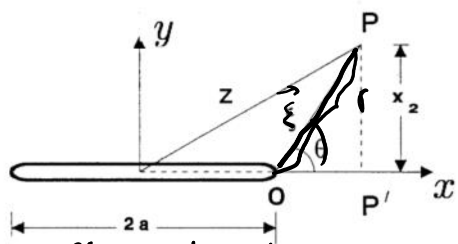
$\xi/(2a)))$

Recall from the last time that

$$Z = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}}$$

$\sigma\sqrt{\pi a}$: K_I

mode I stress intensity factor for mid crack in an infinite domain

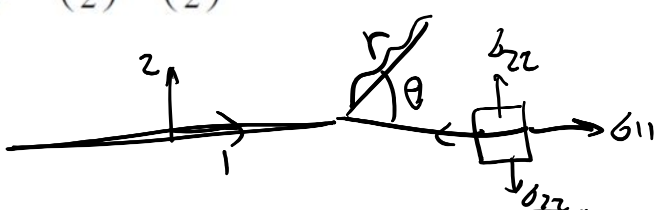


$$\sigma_{11}: \quad \sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

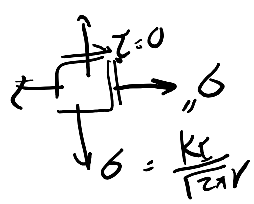
$$\sigma_{22}: \quad \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

stress solutions around the crack tip



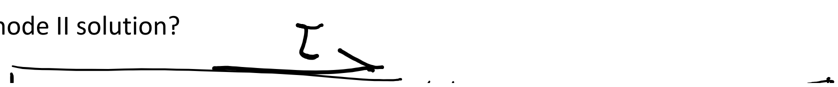
$$\boxed{\begin{aligned}\sigma_{11}(r,\theta=0) &= \sigma_{22}(r,\theta=0) = \frac{K_I}{\sqrt{2\pi r}}\\ \sigma_{12}(r,\theta=0) &= 0\end{aligned}}$$



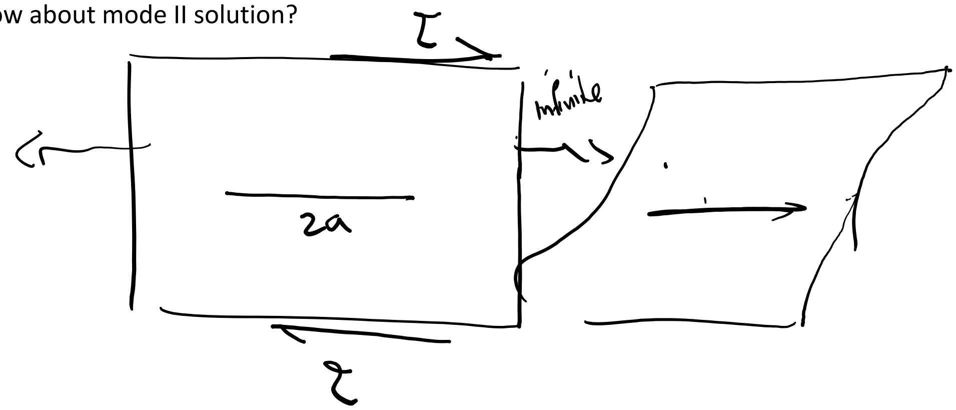
hydrostatic tension

$\sigma_{11}, \sigma_{22} \to \infty$ as $r \to 0^+$ like $\frac{1}{\sqrt{r}}$

How about mode II solution?



How about mode II solution?



Mode II problem

Boundary conditions

$$\boxed{\begin{array}{l}(x,y)\to\infty : \sigma_{xx}=\sigma_{yy}=0, \tau_{xy}=\tau \\ |x|<a, y=0 : \sigma_{yy}=\tau_{xy}=0\end{array}}$$

Stress function

τ

$$\boxed{Z = \frac{-i\tau z}{\sqrt{z^2-a^2}}}$$

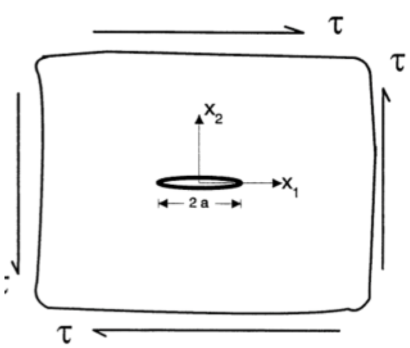
recall for Mode I

$$Z = \frac{\sigma z}{\sqrt{z^2-a^2}}$$

Again, we can look at the asymptotic expression of the solution around the crack tip:

Stress function

$$\boxed{Z = -\frac{i\tau z}{\sqrt{z^2-a^2}}}$$



$$\boxed{\begin{aligned}\sigma_{xx} &= -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\\ \sigma_{yy} &= \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\\ \tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\end{aligned}}$$

mode II SIF

$$\boxed{K_{II} = \tau\sqrt{\pi a}}$$

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Compare with Mode I
$K_I = \sigma\sqrt{\pi a}$

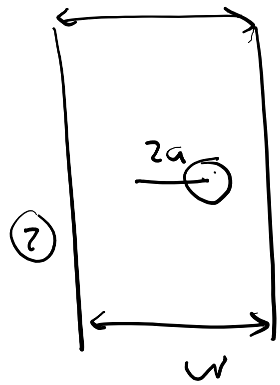
Why asymptotic expressions are useful?

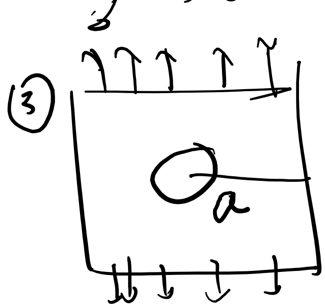
Why asymptotic expressions are useful?

$Z(z) = \frac{\sigma z}{\sqrt{z^2-a^2}}$

$\sigma_{22} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z'$

$\vdots$





Q1: is the global stress solution we obtained for ① useful for ②, ③, ... ? No

Q2: How about the asymptotic solution around the crack?

Around the crack tip they all have the same asymptotic expansion for stress, displacement, and strain solutions.

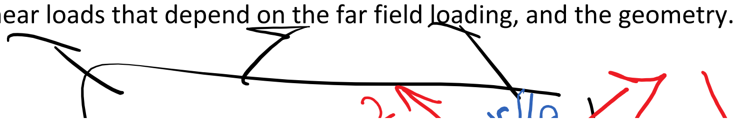
$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} f^{I}_{22}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f^{II}_{22}(\theta)$$

$K_I = \sqrt{\pi a}\,\sigma \qquad K_{II} = \tau\sqrt{\pi a}$

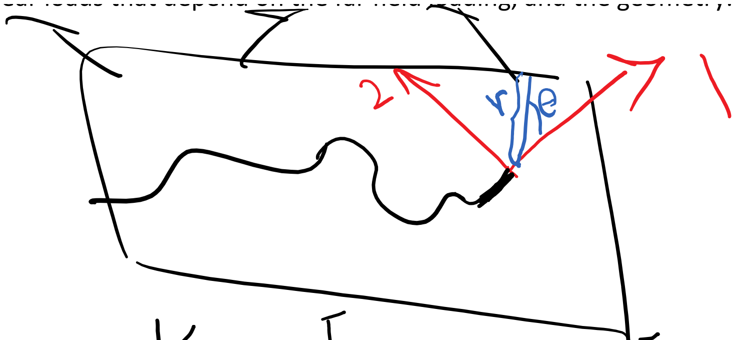
σ & τ that are sort of active "far field" stresses around the crack are functions of Actual loading & geometry

$$\sigma_{22} = \frac{K_I(P_1, P_2, F, \dots; L, W; \text{crack geometry})}{\sqrt{2\pi r}} f^{I}_{22}(\theta) + K_{II} \dots f^{II}_{22}(\theta)$$

Around the crack tip we have the same asymptotic expansion, because close to it we have some effective "relatively far field" normal and shear loads that depend on the far field loading, and the geometry.



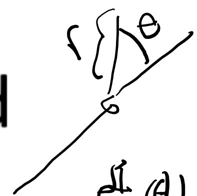
"far-field" normal and shear loads that depend on the far-field loading, and the geometry.



$$\sigma_{11}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{11}^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{11}^{II}(\theta)$$

so the only parameters that change the in-plane stress solutions around the crack tip are SIFs K_I, K_{II}

asymptotic stress field



Westergaards, Sneddon etc.

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \overbrace{\cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]}^{f_{11}^{I}(\theta)}$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]}_{f_{22}^{I}(\theta)}$$

$$\tau_{12} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}_{f_{12}^{I}(\theta)}$$

(mode I)

$$\sigma_{11} = -\frac{K_{II}}{\sqrt{2\pi r}} \overbrace{\sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)}^{f_{11}^{II}(\theta)}$$

$$\sigma_{22} = \frac{K_{II}}{\sqrt{2\pi r}} \underbrace{\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}}_{f_{22}^{II}(\theta)}$$

$$\tau_{12} = \frac{K_{II}}{\sqrt{2\pi r}} \underbrace{\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)}_{f_{12}^{II}(\theta)}$$

(mode II)

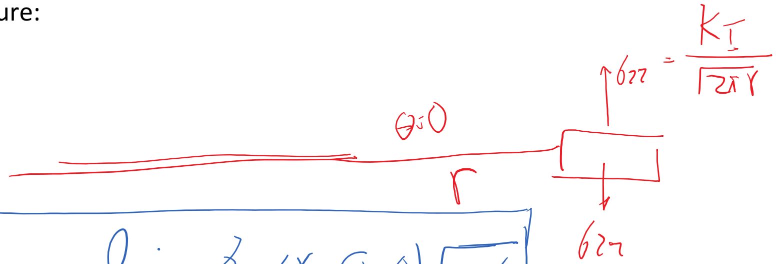
In general, for arbitrary stress component we can write

$$\sigma_{ij}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{II}(\theta) + \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{III}(\theta)$$

All we have to do is to find KI, KII, KIII around a crack tip.

How do we derive SIFs? We will discuss that in more detail later, but for the moment, let's look at this equation:

Pure mode I fracture:



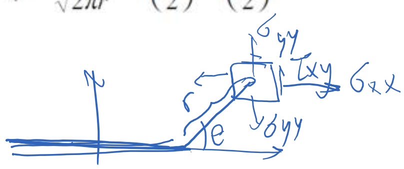
$$\boxed{K_I = \lim_{r \to 0} \sigma_{zz}(r, \theta = 0)\sqrt{2\pi r}}$$

Other asymptotic expansions:
Mode I:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

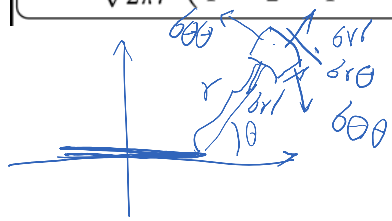
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$



$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$



Principal stresses

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)$$

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right)$$

$$\sigma_3 = \begin{cases} 0 & \text{plane stress} \\ \dfrac{2\nu K_I}{\sqrt{2\pi r}}\cos\dfrac{\theta}{2} & \text{plane strain} \end{cases}$$

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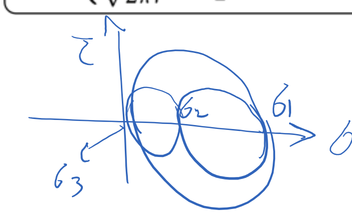
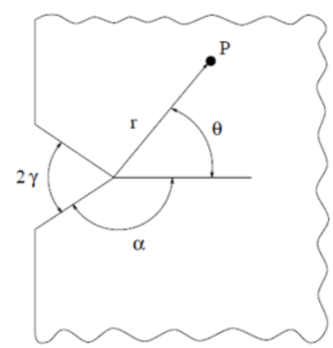


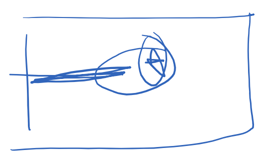
Second example with an exact solution:

Crack solution using V-Notch

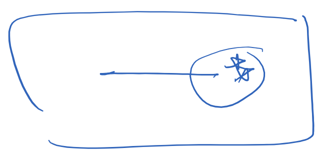


— $\alpha = \pi$ we are back to a crack solution

Single edge notch crack



— This example demonstrates that the asymptotic stress in (☆) is identical to



as expected.

— $\alpha < \pi$ we get exciting mode I & II solutions, some of which are singular. (HW assignment)

The geometry and loading lend themselves to polar coordinate, so it's the easiest to solve this problem in the polar coordinate system.

Polar coordinate

$$\psi(r,\theta) \qquad \Delta\Delta\psi = 0$$

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad (1)$$

Let's choose (2) $\psi(r,\theta) = r^{\lambda+1} F(\theta) \rightarrow \sigma_{ij} \propto r^{\lambda}$

λ: power of singularity for stress

plug (2) in $\Delta\Delta\psi = 0 \rightarrow$

$$\frac{\partial^4 F(\theta)}{\partial \theta^4} + 2(\lambda^2+1)\frac{\partial^2 F}{\partial \theta^2} + (\lambda^2-1)^2 F(\theta) = 0$$

$$F(\theta) = e^{m\theta}$$

$$\left[m^4 + 2(\lambda^2+1)\, m^2 + (\lambda^2-1)^2\right] e^{m\theta} = 0 \Rightarrow$$

$$\left[m^2 + (1-\lambda)^2\right]\left[m^2 + (1+\lambda)^2\right] = 0$$

$$\rightarrow \quad m = \pm i(1-\lambda)$$

4 solutions $\quad \& \pm i(1+\lambda)$

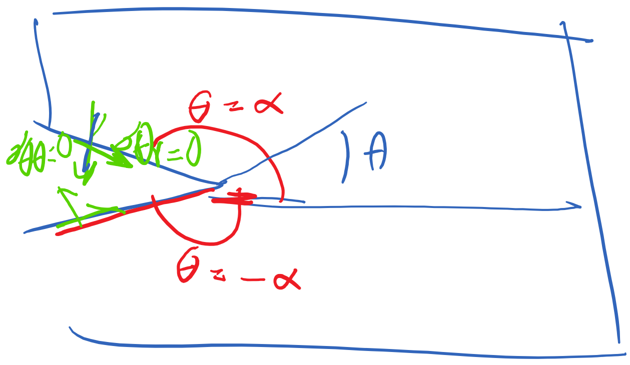
Eventually, the solution F(theta) is a combination of the four following angle functions

$$F(\theta) \in \left\{ \cos(\lambda-1)\theta,\ \cos(\lambda+1)\theta,\ \sin(\lambda-1)\theta,\ \sin(\lambda+1)\theta \right\}$$

So, the following stress function in polar coordinate is bar harmonic:

$$\Phi(r,\theta) = r^{\lambda+1}\underbrace{\left[A\cos(\lambda-1)\theta + B\cos(\lambda+1)\theta + C\sin(\lambda-1)\theta + D\sin(\lambda+1)\theta\right]}_{F(\theta,\lambda)}$$

To further narrow down the form of stress function (hence eventually the actual stress and displacement solutions), we need to use the boundary conditions:



- Stress values

$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = r^{\lambda-1}\lambda(\lambda+1)F(\theta)$$
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = r^{\lambda-1}[-\lambda F'(\theta)]$$

- Boundary conditions

$$\begin{aligned} \sigma_{\theta\theta}\,|_{\theta=\pm\alpha} &= 0 \\ \sigma_{r\theta}\,|_{\theta=\pm\alpha} &= 0 \end{aligned} \Rightarrow$$

The four equations are:

$$F(\alpha) = F(-\alpha) = F'(\alpha) = F'(-\alpha) = 0 \Rightarrow$$

$$\boxed{\begin{bmatrix} \cos(\lambda-1)\alpha & \cos(\lambda+1)\alpha & 0 & 0 \\ \omega\sin(\lambda-1)\alpha & \sin(\lambda+1)\alpha & 0 & 0 \\ 0 & 0 & \sin(\lambda-1)\alpha & \sin(\lambda+1)\alpha \\ 0 & 0 & \omega\cos(\lambda-1)\alpha & \cos(\lambda+1)\alpha \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0}$$

$\omega = \frac{\lambda-1}{\lambda+1}$ $\quad$ $\det\left(\;\right) = 0$

The only way not to have trivial zero solution is to have the determinant of this 4x4 matrix to be zero

So, we get:

- Eigenvalues and eigenvectors for nontrivial solutions

$$\boxed{\begin{aligned} \sin 2\lambda_n\alpha + \lambda_n \sin 2\alpha &= 0 \\ \sin 2\xi_n\alpha - \xi_n \sin 2\alpha &= 0 \end{aligned}}$$

← Mode I

← Mode II

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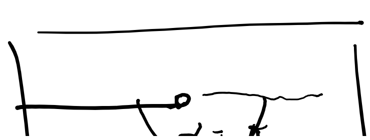
You'll play with these two equations for mode one and two to get the stress singularity.

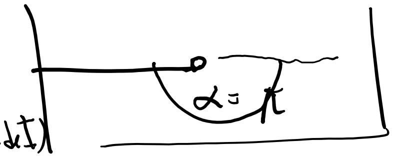
λ_n's & ξ_n's are the solutions for λ in

$\psi(r,\theta) = r^{\lambda+1} F(\theta)$

Example : sharp crack



Example : sharp crack



$\sin 2\lambda_n(\pi) + \lambda_n \cancel{\sin 2\pi} = 0$ (mode I)

$\sin 2\xi_n(\pi) - \xi_n \cancel{\sin 2\pi} = 0$ mode II

for mode I we get $\sin 2\pi\lambda_n = 0 \rightarrow$

$2\pi\lambda_n = n\pi \longrightarrow \lambda_n = \frac{n}{2} = \{ \ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots \}$

power of stress singularity

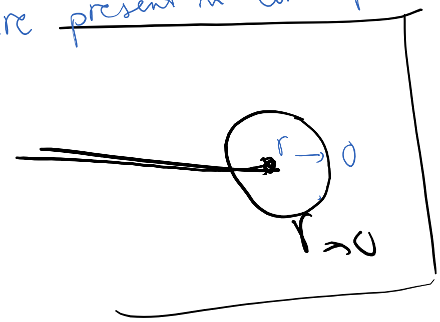
$\sigma \propto \frac{1}{r^{\frac{3}{2}}}$ $\sigma \propto \frac{1}{r\sqrt{r}}$ $\sigma \propto \frac{1}{\sqrt{r}}$ $\sigma = \sqrt{r}, \ldots$

why we end up with only $\frac{1}{\sqrt{r}}$ term for σ

$\boxed{\sigma \propto \frac{1}{r\sqrt{r}}}$

proportional to

Why not this or stronger singularities than $\frac{1}{\sqrt{r}}$ are present in our expansions?



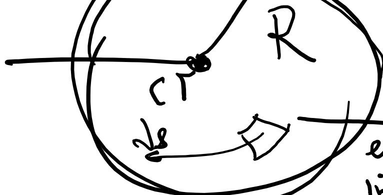
$\sigma \propto \frac{1}{\sqrt{r}}$ $\longrightarrow \infty$ $r \rightarrow 0$

$\sigma \propto \sqrt{r}$ $\longrightarrow 0$ as $r \rightarrow 0$

more important around the crack tip than

$\frac{1}{1-x} = 1 + x + x^2 + \ldots$ $|x|<1$ for $|x|<1$

$\mathcal{E} = \int_0^R \underbrace{\frac{1}{2}\sigma : \varepsilon}_{e} \, dS$ $\varepsilon = C^{-1}\sigma$



look at energy in the material around the CT with radius R

$\int_0^R \underbrace{r^\lambda}_{\text{singularity}} r^\lambda \, r \, dr \, d\theta =$

of stress

$$\propto \int_0^{\rho} r^{2\lambda+1}\, dr\, d\theta$$

For the energy to be finite ...

$$2\lambda+1 \geq 0 \rightarrow \boxed{\lambda \geq \frac{-1}{2}}$$

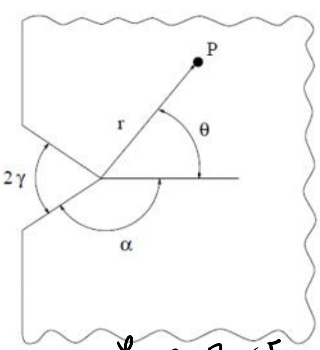
So stress singularity cannot be stronger than -0.5 otherwise, internal energy around the crack tip will be infinite which makes no sense.

Crack solution using V-Notch

- Sharp crack

$$\alpha = \pi \Rightarrow \begin{array}{rcl} \sin(2\pi\lambda_n) & = & 0 \\ \sin(2\pi\xi_n) & = & 0 \end{array} \Rightarrow$$

$\lambda_n = \frac{n}{2}$ with $n = 1, 3, 4, \cdots$ (n = 2 constant stress)



- After having eigenvalues, eigenvectors are obtained from

Mode I
$$\begin{array}{rcl} A_n \cos(\lambda_n - 1)\alpha + B_n \cos(\lambda_n + 1)\alpha & = & 0 \\ A_n \omega \sin(\lambda_n - 1)\alpha + B_n \sin(\lambda_n + 1)\alpha & = & 0 \end{array}$$

Mode II
$$\begin{array}{rcl} C_n \sin(\xi_n - 1)\alpha + D_n \sin(\xi_n + 1)\alpha & = & 0 \\ C_n \omega \cos(\xi_n - 1)\alpha + D_n \cos(\xi_n + 1)\alpha & = & 0 \end{array}$$

values of A, B (I) C, D (II) are obtained by det( )=0 → A=B → use K_I

- First term of stress expansion

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

- The above terms are the leading terms of the stress solutions around the crack tip for mode I and II.
- We also have constant terms, $\sqrt{r}$ terms, ..., but these terms are much smaller than the above terms around the crack tip.

Notice the leading terms are identical to the asymptotic terms we obtained before from stress function for a mid-crack in an infinite domain:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$

Displacement field around the crack tip:

Mode I: displacement field

$$Z(z) = \frac{K_I}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

asymptotic function that provides stress&displacement solution

$$\boxed{Z(z) = \frac{K_I}{\sqrt{2\pi\xi}}} \qquad \bar{Z} = \int Z(z)dz$$

$$\tilde{Z}(z) = 2\frac{K_I}{\sqrt{2\pi}}\xi^{1/2} = 2K_I\sqrt{\frac{r}{2\pi}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

Recall

$$\boxed{\begin{aligned} 2\mu u &= \frac{\kappa-1}{2}\mathrm{Re}\,\tilde{Z} - y\mathrm{Im}Z \\ 2\mu v &= \frac{\kappa+1}{2}\mathrm{Im}\,\tilde{Z} - y\mathrm{Re}Z \end{aligned}}$$

$r\sin\theta$

$$z = \xi + a$$
$$\xi = re^{i\theta}$$
$$e^{-ix} = \cos x - i\sin x$$

Displacement field

$$\boxed{\begin{aligned} u &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 + 2\sin^2\frac{\theta}{2}\right) \\ v &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\underbrace{\kappa+1}_{} - 2\cos^2\frac{\theta}{2}\right) \end{aligned}}$$

1

Kolosov coef. κ

$$\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

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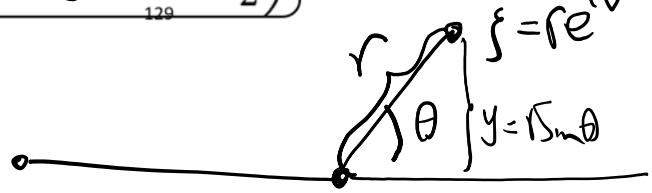
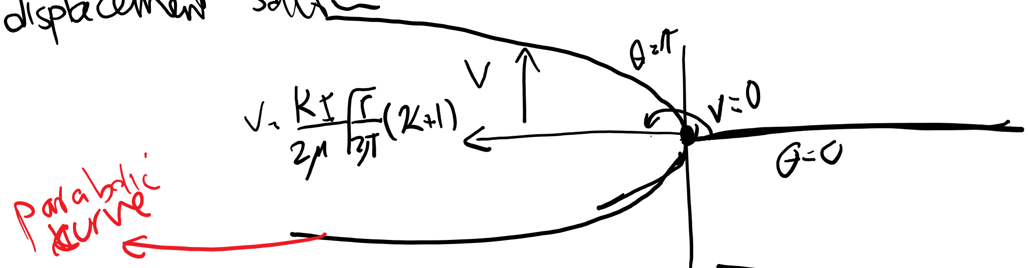


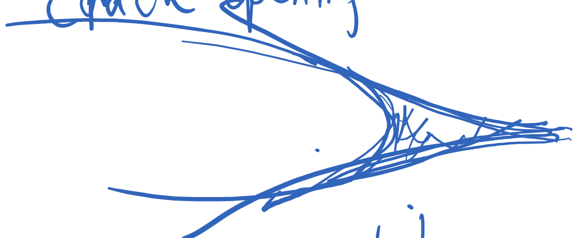
displacement solution

G



behind the crack v. displacement $\propto \sqrt{r}$

real crack opening



removes stress singularity

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