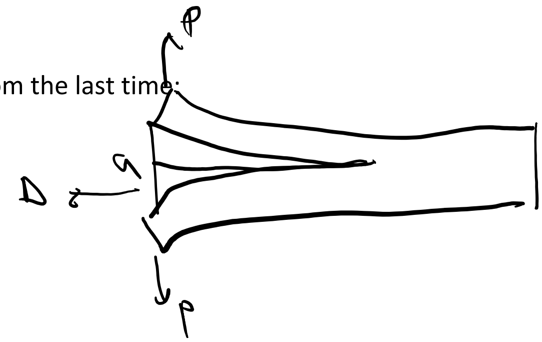
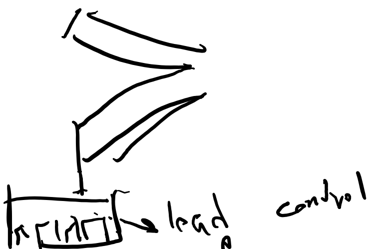
Continuation of the same problem from the last time:



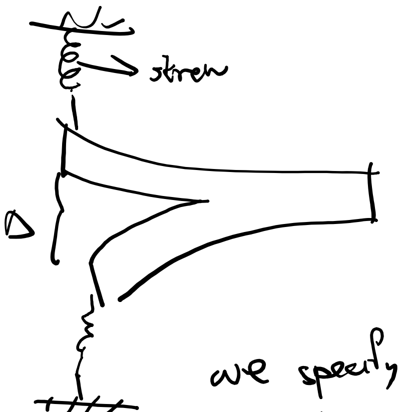
$$\Delta = \frac{2}{3}\frac{Pa^3}{EI} \quad (\star)$$

$$\overrightarrow{P}\ \frac{C}{3EI} = \Delta = ?a^3 \longrightarrow \quad G = \frac{P^2 a^2}{BEI}$$

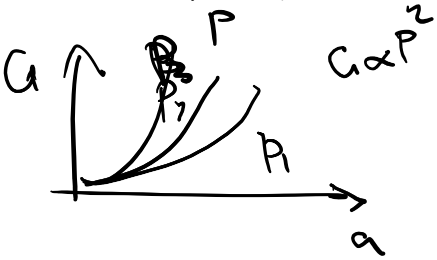
Displacement control:



we specify P



we specify Δ



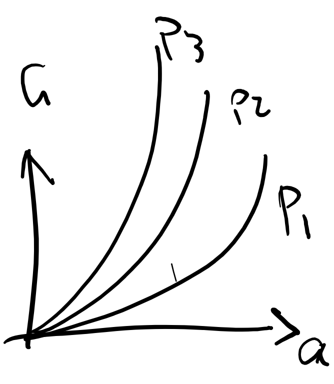
$G \propto P^2$

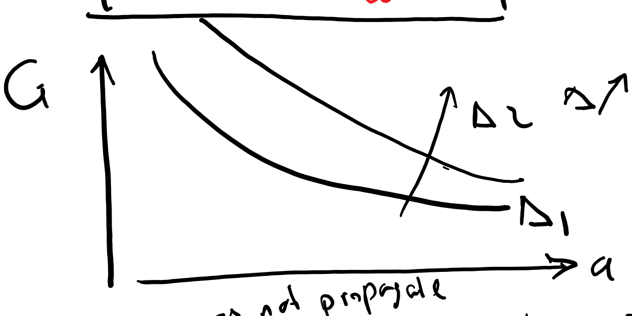
$$\left.\begin{array}{l} G = \dfrac{P^2 a^2}{BEI} \\ P = \dfrac{3}{2}\dfrac{EI\Delta}{a^3} \text{ from } (\star) \end{array}\right\} \longrightarrow$$

$$G = \frac{9}{4}\underbrace{\frac{(EI)^2\Delta^2}{a^6}}_{P^2}\frac{a^2}{EI}$$

$$\boxed{G(a,\Delta) = \frac{9EI\Delta^2}{4Ba^4}}$$

Displacement control



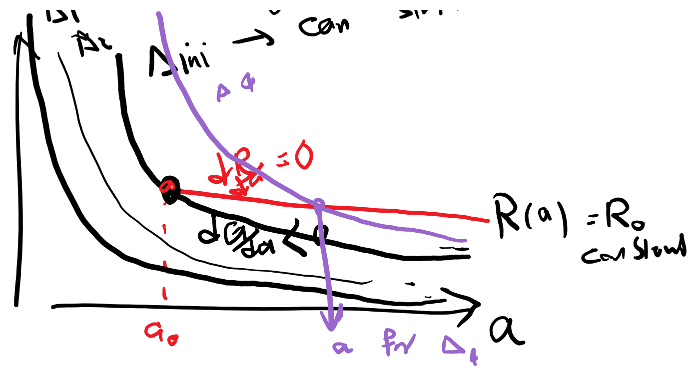


Δ1 does not propagate

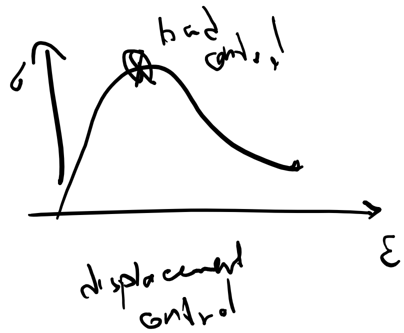
crack can start to propagate

Δ1 Δ2 | Δ_{ini} →

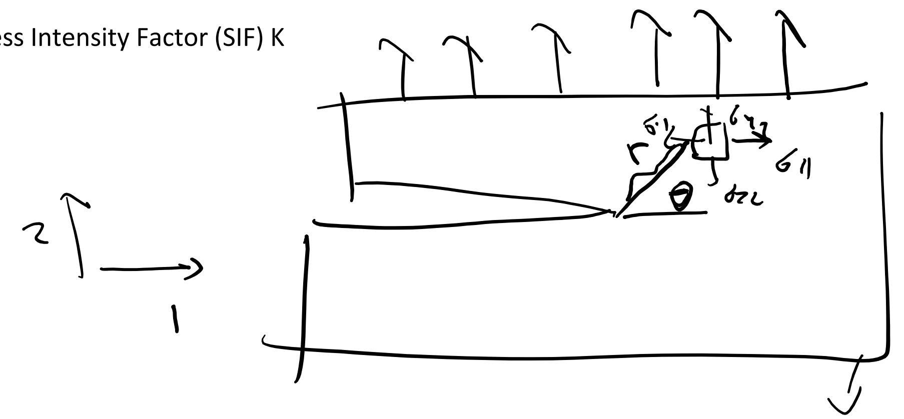
Δ4



similar to ε vs σ



Stress Intensity Factor (SIF) K



$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \Sigma_{22}^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} \Sigma_{22}^{II}(\theta) + \frac{K_{III}}{\sqrt{2\pi r}} \Sigma_{22}^{III}(\theta)$$

K_I, K_{II}, K_{III} are modes I, II, III SIFs & r, θ polar coordinates w.r.t. crack tip

We want to derive this equation.

Solution approach will be stress (Airy) function.

Typical approach for solving elastodynamics (statics)

- Balance law → $\nabla \cdot \sigma + \rho b = \rho \ddot{u}$

  EOM equation of motion

- constitutive equation $\quad \sigma = \mathbb{C}\, \varepsilon$

  $\mathbb{C}$: 4th order elasticity tensor

- compatibility $\quad \varepsilon = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right)$

$\rightarrow$

$$\nabla \cdot \mathbb{C} \nabla u + \rho b = \rho \ddot{u} \qquad \text{elastodynamics}$$

OR

$$\boxed{\nabla \cdot \mathbb{C} \nabla u + \rho b = 0} \qquad \text{elastostatics}$$

Example for $\mathbb{C}$

Constitutive equation

Hook's law $\quad \sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl} \qquad i,j,k,l = 1,2,3$

Isotropic

3D $\quad T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij} \quad \text{or} \quad \mathbf{T} = \lambda \mathbf{I}_E + 2\mu \mathbf{E}$

2D (plane strain)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ \nu & \nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

2D (plane stress)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$p = \rho \mathbf{v}$

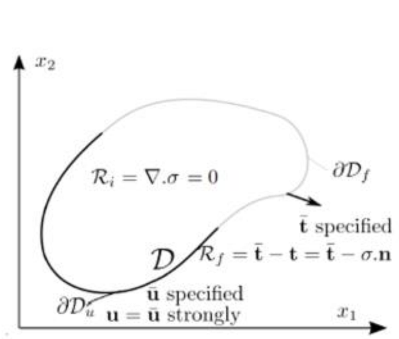
Displacement approach

$$\sigma_{ij,j} = 0$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk,j}\right)$$

$$\frac{E}{1+\nu}\left(\varepsilon_{ij,j} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk,j}\right) = 0$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$



$$★\ \boxed{(\lambda+\mu)\nabla\nabla.\mathbf{u} + \mu\nabla.\nabla\mathbf{u} = 0} \quad \text{or} \quad (\lambda+\mu)u_{j,ji} + \mu u_{i,jj} = 0$$

2nd order PDE for vector $\vec{u}$

BC's

solve u from ★

$$u \longrightarrow \varepsilon = \tfrac{1}{2}(\nabla u + \nabla u^T) \rightarrow \sigma = C\varepsilon$$

Displacement approach.

We can go in the other direction using the stress (Airy) function approach.

Stress function approach

What are Airy stress function approach?

Use of stress function ⇨

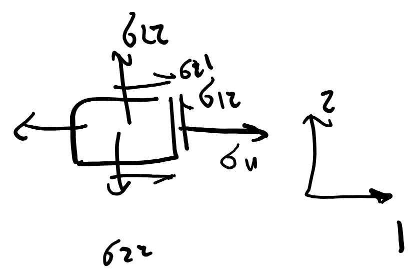
Balance of linear momentum is automatically satisfied (no body force, static)

$$\boxed{\psi(x_1, x_2) \quad \rightarrow \sigma_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk}} \quad (1)$$

arbitrary function (called Airy or stress function)

in 2D

$i=1, j=1$



$$\sigma_{11} = -\psi_{,11} + \delta_{11}(\psi_{,kk})$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \qquad \delta \text{ - kronecker}$$

$$\psi_{,kk} = \sum_{k=1}^{d} \psi_{,kk} = \psi_{,11} + \psi_{,22}$$

$d \rightarrow$ dimension

$\delta_{ij} = \begin{cases} \\ 0 & i \neq j \end{cases}$ δ - kronecker

$\psi_{,kk} = \sum_{k=1}^{d} \psi_{,kk} = \psi_{,11} + \psi_{,22}$ ← d → dimension

$i=1, j=1$: $\sigma_{11} = -\psi_{,11} + 1(\psi_{,11} + \psi_{,22}) = \psi_{,22}$

$i=1, j=2$: $\sigma_{12} = -\psi_{,12} + \underbrace{\delta_{12}}_{0}(\psi_{,11} + \psi_{,22}) = -\psi_{,12} = -\psi_{,21} = \sigma_{21}$

$i=2, j=2$: $\sigma_{22} = -\psi_{,22} + \underbrace{\delta_{22}}_{1}(\psi_{,11} + \psi_{,22}) = \psi_{,11}$

So for 2D stresses are obtained as

$$\psi \longrightarrow \boxed{\begin{aligned} \sigma_{11} &= \psi_{,22} \\ \sigma_{22} &= \psi_{,11} \\ \sigma_{12} &= -\psi_{,12} \end{aligned}} \quad (2)$$

Let's check EOM for

— Statics $\dot{u}=0$ ($\ddot{u}=0$)

— No body force ($b=0$)

$\nabla \cdot \sigma = 0$ in 2D this is

$$\nabla \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = 0 \qquad \sigma_{ij,j} = 0 \quad i=1,2$$

$$\boxed{\begin{aligned} &E1) \quad \sigma_{11,1} + \sigma_{12,2} = 0 \\ &E2) \quad \sigma_{21,1} + \sigma_{22,2} = 0 \end{aligned}} \quad (3)$$

Let's check if $\sigma_{11}, \sigma_{22}, \sigma_{12}$ from (2) satisfy (3)

$\sigma_{11} = \psi_{,22}$, $\sigma_{22} = \psi_{,11}$, $\sigma_{12} = -\psi_{,12}$

E1) $(\underbrace{\psi_{,22}}_{\sigma_{11}})_{,1} + (-\psi_{,12})_{,2} = \psi_{,221} - \psi_{,122} = 0$:)

E2) $(-\psi_{,12})_{,1} + (\underbrace{\psi_{,11}}_{\sigma_{22}})_{,2} = -\psi_{,121} + \psi_{,112} = 0$:)

so stresses obtained from (2)

automatically satisfy EOM ($\rho b = 0$, $\ddot{u} = 0$)

---

1. Choose ψ
2. Compute stresses from (2)
3. Compute strains from

$$\sigma = C\varepsilon \longrightarrow \varepsilon = D\sigma$$

D 4th order compliance

4. Compute displacement by integrating strain

$$\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$$

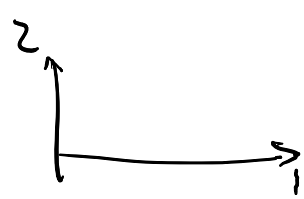
integrate

$\int \longrightarrow$ u = known :)

Maybe not so fast, …

Example:

$$\psi = x_1^4 \quad \text{in 2D}$$



$$\left\{\begin{aligned} \sigma_{11} &= \psi_{,22} = 0 \\ \sigma_{22} &= \psi_{,11} = 12x_1^2 \\ \sigma_{12} &= -\psi_{,12} = 0 \end{aligned}\right.$$

$\sigma_{11,1} + \sigma_{12,2} = 0$

$\sigma_{21,1} + \sigma_{22,2} = 0$

$\sigma \longrightarrow \varepsilon$ let's assume $\nu = 0$, $E = 1$

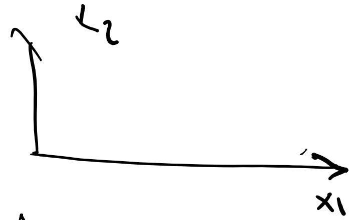
$\varepsilon_{11} = \frac{\sigma_{11}}{E} = 0$

$\varepsilon_{22} = \frac{\sigma_{22}}{E} = 12x_1^2$

$$\epsilon_{12} = \frac{\sigma_{12}}{G} = 0$$

$\varepsilon \longrightarrow u$

$$\begin{cases} \varepsilon_{11} = u_{1,1} = 0 & \text{e1)} \\ \varepsilon_{22} = u_{2,2} = 12 x_1^2 & \text{e2)} \\ \varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = 0 & \text{e3)} \end{cases}$$



e1 $\longrightarrow$ $\boxed{u_1(x_1, x_2) = f(x_2)}$ e4)

e2 $u_{2,2} = 12 x_1^2 \longrightarrow u_2 = \int 12 x_1^2 \, dx_2 + g(x_1)$

$$\boxed{u_2 = 12 x_1^2 x_2 + g(x_1)} \text{ e5)}$$

plug e4, e5 in e3

$$u_{1,2} + u_{2,1} = \frac{\partial f(x_2)}{\partial x_2} + \frac{\partial}{\partial x_1}\left[12 x_1^2 x_2 + g(x_1)\right] = 0$$

$$\boxed{\underbrace{f'(x_2)}_{\text{only a function of } x_2} + \underbrace{24 x_1 x_2}_{\text{function of } x_1 \& x_2} + \underbrace{g'(x_1)}_{\text{only a function of } x_1} = 0}$$

this is impossible to satisfy so

$$\varphi = x_1^4 \text{ is not valid!}$$

Problem is from here

$$\epsilon = \frac{(\nabla u + \nabla u^T)}{2} \quad \text{needs to be}$$

integrated

2D

$$\begin{cases} \epsilon_{11} = u_{1,1} \\ \epsilon_{22} = u_{2,2} \end{cases}$$

$$\left.\begin{matrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{matrix}\right\} \longrightarrow \begin{matrix} u_1 \\ u_2 \end{matrix}$$

$$\begin{cases} \epsilon_{11} = u_{1,1} \\ \epsilon_{22} = u_{2,2} \\ \epsilon_{12} = \frac{1}{2}(u_{1,2}+u_{2,1}) \end{cases} \qquad \left.\begin{matrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{matrix}\right\} \longrightarrow \begin{matrix} u_1 \\ u_2 \end{matrix}$$

In 2D the problem is that we have 3 equations and 2 unknowns (u1, and u2). This cannot always be solved!

If we satisfy the following compatibility conditions, then we can integrate strain to derive displacement:

Can we always obtain u by integration? No

3 displacements (unknowns)
6 strains (equations)

Need to satisfy strain compatibility condition(s)

$$\boxed{\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0.}$$

In 2D this is equivalent to only 1 equation:

$$\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0$$

Why?

assume we have a valid solution

$$\begin{aligned} \epsilon_{11} &= u_{1,1} \\ \epsilon_{22} &= u_{2,2} \\ \epsilon_{12} &= \frac{1}{2}(u_{1,2}+u_{2,1}) \end{aligned} \longrightarrow \begin{aligned} &\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = \\ &u_{1,122} + u_{2,211} - u_{1,212} - u_{2,112} = 0 \end{aligned}$$

So, basically all we need to do is to satisfy the compatibility condition

$$\varepsilon_{ij} = \frac{1+\nu}{E}\{-\psi_{,ij} + (1-\nu)\delta_{ij}\psi_{,kk}\}$$ $\{(\sigma_{11} = \psi_{,22} \ldots) +$ Constitution

$$2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = 0$$ compatibility

$$\underbrace{(\psi_{,11} + \psi_{,22})}_{\Delta\psi}{}_{,11} + (\psi_{,11} + \psi_{,22})_{,22} = 0$$

$$(\Delta\psi)_{,11} + (\Delta\psi)_{,22} = 0$$

$$\rightarrow \boxed{\Delta\Delta\psi = 0 \qquad \nabla^2\nabla^2\psi = 0} \;\star$$

$$\rightarrow \boxed{\Delta\Delta\psi = 0 \text{ or } \nabla^2\nabla^2\psi = 0} \star$$

another way this equation is written

- A function for which the Laplacian is zero is called Harmonic.
- A function for which * is satisfied is called biharmonic.

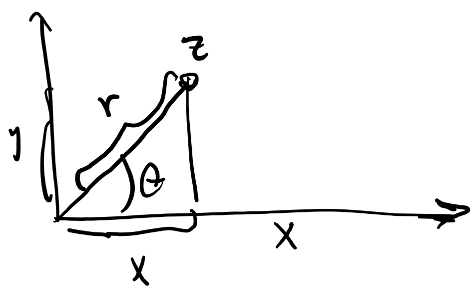
Any harmonic function is also biharmonic function but obviously not the other way around.

Compare the displacement and stress approaches:

Displacement approach	Stress approach
$\nabla \cdot C\nabla u + \rho b = 0$ (2nd order vector PDE; choose $b=0$)	$\nabla^2 \cdot \nabla^2 \psi = 0$ (4th order; Scalar PDE; Classical PDE)
solve u ↓	↓
u	σ (2D: $\sigma_{11} = \psi_{,22}$, $\sigma_{22} = \psi_{,11}$, $\sigma_{12} = -\psi_{,12}$)
↓	↓
$\varepsilon = \frac{1}{2}(\nabla u + \nabla u^t)$	$\varepsilon = C^{-1}\sigma$
↓	↓
$\sigma = C\varepsilon$	u obtained by integrating $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^t)$

The good thing is that there are a lot already existing harmonic functions

$$z = x + iy = re^{i\theta}$$



Ex.

$$f(z) = z^2$$
$$= (x + iy)^2 = (x^2 - y^2) + i(2xy)$$
$$= \underbrace{U(x,y)}_{\text{real}} + i\underbrace{V(x,y)}_{\text{imag.}}$$

U & V are harmonic!

$\Delta(x^2 - y^2) = 2 - 2 = 0$

$\Delta\, 2xy = 0 + 0 = 0$

$Re\, z^3,\ Im\, z^3, \ldots$

Re & Im of any complex function are harmonic

$f(z) = f(x,y) = U(x,y) + i\, V(x,y)$

$\rightarrow \quad f_{,xx} + f_{,yy} = (\underbrace{U_{,xx} + U_{,yy}}_{\Delta U}) + i(\underbrace{V_{,xx} + V_{,yy}}_{\Delta V})$

$f_{,x} = \dfrac{\partial f}{\partial z} \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial z}$ $\qquad z = x + iy$

$f_{,xx} = \dfrac{\partial^2 f}{\partial z^2} = f''$ $\qquad$ similarly

$f_{,y} = \dfrac{\partial f}{\partial z} \dfrac{\partial z}{\partial y} = (i) \dfrac{\partial f}{\partial z}$ $\qquad x + iy$

$f_{,yy} = (i)^2 \dfrac{\partial^2 f}{\partial z^2} = -f''$ $\qquad$ one more time

$f'' + -f'' = 0 = \Delta U + i\, \Delta V$

$$\boxed{\begin{array}{c}\Delta U = 0 \\ \Delta V = 0\end{array}} \qquad \begin{array}{l} U = Re\, f(z) \\ V = Im\, f(z) \end{array}$$

- Any biharmonic solution can be expressed by Kolonov-Muskhelishvili complex potentials, ϕ, χ:

$$\boldsymbol{\Psi}(x_1, x_2) = \mathrm{Re}\left[\bar{z}\phi + \chi\right]$$