2020/08/31

Monday, August 31, 2020 2:46 PM

4. Linear Elastic Fracture Mechanics (LEFM)

Discrepancy between theoretical and experimental values of material strengths:

If the force between two particles is governed by a potential the force is derived from:

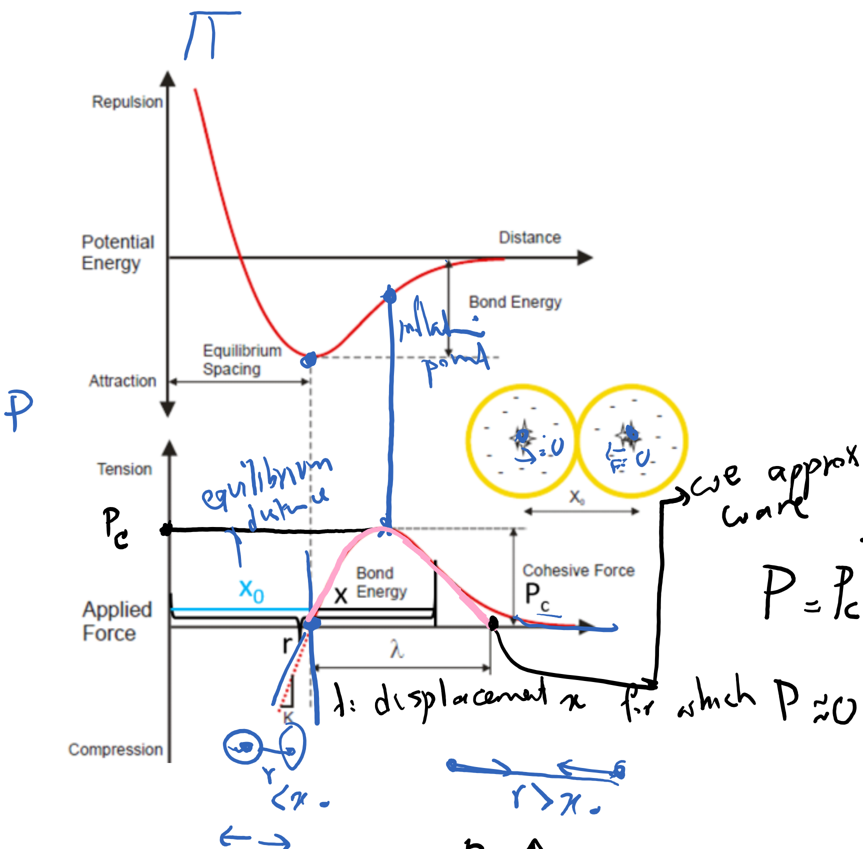
$$\vec{P} = -\frac{\partial \Pi}{\partial \vec{r}}$$

($\vec{P}$: force vector; Π: potential; $\vec{r}$: position vector)

in 1D

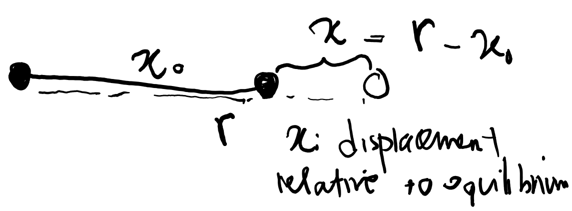
$P > 0$ tensile

$$P = \frac{d\Pi}{dr}$$



we approximate this part by a sine curve

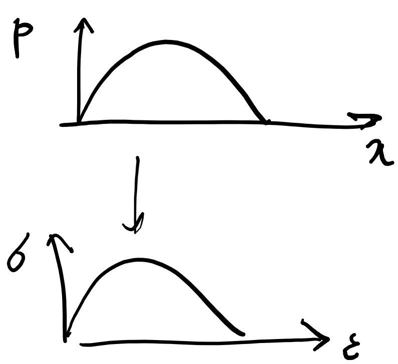
$$P = P_c \sin\left(\frac{\pi x}{\lambda}\right)$$



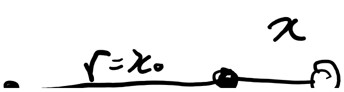
- $x = 0$ ($r = x_0$) $P = 0$
- $x = \lambda \rightarrow P = 0$

$$P = P_c \sin\left(\frac{\pi x}{\lambda}\right)$$

$P \rightarrow \sigma$, $x \rightarrow \varepsilon$: want to make these changes

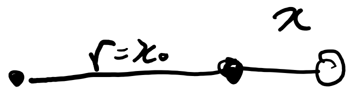


I)



I)

$x \rightarrow \varepsilon$

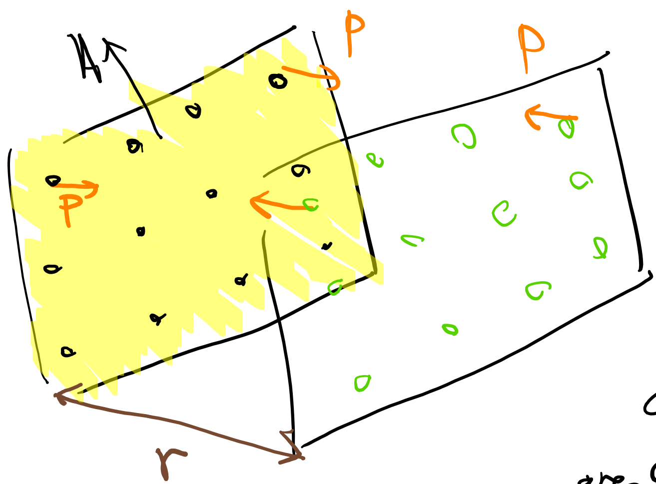


$$\boxed{x = x_0 \varepsilon} \quad (1)$$

$$\varepsilon = \frac{\text{new length} - \text{old length}}{\text{old length}} = \frac{x}{x_0}$$

II) $P \rightarrow \sigma$

two parallel atom planes



all equal forces

$F = \sum P = nP$

n → number of atoms in the plane intersection

e.g. $n = 12$

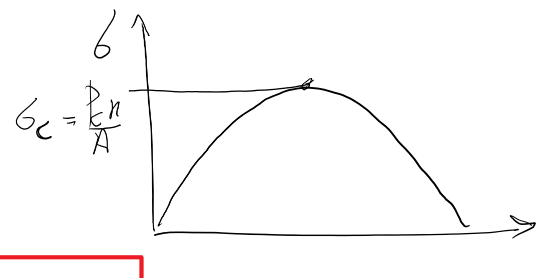
$\sigma = \frac{F}{A}$

area ← A

$$\rightarrow \boxed{\sigma = \frac{Pn}{A}} \quad (2)$$

(1) & (2) & $P \approx P_c \sin\left(\frac{\pi x}{\lambda}\right)$

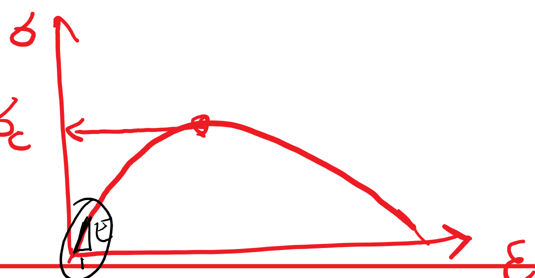
$$\rightarrow \sigma = \frac{P_c n}{A} \sin\left(\frac{\pi x_0}{\lambda} \varepsilon\right)$$



$$\sigma = \sigma_c \sin\left(\frac{\pi x_0}{\lambda} \varepsilon\right)$$

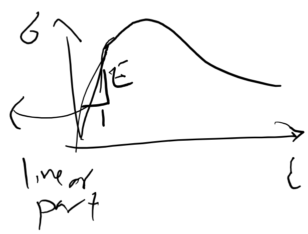
$\varepsilon - \sigma$ relation

strength ← σ_c



small strain

Young's modulus E



$$E = \frac{d\sigma}{d\varepsilon} \quad (\varepsilon = 0)$$

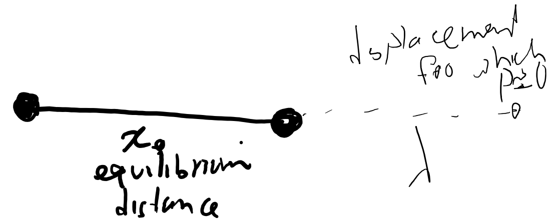
$$\frac{d\sigma}{d\varepsilon} = \sigma_c \frac{\pi x_0}{\lambda} \cos\left(\frac{\pi x_0}{\lambda} \varepsilon\right) \quad @ \quad \varepsilon = 0$$

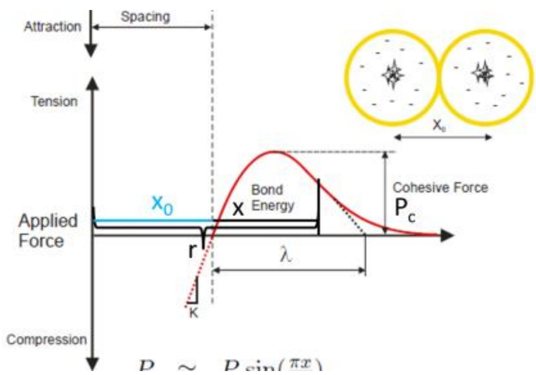
$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\dots} = E = \sigma_c \pi x_0$$

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0} = E \approx \sigma_c \frac{\pi x_0}{\lambda}$$

$\longrightarrow$ (3)

$$\boxed{\sigma_c = \frac{E}{\pi}\frac{\lambda}{x_0}}$$





generally

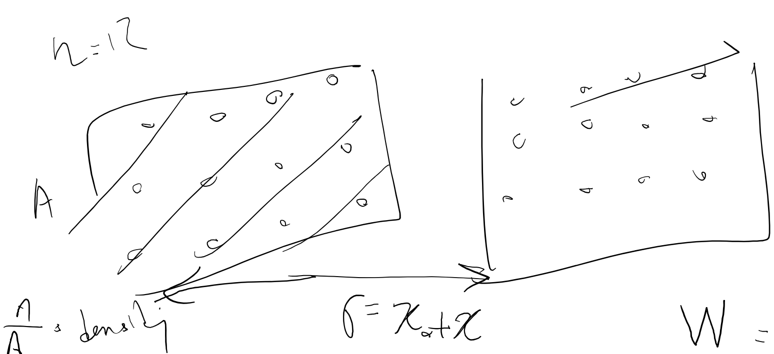
$\lambda \approx x_0$ $\frac{\lambda}{x_0} = O(1)$

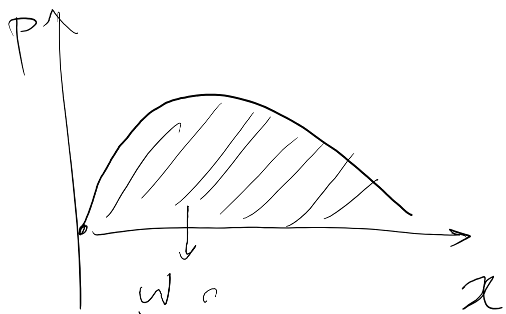
$$\boxed{\sigma_c \approx \frac{E}{\pi}}$$ (4)

steel $E = 200$ GPa $\longrightarrow$ expect $\sigma_c \approx 60$ GPa

example $\sigma_c \gg \sigma_y$ $\sigma_y = 250$ MPa

Aside from E, we want to derive fracture toughness from the atomistic model:





$$\underset{\text{work}}{W} = \int P\,dx = \int_0^\lambda P_c \sin\left(\frac{\pi x}{\lambda}\right) dx$$

$\longrightarrow$ $\underset{\text{work of breaking 1 atom bond}}{W} = \frac{2\lambda P_c}{\pi}$

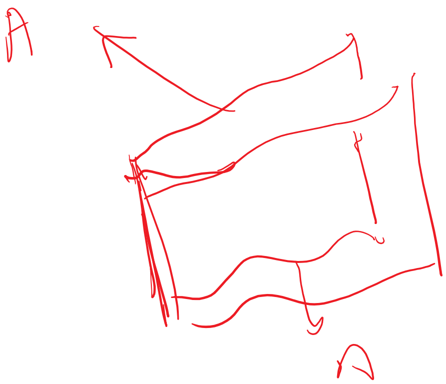
work of breaking n atoms $= \frac{2n\lambda P_c}{\pi}$

work of breaking atomistic bonds / surface $= 2\gamma_s$ (because crack has two surfaces) work of breaking atomistic bonds / surface = $2\gamma_s$

$= \frac{2\lambda}{\pi}\left(\frac{\pi P_c}{A}\right)$ // σ_c

fracture toughness

"energy needed to break atomic bonds per surface area"



$\frac{\lambda}{\pi}\sigma_c = \gamma_s \longrightarrow$

$$\sigma_c = \frac{\pi \gamma_s}{\lambda} \quad (5)$$

already had

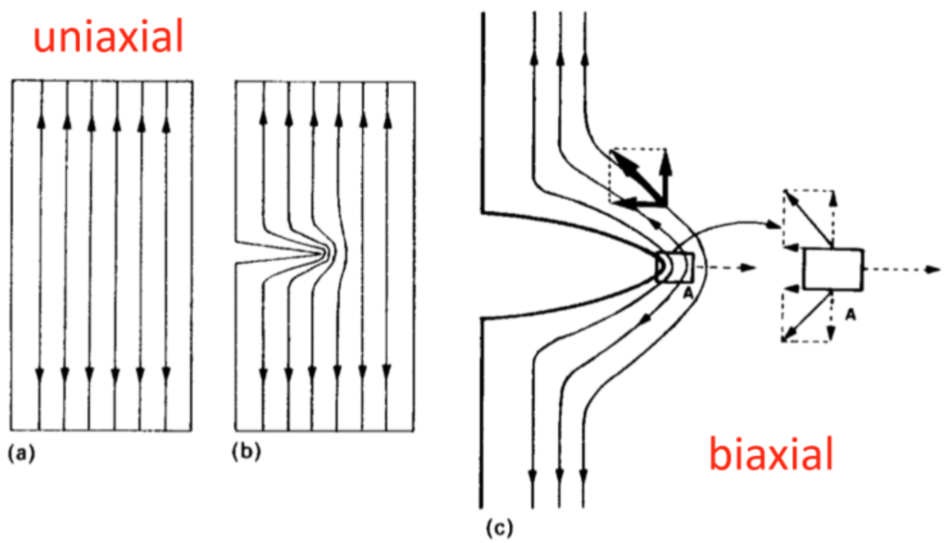
$$\sigma_c = \frac{E}{\pi}\frac{\lambda}{x_0}$$

multiply these by each other →

$$\sigma_c^2 = \frac{\gamma_s E}{x_0} \quad (6)$$

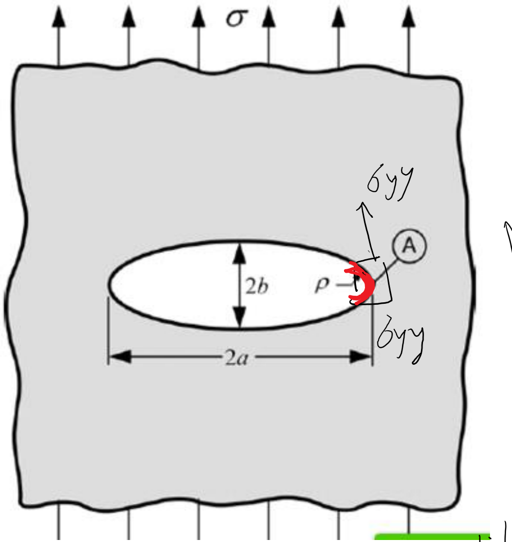
$$\sigma_c = \sqrt{\frac{E\gamma_s}{x_0}} \qquad \sigma_c = \frac{E}{\pi}\frac{\lambda}{x_0}$$

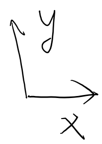
Stress concentration (cont.)



Elliptic hole

Inglis, 1913, theory of elasticity





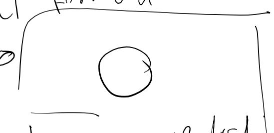
$$(\sigma_{yy})_A = \sigma\left(1 + \frac{2a}{b}\right)$$

$$\rho = \frac{b^2}{a}$$

as $b \to 0$, $(\sigma_{yy})_A \to \infty$

useful formula

$a = b$



stress concentration $= 3$

$$(\sigma_{yy})_A = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$

as $\rho \to 0$ approaching a sharp crack

$$(\sigma_{yy})_A \approx 2\sigma\sqrt{\frac{a}{\rho}}$$

$(\sigma_{yy})_A$: σ_{max}; σ: far field

we increase σ until atomic bonds break somewhere in the specimen. (A)

$$\sigma_c = 2\sigma_f\sqrt{\frac{a}{\rho}}$$

at failure

σ_f: continuum strength — what we experimentally measure as strength

(7)
$$\sigma_f = \sqrt{\frac{\rho}{4a}}\;\sigma_c$$

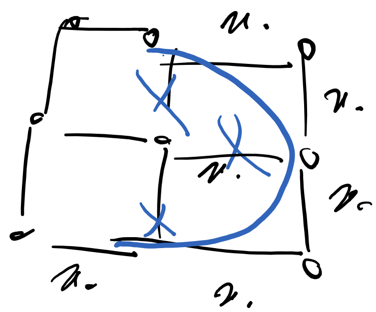
Recall $\sigma_c = \frac{E}{\pi}\frac{\lambda}{x_0}$ or $\sqrt{\frac{E\gamma_s}{x_0}}$

as $\rho \to 0$ $\sigma_f \to 0$

but there is a lower limit to ρ.

The sharpest crack has $\rho \approx x_0$

$\rho \approx x_0$



$$\sigma_f = \sqrt{\frac{x_0}{4a}}\;\sigma_c$$

sharpest crack possible

recall $\sigma_c = \sqrt{\frac{E\gamma_s}{\ldots}}$

recall $\boxed{\sigma_c = \sqrt{\frac{E\gamma_s}{x_0}}}$

$$\rightarrow \quad \sigma_f = \sqrt{\frac{x_0}{4a}}\sqrt{\frac{E\gamma_s}{x_0}} = \sqrt{\frac{E\gamma_s}{4a}}$$

Summary $\quad a \gg x_0$

$$\sigma_f = \sqrt{\frac{E\gamma_s}{4a}} \ll \sigma_c = \sqrt{\frac{E\gamma_s}{x_0}}$$

Continuum strength $\qquad\qquad$ atomistic strength

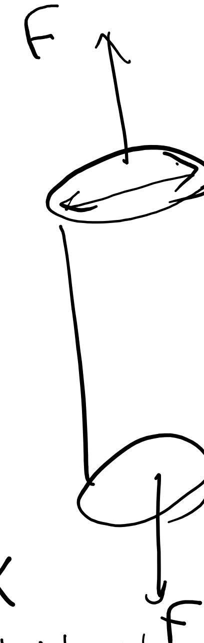
FM was developed during WWI by English aeronautical engineer A. A. Griffith to explain the following observations:

- The stress needed to fracture bulk glass is around 100 MPa
- The theoretical stress needed for breaking atomic bonds is approximately 10,000 MPa
- experiments on glass fibers that Griffith himself conducted: the fracture stress increases as the fiber diameter decreases => Hence the uniaxial tensile strength, which had been used extensively to predict material failure before Griffith, could not be a specimen-independent material property.

Griffith's size effect experiment

TABLE 1.1. Strength of glass fibers according to Griffith's experiments.

Diameter (10^{-3} in)	Breaking stress (lb/in^2)	Diameter (10^{-3} in)	Breaking stress (lb/in^2)
40.00	24 900	0.95	117 000
4.20	42 300	0.75	134 000
2.78	50 800	0.70	164 000
2.25	64 100	0.60	185 000
2.00	79 600	0.56	154 000
1.85	88 500	0.50	195 000
1.75	82 600	0.38	232 000
1.40	85 200	0.26	332 000
1.32	99 500	0.165	498 000
1.15	88 700	0.130	491 000



~20X

σ_c → atomistic-based strength

$$\sigma_{th} = \sqrt{\frac{E\gamma}{a_0}}$$

	a_0 [m]	E [GPa]	σ_{th} [GPa]	σ_b [MPa]	σ_{th}/σ_b
glass	$3 * 10^{-10}$	60	14	170	82
steel	10^{-10}	210	45	250	180
silica fibers	10^{-10}	100	31	25000	1.3
iron whiskers	10^{-10}	295	54	13000	4.2
silicon whiskers	10^{-10}	165	41	6500	6.3
alumina whiskers	10^{-10}	495	70	15000	4.7
ausformed steel	10^{-10}	200	45	3000	15
piano wire	10^{-10}	200	45	2750	16.4

→ continuum strength (σ_f)

bulk materials 100 1000

Griffith's verification experiment

- Glass fibers with artificial cracks (much larger than natural crack-like flaws), tension tests

	Crack Length, $2a$ mm	Measured Strength, σ_f MPa	$\sigma_f\sqrt{a}$ MPa$\sqrt{m}$
sample 1	3.8	6.0	0.26
sample 2	6.9	4.3	0.25
sample 3	13.7	3.3	0.27
sample 4	22.6	2.5	0.27

(Data from the Griffith experiment)

$\sigma_f = \sqrt{\frac{E\gamma_s}{4a}}$ $\qquad$ $\sigma_f\sqrt{a} = \sqrt{\frac{E\gamma_s}{4}}$ constant

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Previous derivation was based on stress concentration and basically was a **stress approach**.

We want to demonstrate the same phenomena using the **energy approach**.

4.1.3. Cause of discrepancy:

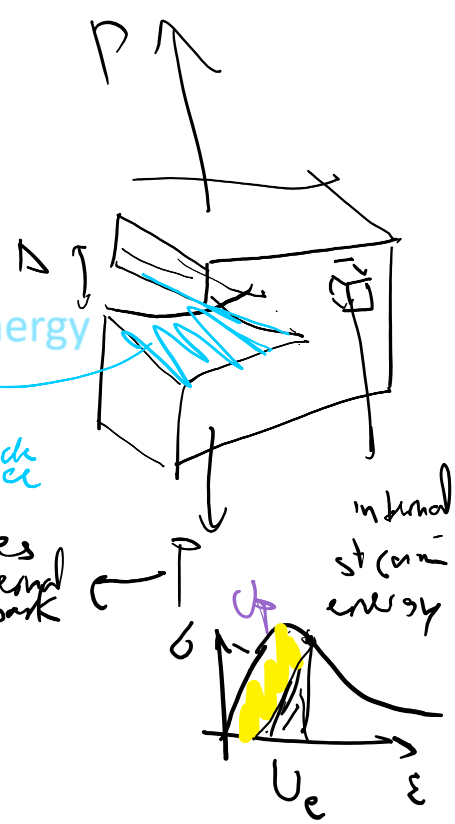
2. Energy approach

external work $\quad$ kinetic energy

$$\dot{W} = \dot{U}_e + \dot{U}_p + \dot{U}_k + \dot{U}_\Gamma$$

internal strain energy $\quad$ surface energy

energy rate to create crack surface



(W) does external work ← P

Brittle fracture: ignore Up term
Quasi-static fracture: ignore dynamic effects

$$\dot{W} = \dot{U}_e + \dot{U}_\Gamma$$

$$\Pi = U_e - W \quad (\rightarrow$$

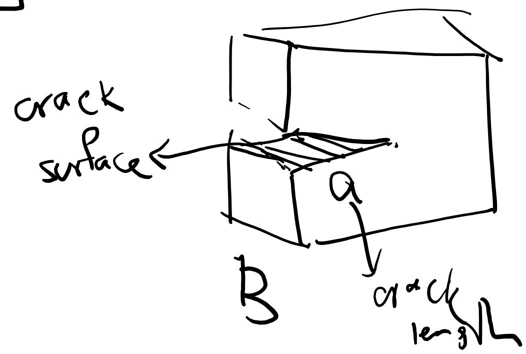
$$\Pi = U_e - W \quad \rightarrow \qquad U_e \quad \varepsilon$$

(Π: potential energy)

$$\boxed{\dot{U}_\Gamma = -\dot{\Pi} = \dot{W} - \dot{U}_e}$$

$$\frac{\partial U_\Gamma}{\partial t} = \frac{\partial U_\Gamma}{\partial a}\left(\frac{\partial a}{\partial t}\right)$$

$\frac{\partial a}{\partial t}$ ↓ crack speed



$$\frac{\partial(\cdot)}{\partial t} = \frac{\partial(\cdot)}{\partial a} V_c$$

$$\boxed{\frac{1}{B}\frac{\partial U_\Gamma}{\partial a} = -\frac{1}{B}\frac{\partial \Pi}{\partial a}}$$

$$A = aB$$

a: crack length; B: plate width

$$\rightarrow \quad \underbrace{\frac{\partial U_\Gamma}{\partial A}}_{2\gamma_s} = -\frac{\partial \Pi}{\partial A} = -\frac{1}{B}\frac{\partial \Pi}{\partial a}$$

$$\boxed{2\gamma_s = -\frac{1}{B}\frac{\partial \Pi}{\partial a} = -\frac{\partial \Pi}{\partial A}}$$