2020/09/21

Monday, September 21, 2020 2:48 PM

Last time, at the end we discussed the form of asymptotic displacement fields around the crack tip

$$Z(z) = \frac{K_I}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

$$Z(z) = \frac{K_I}{\sqrt{2\pi\xi}} \qquad \bar{Z} = \int Z(z)dz$$

$$\tilde{Z}(z) = 2\frac{K_I}{\sqrt{2\pi}}\xi^{1/2} = 2K_I\sqrt{\frac{r}{2\pi}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

Recall

$$\boxed{\begin{aligned} 2\mu u &= \frac{\kappa-1}{2}\mathrm{Re}\,\tilde{Z} - y\mathrm{Im}Z \\ 2\mu v &= \frac{\kappa+1}{2}\mathrm{Im}\,\tilde{Z} - y\mathrm{Re}Z \end{aligned}}$$

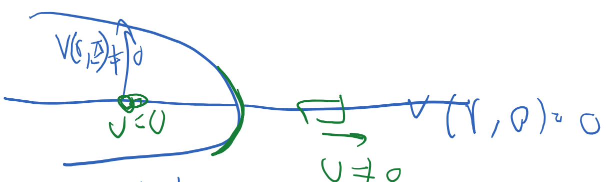
$$z = \xi + a$$
$$\xi = re^{i\theta}$$
$$e^{-ix} = \cos x - i\sin x$$

Displacement field

$$\boxed{\begin{aligned} u &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 + 2\sin^2\frac{\theta}{2}\right) \\ v &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 - 2\cos^2\frac{\theta}{2}\right) \end{aligned}}$$

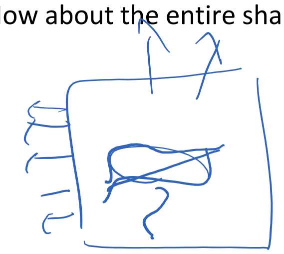
Kolosov coef. κ

$$\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

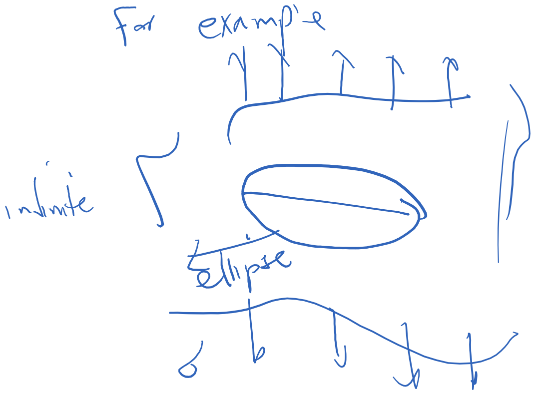


parabolic displacement around the crack tip

How about the entire shape of crack opening? Unfortunately, this changes from problem to problem.



For example



$y = 0, \quad a \le x \le a$

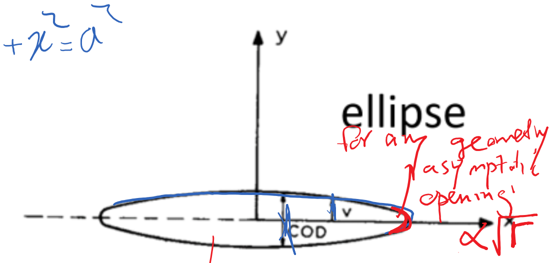
$$2\mu v = \frac{\kappa+1}{2}\mathrm{Im}\tilde{Z} - y\mathrm{Re}Z \longrightarrow v = \frac{\kappa+1}{4\mu}\mathrm{Im}\tilde{Z}$$

$$Z(z) = \frac{\sigma \cancel{z}\,x}{\sqrt{\cancel{z}\,x^2 - a^2}} \longrightarrow \tilde{Z}(z) = \sigma\sqrt{x^2 - a^2}$$

$$-a \le x \le a \quad i = \sqrt{-1} \longrightarrow \tilde{Z}(z) = i(\sigma\sqrt{a^2 - x^2})$$

$$\boxed{v = \frac{\kappa+1}{4\mu}\sigma\sqrt{a^2 - x^2}}$$

$\left(\frac{v}{A}\right)^2 + x^2 = a^2$



for any geometry asymptotic opening $\propto \sqrt{r}$

overall opening shape changes problem to problem

Crack Opening Displacement

$$\mathrm{COD} = 2v = \frac{\kappa+1}{2\mu}\sigma\sqrt{a^2 - x^2}$$

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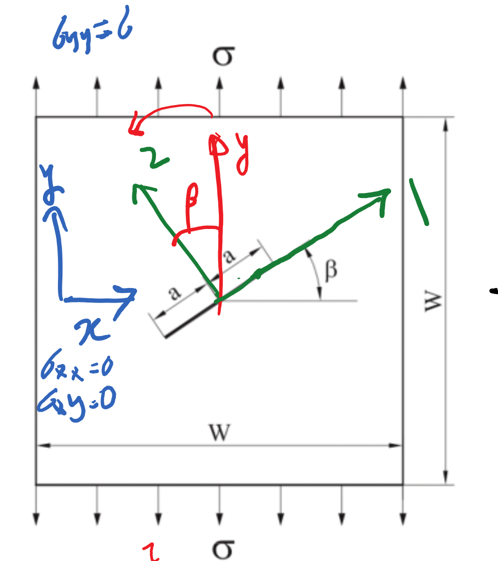
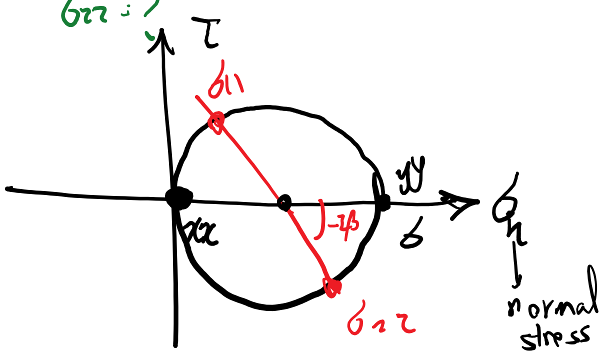
General discussion on how to compute SIF

far field values for

$\sigma_{12} = ? \longrightarrow$ Mode II

$\sigma_{22} = ? \longrightarrow$ Mode I

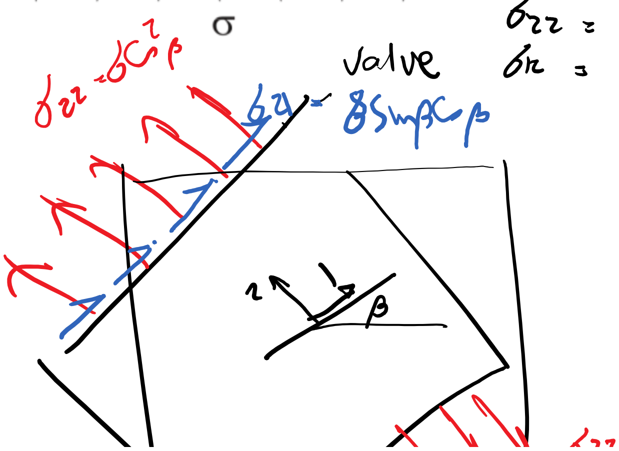




$\sigma_{11} = \frac{\sigma}{2}(1 - \cos 2\beta) = \sigma\sin^2\beta$

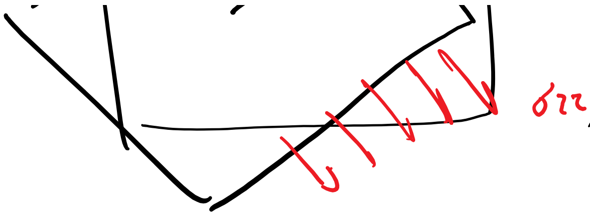
$\sigma_{22} = \frac{\sigma}{2}(1 + \cos 2\beta) = \sigma\cos^2\beta$

$\sigma_{12} = \frac{\sigma}{2}\sin 2\beta = \sigma\sin\beta\cos\beta$



$$K_I = \underbrace{\left(\text{far field normal stress}\right)\sqrt{\pi a}}_{\text{an infinite plate}}$$

$$= \sigma_{22}\sqrt{\pi a} = \cos^2\beta\,\sigma\sqrt{\pi a}$$



$= \sigma_{zz}\sqrt{\pi a} = \cos^2\beta\,\sigma\sqrt{\pi a}$

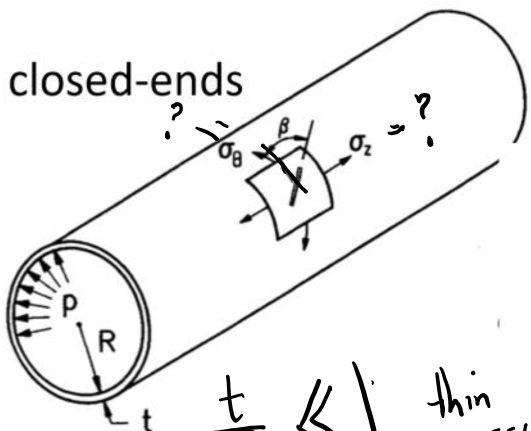
$K_{II} = \left(\text{far field shear stress}\right)\sqrt{\pi a}$

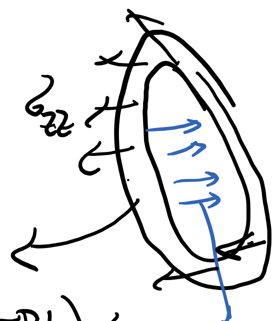
$= \sigma_{z1}\sqrt{\pi a} = \sin\beta\cos\beta\,\sigma\sqrt{\pi a}$

- A key assumption is that we are modeling the crack as if it's in an infinite domain, whereas this is obviously not true. However, if the crack size a is much smaller than the plate width (W) and height (H), this is a decent approximation.

Another example:

Cylindrical pressure vessel with an inclined through-thickness crack

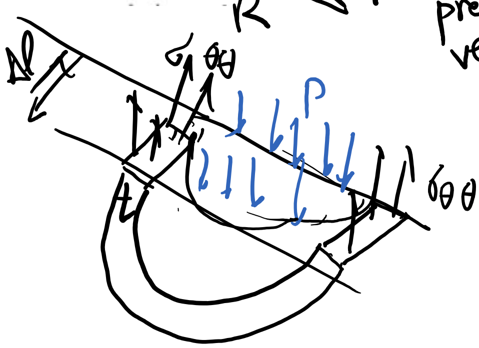




$\underbrace{(2\pi R t)}_{\text{area}}\sigma_{zz} = \underbrace{(\pi R^2)}_{\text{area}} p$

$\frac{t}{R} \ll 1$ thin pressure vessel

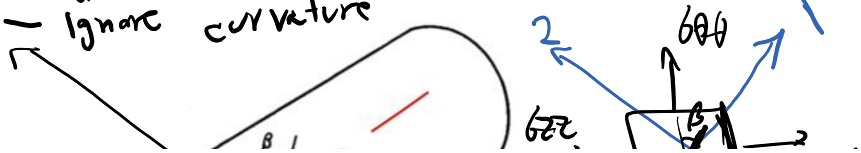
$\rightarrow \boxed{\sigma_{zz} = p\frac{R}{2t}}$

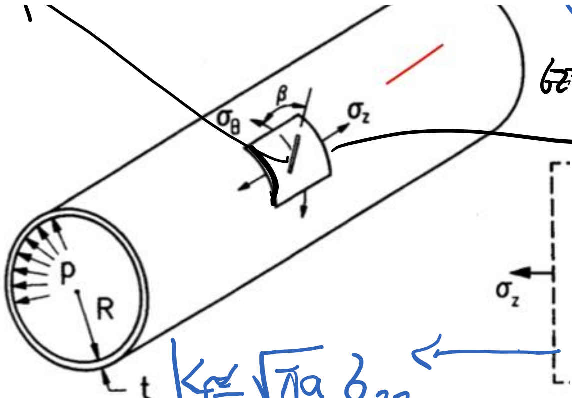


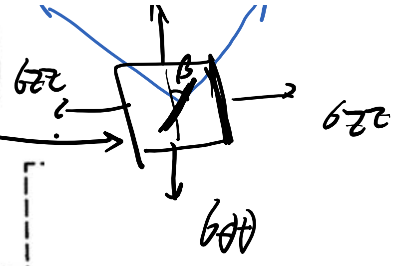
$\sigma_{\theta\theta}\underbrace{(\Delta l\, t \times 2)}_{\text{area}} = p\underbrace{(2R\Delta l)}_{\text{area}}$

$\boxed{\sigma_{\theta\theta} = p\frac{R}{t} = 2\sigma_{zz}}$

short crack with orientation β as shown
- assume it's in infinite domain
- ignore curvature







$K_I = \sqrt{\pi a}\,\sigma_{22}$

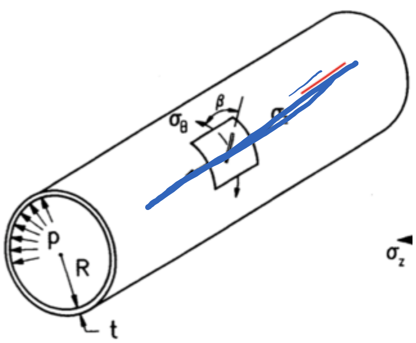
$\sigma_{22} = \underbrace{\sigma_{zz}}_{\frac{PR}{2t}} \cos^2\beta + \underbrace{\sigma_{\theta\theta}}_{\frac{PR}{t}} \sin^2\beta$

$$\boxed{\begin{aligned} K_I &= \frac{PR}{2t}\left(1+\sin^2\beta\right) \\ K_{II} &= \frac{PR}{2t}\sin\beta\cos\beta \end{aligned}}$$

$|\sigma_{21}| = -\sigma_z \sin\theta\cos\theta + \sigma_\theta \sin\theta\cos\beta$

$= \frac{PR}{2t}\sin\theta\cos\theta$

For what orientation we have the maximum KI?



$\beta = \frac{\pi}{2}$

why over cooked hotdog breaks along its axis on the surface

How do we computer SIF?

- SIF handbooks
- Computationally (FEM, …)
- Experimentally (e.g. photo sensitive materials, photoelasticity)

Computation of SIFs

- Analytical methods (limitation: simple geometry)

  - superposition methods
  - weight/Green functions

- Numerical methods (FEM, BEM, XFEM)

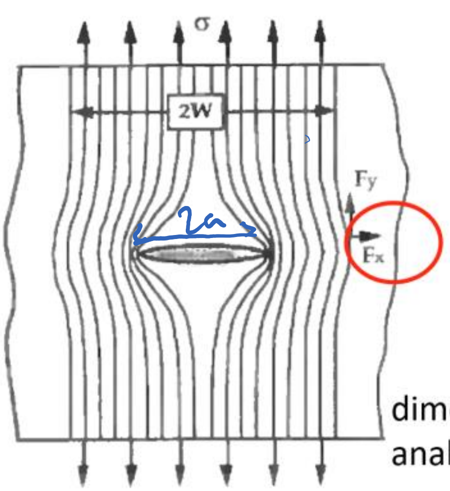
  numerical solutions -> data fit -> **SIF handbooks**

- Experimental methods
  - photoelasticity

SIF for finite size samples

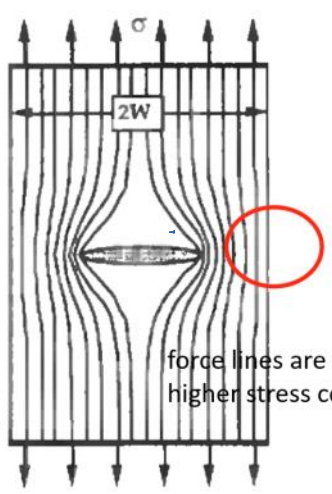
Exact (closed-form) solution for SIFs: simple crack geometries in an **infinite** plate.

Cracks in finite plate: influence of external boundaries cannot be neglected -> generally, no exact solution



(a) Infinite plate

dimensional analysis



force lines are compressed->> higher stress concentration

(b) Finite plate

finite problem

must be larger than infinite domain

(a) Infinite plate

infinite problem

$$K_I = \sigma\sqrt{\pi a}$$

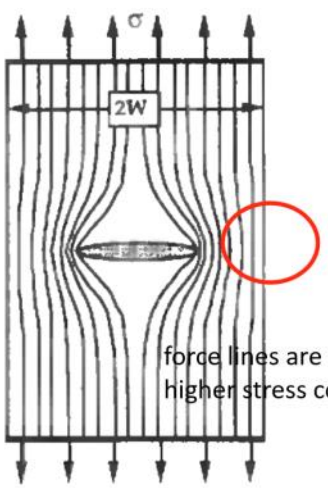
(b) Finite plate

must be larger than infinite domain

$$K_I = f\,\sigma\sqrt{\pi a}$$

geometry correction factor

For this particular problem, we have the analytical form of f:



force lines are compressed->> higher stress concentration

(b) Finite plate

$$\underbrace{K_I = f(a,W)}_{\text{geometry correction factor}}\,\sigma\sqrt{\pi a}$$

$$f(a,W) = f\left(\frac{a}{W}\right) = \frac{1}{\sqrt{\cos\frac{\pi a}{W}}}$$

short crack $\frac{a}{W} \to 0$, $f \to 1$ infinite domain limit

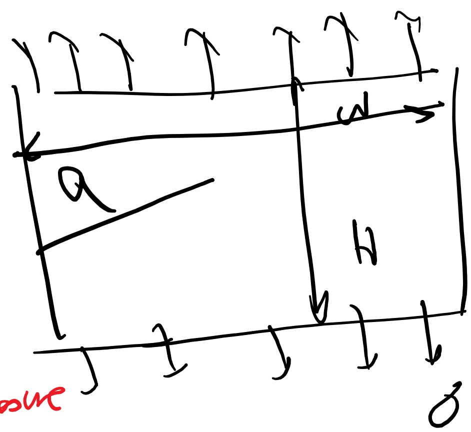
$$\frac{a}{W} \to \frac{1}{2} \to f \to \infty$$

Formulas for SIF are tabulated generally using the following convention:

$$K_I = f^I(\text{geometry})\,\sigma_{max}\sqrt{\pi a}$$

geometry: $\frac{a}{H}, \frac{a}{W}, \ldots$

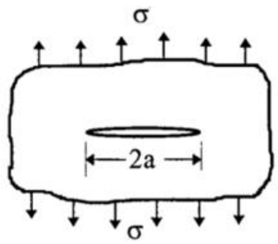
σ_{max}: stress measure on the crack plane or a far field stress measure



A few examples:

Geometry	Stress Intensity Factor

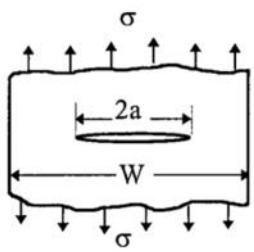
1. Crack in an infinite body



$$K_I = \sigma\sqrt{\pi a}$$

$f = 1$

2. Centre crack in a strip of finite width



we just had this above

$$\frac{1}{\sqrt{\cos\frac{\pi a}{W}}} = f$$

$$K_I = \sqrt{\sec\frac{\pi a}{W}}\,\sigma\sqrt{\pi a}$$

secant function

$$\sec\theta = \frac{1}{\cos\theta}$$

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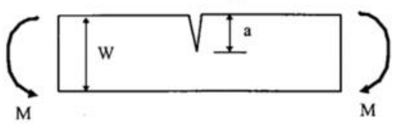
5. Edge crack in a beam of width B subjected to bending

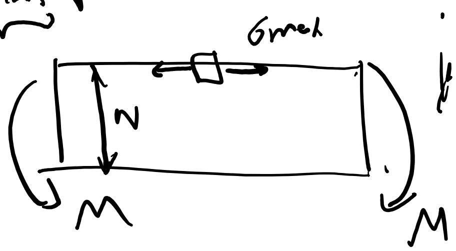


$$K_I = f\left(\frac{a}{W}\right)\sigma\sqrt{\pi a} \quad \text{where } \sigma = \frac{6M}{BW^2}$$

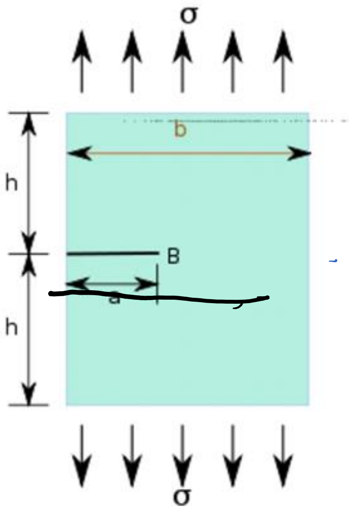
a/W	$f(a/W)$
0.1	1.044
0.2	1.055
0.3	1.125
⋮	⋮

$$K_I = f\left(\frac{a}{W}\right)\underbrace{\sigma_{max}}\sqrt{\pi a}$$

$$\sigma_{max} = \frac{\frac{W}{2}M}{EI} = \frac{6M}{BW^2}$$



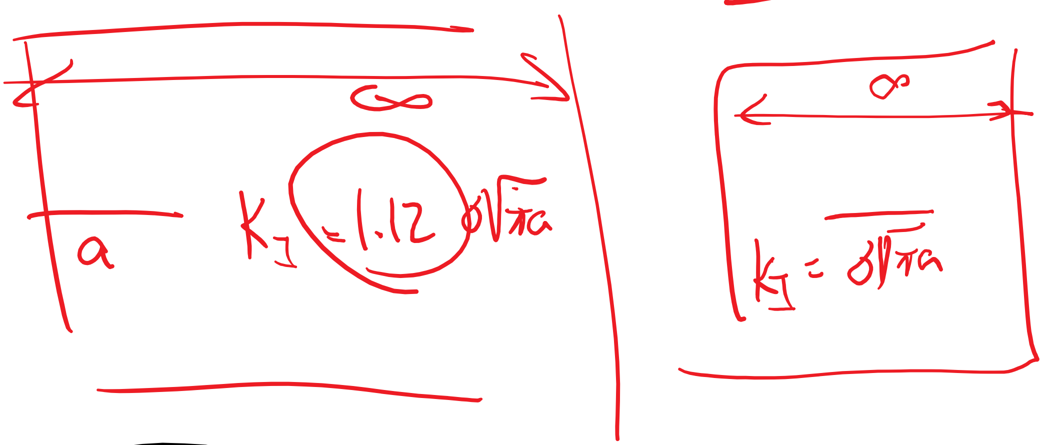
Another example:

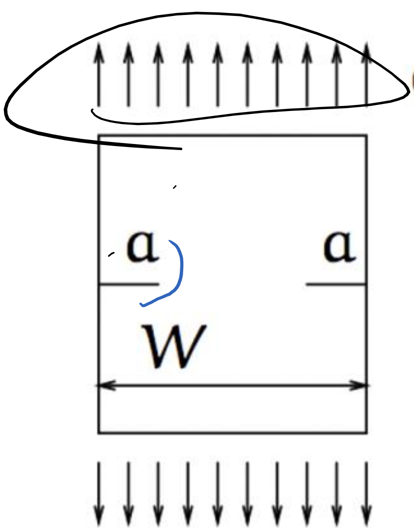


$h/b \geq 1$ and $a/b \leq 0.6$

$$K_I = \sigma\sqrt{\pi a}\left[1.12 - 0.23\left(\frac{a}{b}\right) + 10.6\left(\frac{a}{b}\right)^2 - 21.7\left(\frac{a}{b}\right)^3 + 30.4\left(\frac{a}{b}\right)^4\right]$$

how about $\frac{a}{b} \to 0$ short crack $\quad K_I = \underline{1.12}\,\sigma\sqrt{\pi a}$

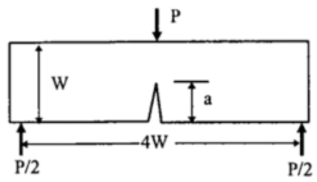




Q: $\frac{a}{W} \to 0 \quad f = ?\ 1.12$

$$K_I = \sigma\sqrt{a}\left[1.12\sqrt{\pi} + 0.76\frac{a}{W} - 8.48\left(\frac{a}{W}\right)^2 + 27.36\left(\frac{a}{W}\right)^3\right]$$

9. Single-edge notch bend (SENB), thickness B

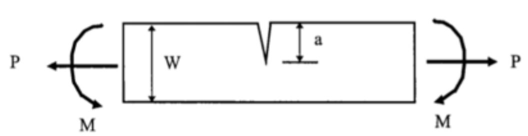


$B = W/2$

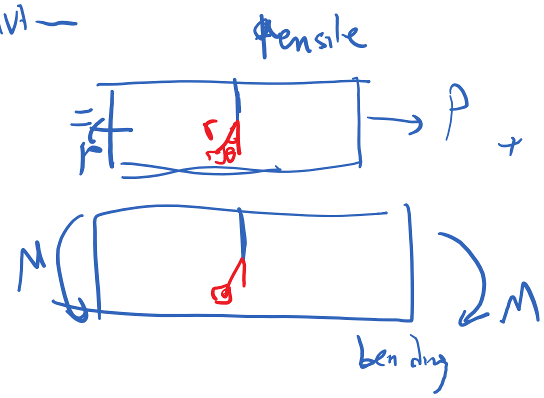
$$K_I = Y\frac{4P\sqrt{\pi}}{B\sqrt{W}}$$

$$Y = 1.63\left(\frac{a}{W}\right)^{1/2} - 2.6\left(\frac{a}{W}\right)^{3/2} + 12.3\left(\frac{a}{W}\right)^{5/2} - 21.3\left(\frac{a}{W}\right)^{7/2} + 21.9\left(\frac{a}{W}\right)^{9/2}$$

Example 1:



solve –



LEFM (**Linear** ....) -> We can use superposition.

$$\sigma_{ij} = \sigma_{ij}^{\text{tensile}} + \sigma_{ij}^{\text{bending}}$$

$$= K_I^{\text{tensile}} \frac{1}{\sqrt{2\pi r}} f_{ij}^{I}(\theta) + K_I^{\text{bending}} \frac{1}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$$

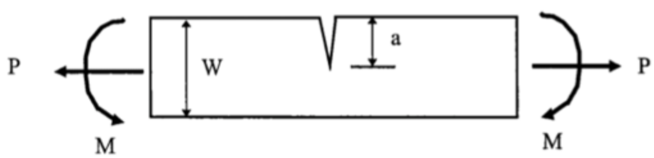
$$= \frac{K_I^{\text{tensile}} + K_I^{\text{bending}}}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$$

$$K_I = K_I^{\text{tensile}} + K_I^{\text{bending}}$$

Determine the stress intensity factor for an edge cracked plate subjected to a combined tension and bending.

$a/W = 0.2$

B thickness

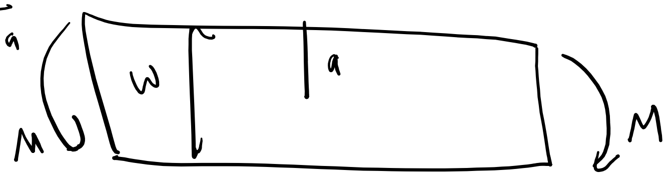


$K_I^{tensile} \approx \left(\frac{P}{W}\right) \quad (1.12)\sqrt{\pi a}$

far field stress

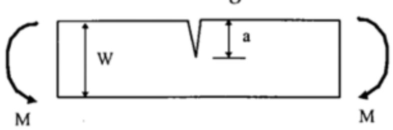
$a/W = 0.2$ small

$K_I^{bending} = 1.055 \times \frac{6M}{BW^2}\sqrt{\pi a}$



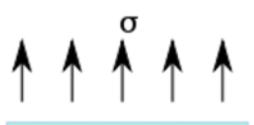
$K_I = 1.12 \frac{P}{BW}\sqrt{\pi a} + 1.055 \frac{6M}{BW^2}\sqrt{\pi a}$

5. Edge crack in a beam of width B subjected to bending



$K_I = f\left(\frac{a}{W}\right)\sigma\sqrt{\pi a}$ where $\sigma = \frac{6M}{BW^2}$

a/W	$f(a/W)$
0.1	1.044
0.2	$\boxed{1.055}$
0.3	1.125
0.4	1.257
0.5	1.500
0.6	1.915



I have shared a zip file with you that has SIF handbooks. Use can use that or tables in Anderson book to computer SIFs for HW assignments.