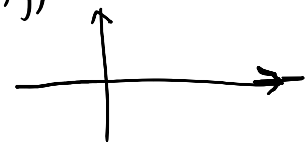
From last time we observed that

complex func. $f(z) = U(x,y) + i\,V(x,y)$

$\Delta U = 0 \quad \Delta V = 0$



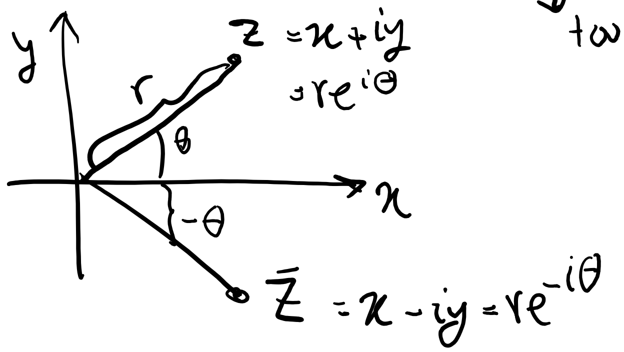
ψ we needed $\Delta\Delta\psi = 0$

- Any biharmonic solution can be expressed by Kolonov-Muskhelishvili complex potentials, ϕ, χ:

$$\Psi(x_1, x_2) = \mathrm{Re}\left[\bar{z}\phi + \chi\right]$$

the entire set of biharmonic functions

two arbitrary complex functions



Message: So the solution to $\Delta\Delta\psi = 0$

Is trivial for many problems by using the theorem above.

Today, we want to derive the exact crack tip solutions using this approach.

The entire stress, strain, and displacement solution

- Any biharmonic solution can be expressed by Kolonov-Muskhelishvili complex potentials, ϕ, χ:

$$\boldsymbol{\Psi}(x_1, x_2) = \mathrm{Re}\left[\bar{z}\phi + \chi\right]$$

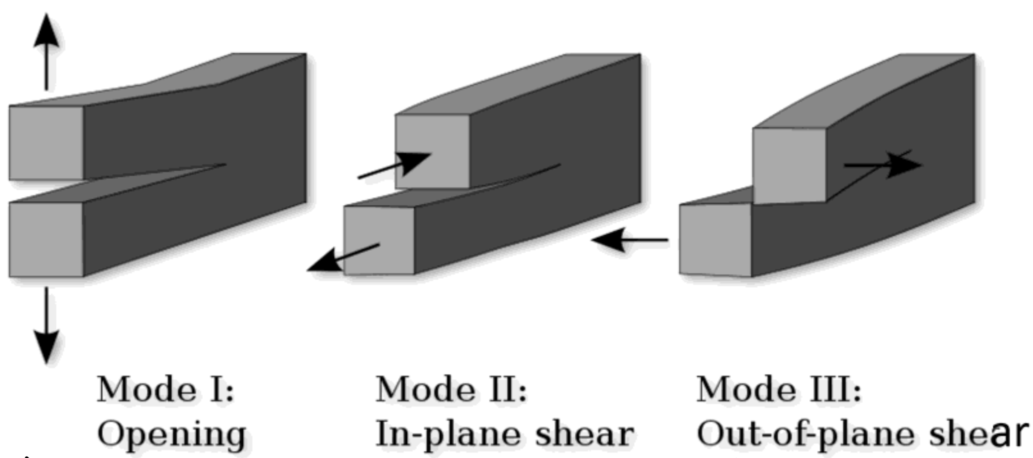
- Stresses are obtained differentiation,

$$\begin{aligned}
\sigma_{11} &= \boldsymbol{\Psi}_{,22} = \mathrm{Re}\left[\phi' - \frac{1}{2}\bar{z}\phi'' - \frac{1}{2}\chi''\right] \\
\sigma_{22} &= \boldsymbol{\Psi}_{,11} = \mathrm{Re}\left[\phi' + \frac{1}{2}\bar{z}\phi'' + \frac{1}{2}\chi''\right] \\
\sigma_{12} &= -\boldsymbol{\Psi}_{,12} = \frac{1}{2}\mathrm{Re}\left[\bar{z}\phi'' + \chi''\right]
\end{aligned}$$

- Displacements are obtained by integration of strains:

$$\begin{aligned}
u_1 &= \mathrm{Re}\left[\kappa\phi - \bar{z}\phi' - \chi'\right] \\
u_2 &= \mathrm{Im}\left[\kappa\phi + \bar{z}\phi' + \chi'\right] \\
\kappa &= \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}
\end{aligned}$$

3 Modes of fracture:



In plane modes

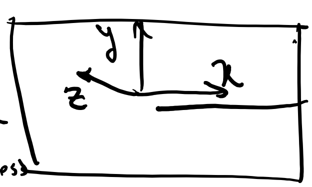
out of plane

$u_x \neq 0$

$u_y \neq 0$

$u_z = 0$ p. strain

$u_z(z)$ p. stress



Inplane $\sigma_{xx} \neq 0$ $\sigma_{yy} \neq 0$ $\sigma_{xy} \neq 0$

$\varepsilon_{xx} = 0$ $\varepsilon_{yy} \neq 0$ $\varepsilon_{xy} \neq 0$

out-of plane

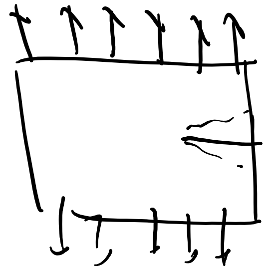
$(\sigma, \varepsilon)_{zx} = 0$

$(\sigma,\varepsilon)_{zy}=0$

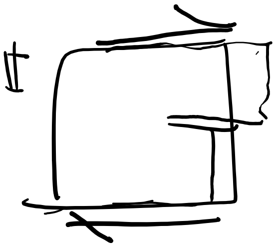
$\varepsilon_{zz}=0$ $\quad\sigma_{zz}=\nu(\sigma_{xx}+\sigma_{yy})$ plane-stain

$\varepsilon_{zz}\neq 0$ $\quad\sigma_{zz}=0$ plane-stress

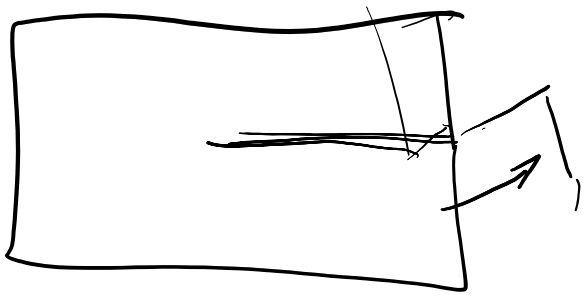
mode I

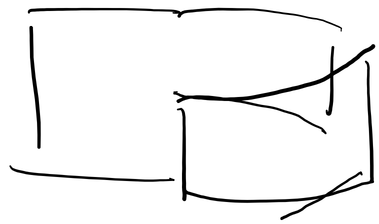


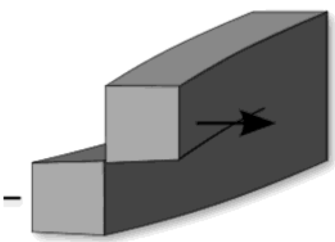
mode II



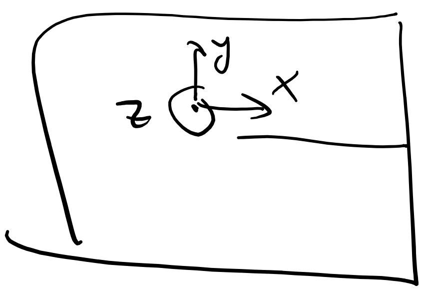
mode III







Mode III:
Out-of-plane shear



simpler problem

$u_x=u_y=0$

$u_z(x,y)$

in-plane stresses $\quad\sigma_{xx}=\sigma_{yy}=\sigma_{xy}=0$

out-of-plane " $\quad\sigma_{xz}=u_{z,x}/2\neq 0$

$\sigma_{yz}=u_{z,y}/2\neq 0$

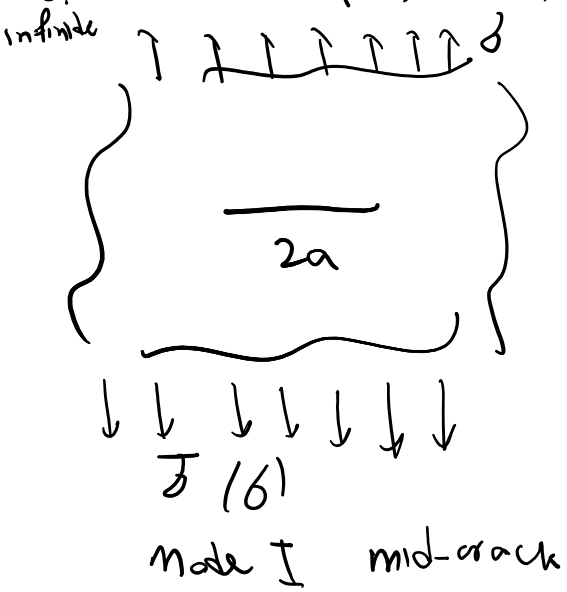
$\sigma_{zz}=0$

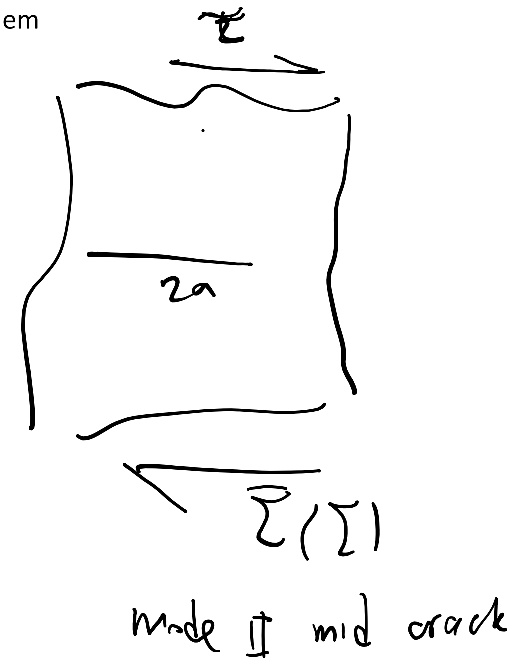
Generally, modes I and II solutions are more complex and difficult to derive.

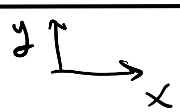
Getting the solutions for Mode I and II mid-crack problem

Generally, modes I and II solutions are more complex and difficult to derive.

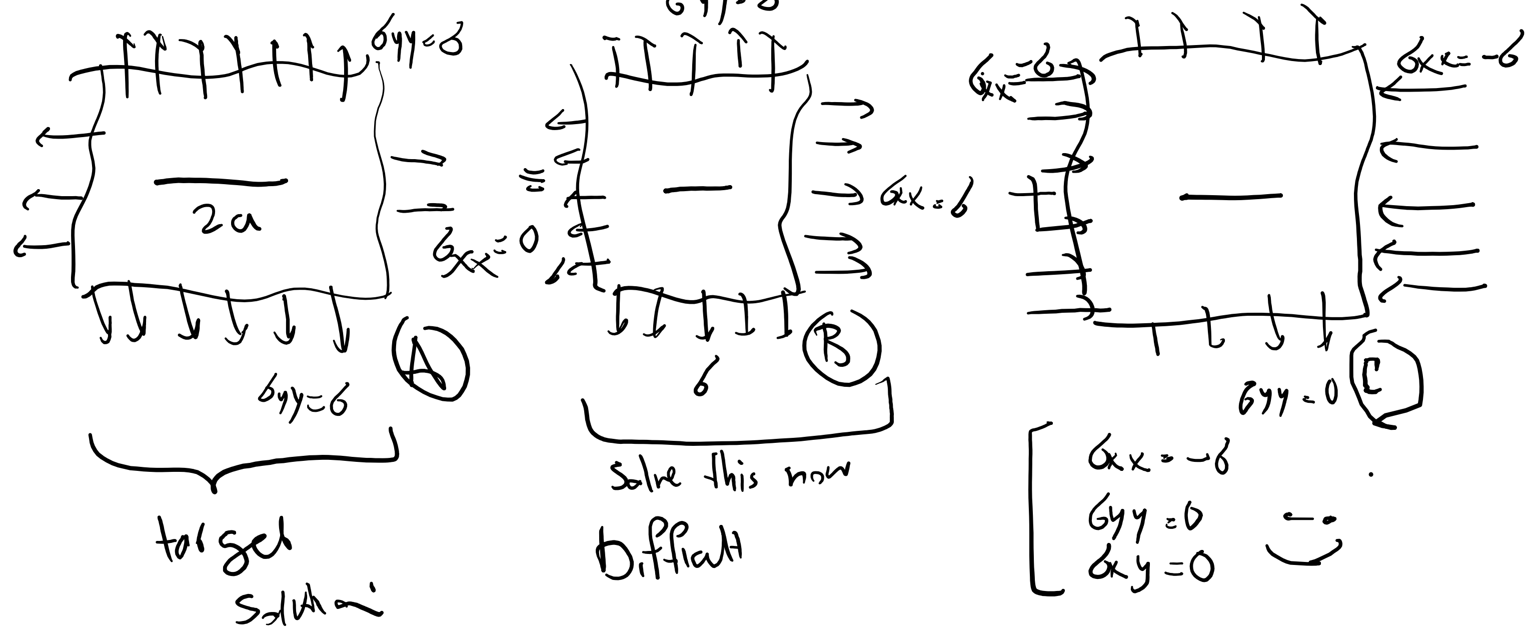
Getting the solutions for Mode I (and II) mid-crack problem



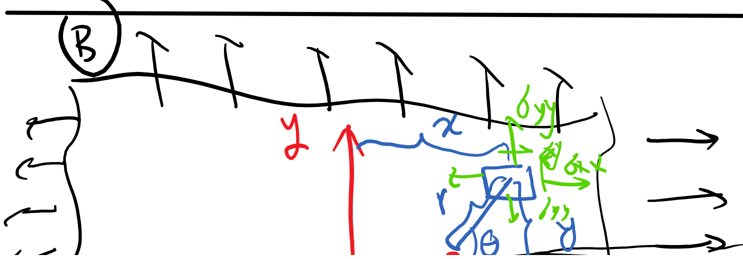




First: Mode I solution



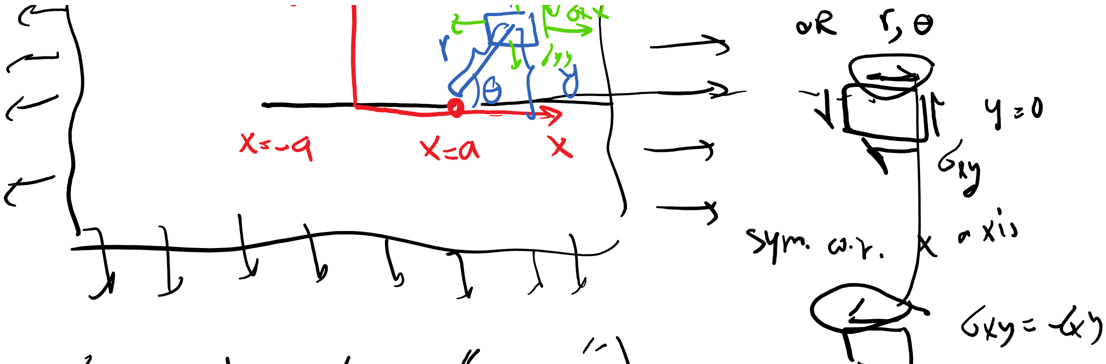
So by solving problem (B) which will be done now and subtracting sxx by -sigma, we get mode I mid-crack solution in an infinite domain



Goal

$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\ (x,y)$

or r, θ



$\boxed{\sigma_{xy}=0 \text{ on } y=0}$

$$\sigma_{xy}=\frac{1}{2}\mathrm{Re}\left(\bar{z}\,\phi''+\chi''\right)$$

on $y=0$ $\quad \underset{x+iy}{z}=\underset{x-iy}{\bar{z}}=x\pm\underbrace{0}_{y}i=x$

on $y=0$ $\quad z=\bar{z}$ so let's try looking for ϕ,χ

$\ni$

$$z\,\phi''+\chi''=0 \rightarrow \chi''=-z\,\phi'' \quad \text{integrate}$$

$$\rightarrow \chi'=-z\,\phi'+\phi \quad \text{integrate again} \rightarrow$$

$$(1)\ \boxed{\chi=-z\,\phi+2\tilde{\phi}}$$

$\tilde{\phi}$: anti-derivative of ϕ

$\tilde{\phi}=\int\phi\,dz$

$(\tilde{\phi})'=\phi$

a good relation between ϕ & χ for which $\sigma_{xy}(x,y=0)$ will be satisfied.

$$\psi=\mathrm{Re}\left(\bar{z}\,\phi+\chi\right)=\mathrm{Re}\left(\bar{z}\,\phi+\underbrace{\{-z\phi+2\tilde{\phi}\}}_{\chi}\right)$$

replace χ from (1)

$$=\mathrm{Re}\left((x-iy)\,\phi-(x+iy)\,\phi+2\,\tilde{\phi}\right)$$

$$=\mathrm{Re}\left(-2iy\,\phi+2\tilde{\phi}\right)$$

$$\rightarrow \boxed{\psi=2\left[y\,\mathrm{Im}\,\phi+\mathrm{Re}\,\tilde{\phi}\right]}\ (2)$$

$$\longrightarrow \boxed{\psi = 2\left[y\, \underbrace{\operatorname{Im}}_{\text{imaginary part}} \phi + \operatorname{Re}\tilde{\phi}\right]} \quad (2)$$

$$\boxed{\begin{aligned} \operatorname{Re} \overset{x+iy}{Z} &= x \\ \operatorname{Im} Z &= y \end{aligned}}$$

now ψ is written in terms of φ instead of φ

φ & χ

and it will satisfy $\sigma_{xy}(x, y=0) = 0$

Use $\boxed{2\phi = \tilde{Z}}$ (3) $\quad (\tilde{Z}' = Z)$

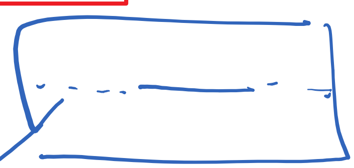
where Z is a complex functi

(2) & (3) $\longrightarrow$

$$\psi = y \operatorname{Im} \tilde{Z} + \operatorname{Re} \tilde{\tilde{Z}} \Longrightarrow$$

$$\boxed{\begin{aligned} \sigma_{11} &= \psi_{,22} = \operatorname{Re} Z - y \operatorname{Im} Z' \\ \sigma_{22} &= \psi_{,11} = \operatorname{Re} Z + y \operatorname{Im} Z' \\ \sigma_{12} &= -\psi_{,12} = -y \operatorname{Re} Z' \end{aligned}} \quad (4)$$

Verify $\underbrace{\sigma_{12}}_{\sigma_{xy}}(x, y=0) = 0$:)



:) $\sigma_{12} = 0$

Focus on the crack surface $(y=0)$

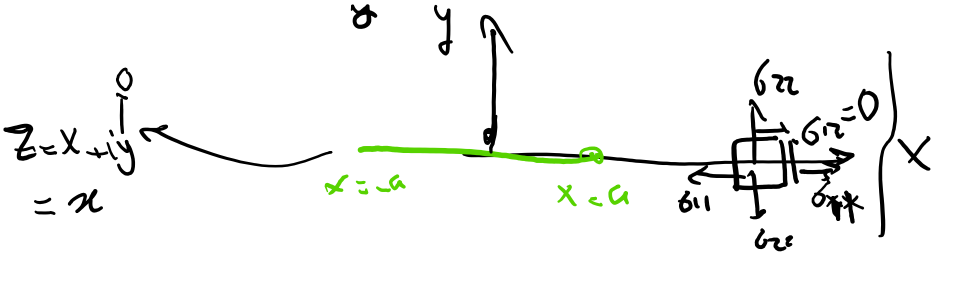
$\sigma_{12} = 0$ ✓

(4) $\boxed{\sigma_{11}(x, y=0) = \sigma_{22}(x, y=0) = \quad y=0}$

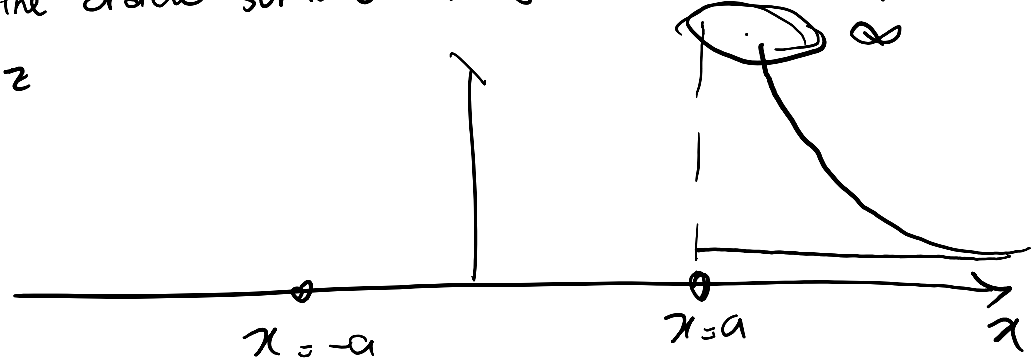
(4)

$$\sigma_{11}(x, y=0) = \sigma_{22}(x, y=0) =$$

$$\mathrm{Re}\,Z \mp \underbrace{0}_{y}\,\mathrm{Im}\,Z' = \mathrm{Re}\,Z(x, \overset{y=0}{y})$$



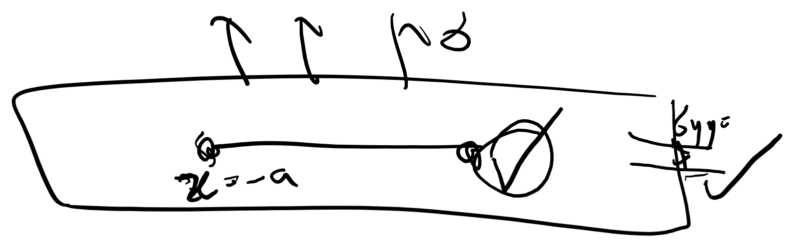
on the crack surface let's work with x instead of z



$\sigma_{22} = \mathrm{Re}\,Z$

let's use

$$Z = \frac{\sigma}{1 - z/a}$$



$$Z(x) = \frac{\sigma}{1 - \frac{a}{x}}$$

$x = z$ on crack surface

$Z(a^+) \to \infty$ ✓

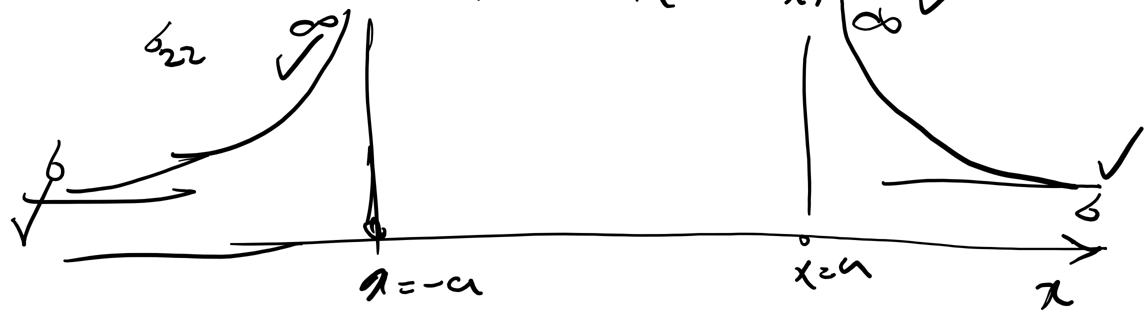
$Z(\infty) = \sigma$ ✓

$$Z(-a) = \frac{\sigma}{1+1} = \frac{\sigma}{2} \neq \infty \quad ✗$$

How about this? sym. w.r.t. $y = 0$

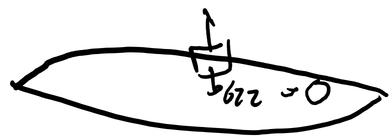
$$Z(x) = \frac{\sigma}{(1 - \frac{a}{x})(1 + \frac{a}{x})}$$

$$Z(x) = \frac{\sigma}{\left(1 - \frac{a}{x}\right)\left(1 + \frac{a}{x}\right)}$$



$$\sigma_{22} = \mathrm{Re}\, Z(x)$$

$$Z(z) = \frac{\sigma}{1 - a^2/z^2}$$

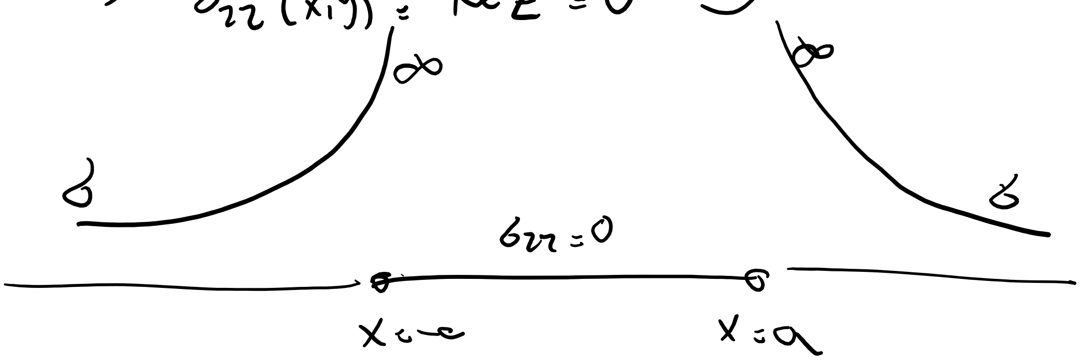


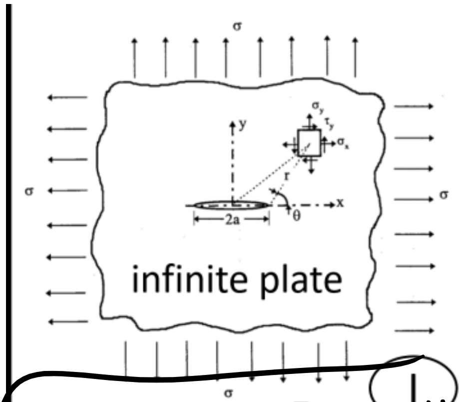
How about if we use

$$\boxed{Z(z) = \frac{\sigma}{\sqrt{1 - a^2/z^2}}}$$

$-a < z = x < a \quad Z(z) = \text{imaginary}$

$\rightarrow \quad \sigma_{22}(x,y) = \mathrm{Re}\, Z = 0$:)





$$\sigma_{xx} = \mathrm{Re}Z - y\mathrm{Im}Z'$$
$$\sigma_{yy} = \mathrm{Re}Z + y\mathrm{Im}Z'$$
$$\tau_{xy} = -y\mathrm{Re}Z'$$

y

entire stress solution

$$\boxed{Z(z) = \frac{\sigma}{\sqrt{1-(a/z)^2}}}$$

$$Z(z) = \frac{\sigma x}{\sqrt{x^2-a^2}} \quad \text{is imaginary}$$

$$\boxed{Z'(z) = -\frac{\sigma a^2}{(z^2-a^2)^{3/2}} \to 0}$$

We can also get the displacement solutions

Kolosov coef. κ

$$\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

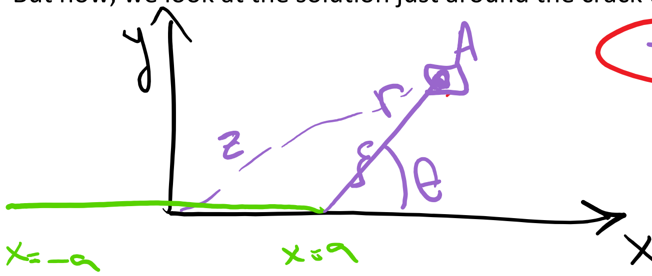
$$\mu = \frac{E}{2(1+\nu)} \quad \text{shear modulus}$$

$$2\mu u = \frac{\kappa-1}{2}\mathrm{Re}\tilde{Z} - y\mathrm{Im}Z$$
$$2\mu v = \frac{\kappa+1}{2}\mathrm{Im}\tilde{Z} - y\mathrm{Re}Z$$

117

The derivation above is very interesting but unfortunately globally is only valid for a single crack in an infinite domain.

But now, we look at the solution just around the crack tip as asymptotic solution:



$$\boxed{Z = \underbrace{\xi}_{\text{small magnitude}} + a}$$

$$\xi = re^{i\theta}$$

$$\boxed{|\xi/a| \ll 1}$$

point A is very close to the crack tip

$$Z(z) = \frac{\sigma}{\sqrt{1-(a/z)^2}} = \frac{\sigma z}{\sqrt{z^2-a^2}} \qquad z = \xi + a$$

$$Z(z) = \frac{\sigma a\,(1+\xi/a)}{\sqrt{\xi^2+2a\xi}} = \frac{\sigma a\,(1+\xi/a)}{\sqrt{2a\xi\left(1+\frac{\xi}{2a}\right)}}$$

$\xi/a \approx 0$, $\frac{\xi}{2a} \approx 0$

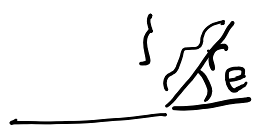
$$Z(z) = Z(\xi) = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}}$$ around the crack tip

we call this stress intensity factor K_I for mid crack specimen

$$\boxed{Z(z) = Z(\xi) \approx \frac{K_I}{\sqrt{2\pi\xi}}}$$

approximate expression for stress around the crack tip

$$\boxed{\sigma_{22} = \text{Re}\, Z - y\, \text{Im}\, Z'}$$



$\xi = re^{i\theta}$

$$Z = \frac{K_I}{\sqrt{2\pi}}\,\xi^{-1/2} \qquad \frac{dZ}{dz} = \frac{dZ}{d\xi}\,\overset{z-a}{\frac{d\xi}{dz}} = \frac{dZ}{d\xi} = \frac{K_I}{\sqrt{2\pi}}\left(-\frac{1}{2}\right)\xi^{-3/2}$$

$y = r\sin\theta$

$$\text{Re}\,Z = \text{Re}\,\frac{K_I}{\sqrt{2\pi}}\left(r^{-1/2}\right)\left(\cos\frac{-\theta}{2} + i\sin\frac{-\theta}{2}\right)$$

$$= \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$\sqrt{2\pi r}$ 2

$\xi^{-3/2} = \frac{1}{r\sqrt{r}} \left(\cos -\frac{3\theta}{2} + i \sin -\frac{3\theta}{2} \right)$

all terms considered

$$Z(z) = \frac{K_I}{\sqrt{2\pi\xi}}, K_I = \sigma\sqrt{\pi a}$$

$$Z(z) = \frac{K_I}{\sqrt{2\pi r}} e^{-i\theta/2} \qquad \xi = re^{i\theta}$$

$$Z'(z) = -\frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\xi^{-3/2} = -\frac{K_I}{2r\sqrt{2\pi r}} e^{-i3\theta/2}$$

Crack tip stress field

Recall

$$\boxed{\begin{aligned} \sigma_{xx} &= \text{Re}Z - y\text{Im}Z' \\ \sigma_{yy} &= \text{Re}Z + y\text{Im}Z' \\ \tau_{xy} &= -y\text{Re}Z' \end{aligned}}$$

$$e^{-ix} = \cos x - i \sin x$$

$$y = r\sin\theta$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$