+\cos\theta)(\kappa-\cos\theta)\right]\\
a_{12} &= \frac{\sin\theta}{16\mu}\left[2\cos\theta-(\kappa-1)\right]\\
a_{22} &= \frac{1}{16\mu}\left[(\kappa+1)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)\right]
\end{aligned}$$

$$\begin{aligned}
\kappa &= \tfrac{3-\nu}{1+\nu} \quad \text{(plane stress)}\\
\kappa &= 3-4\nu \quad \text{(plane strain)}
\end{aligned}$$

342

Comparison of these three criteria:

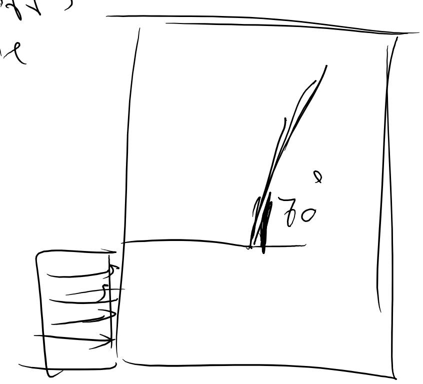


K_{II}/K_{I}

$K_{II}=0$ $\quad$ $K_{II} < K_I$ $\quad$ $K_{II}/K_I = 1$

K_I dominant

Kalthoff's example



Mixed mode criteria: Observations

(i) What

Mixed mode criteria: Observations

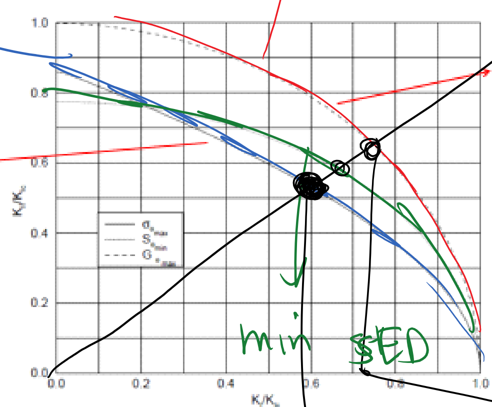
(i) What direction

1. First crack extension θ_0 is obtained followed by on whether crack extends in θ_0 direction or not.
2. Strain Energy Density (SED) and Maximum Circumferential Tensile Stress require an r_0 but the final crack propagation locus is independent of r_0.
3. SED theory depends on Poisson ratio ν.
4. All three theories give identical results for **small ratios of K_{II}/K_I** and diverge slightly as this ratio increases
5. Crack will always extend in the direction which attempts to minimize K_{II}/K_I.
6. For practical purposes during crack propagation all three theories yield very similar paths as from 4 and 5 cracks extend mostly in mode I where the there is a better agreement between different criteria

(ii) Does it propagate?

max energy release rate

max circumferential stress



Least conservative

Most conservative

min SED

Small KI

$$\frac{K_{II}}{K_I} = \text{known}$$

Max K_I least conservative